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ON THE DOMINATION AND INDEPENDENT SETS OF $G^M_{m,n}$ GRAPH

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ABSTRACT. In graph theory, the theory of domination has several applications in various fields of science and technology, which is considered as a turn up field of research. In real life, it is extremely important in fields like network desigh, wireless sensor networks,logistics, mobile computing, telecommunication and others, Problems with facility location, communication or electrical network monitoring can lead to dominance. Undirected graphs is one of the most excellent models in connection with distributed computation and parellel processing. A set $S \subset V$ is said to be a dominating set of a graph G if every vertex in $V - S$ is adjacent to atleast one vertex in S. The domination number $\gamma(G)$ of the graph G is the minimum cardinality of a dominating set of G. An independent dominating set $S \subset V$ is exists if no edges in the induced subgraph $\langle S \rangle$ and the independent dominating number $\gamma_i(G)$ is the minimum cardinality of an independent dominating set of G.

in this paper, some results on dominating sets and independent dominating sets of $G^M_{m,n}$ graph on a finite sebset of natural numbers are presented and the domination numbers are obtained for various values of m, n.

1. INTRODUCTION

A graph (V, E) is a mathematical construction that may be seen as a collection of edges. In a graph G, two vertices are said to be adjacent if an edge links them.

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Otherwise, they are said to be non-adjacent.We represent by $V(G)$ and $E(G)$ to the number of vertices and edges in a graph, respectively. The theory of dominance was first discussedby Berge in his book, published in 1958 [1], where the term "coefficient of external stability" was used to refer to a graph's domination number.Another book on graph theory was written by Ore [3] in 1962 in which the idea of domination and the terms "domination set" and "domination number"with the notation "$d(G)$" for the first time, are presented.

Let $G(V, E)$ be a graph, a set $\mathcal{S} \subset V$ is said to be a dominating set of G if for every $x \in V$ is an element of $\mathcal{S}$.if no vertex can be removed from a dominating set without compromising its dominance property,it is said to be a minimal dominating set (MDS) of G.The dominating number $\gamma(G)$of a graph G is the minimum cardinality of a dominating set in G.

A dominating set I is said to be an independent dominating set of G if no two vertices of I are adjacent to each other and the induced subgraph $\langle I \rangle$ has no edges.The independent domination number of G is the minimum cardinality of an independent dominating set of G and is denoted by $\gamma_i(G)$.

Let m, n be positive integers.The graph $G^M_{m,n}$ is a graph with vertex set $V = \{1, 2, 3, \ldots, n\}$ and the edge set $E = \{(x, y)/m \nmid x.y, where x, y \in V\}$

In this paper, results on dominating and independent dominating sets of $G^M_{m,n}$ graph are presented and the respective domination numbers are obtained for various values of m, n

2. THE $G^M_{m,n}$ GRAPH AND ITS PROPERTIES

when Caley built the digraph of a group in 1978, he opened the door for the development of many other semigroup graphs, including the divisibility graph,power graph,annihilator graph,and others. Chakrabarty [4] developed the concept of an undirected graph on a finite subset of natural numbers and it is denoted by $G^M_{m,n}$.

Definition 2.1. *for any two natural numbers m, n a graph $G^M_{m,n} = (V, E)$ is defined as a graph whose vertex set $V = \{1, 2, 3, \ldots, n\}$ where $x, y \in V$ are adjacent if and only if x.y is not divisible by m.*

if $m = 1$, then the graph $G^M_{m,n}$ is null graph some of the properties of $G^M_{m,n}$ given by Ivy [4] are

Lemma 2.1. *for $1 < m \leq n, G^M_{m,n}$ graph is disconnected.*

Lemma 2.2. *the graph $G^M_{m,n}$ is connected if $m > n$.*

Lemma 2.3. *The graph $G^M_{m,n}$ is complete when $n < m \leq n(n-1)$ and m is prime.*

3. DOMINATION IN A GRAPH $G^M_{m,n}$

A book on Graph theory was written by Berge [1] in 1958 in which the concept of domination number was introduced and the domination number was considered as the coefficient of external stability and used $d(G)$ for the notation of domination number of a graph G. Later Cockayne.et.al [5] gave the notation $\gamma(G)$ for the domination number of a Graph G. Nowakowski.et.al [7] studied the behaviour of the domination number on product of graphs.A unique minimum dominating set has been studied by Gunther et al [8].

Definition 3.1. *In a graph $G(V, E)$, a set $D \subset V$ is called as a dominating set if every vertex in V is an element in D or adjacent to an element in D. A dominating set of a graph G is called as a* ***Minimal dominating set*** *of G if no vertex is removed without destroying its dominance property.*

Definition 3.2. *The minimun cardinality of a dominating set in G is called as domination number of G and it is denoted by $\gamma(G)$. If $|D| = \gamma(G)$ then the dominating set D is called a minimum dominting set of G.*

Theorem 3.1. *If $1 < m \leq n$ and m is prime, then $\gamma(G^M_{m,n}) = [\frac{n}{m}] + 1$.*

Proof. Let $G^M_{m,n}$ be a graph with vertex set $V = \{1, 2, \ldots, n\}$, where $1 < m \leq n$ and m be a prime. Consider two distincts sets $S_1 = \{x \in V : gcd(x, m) = m\}$ and $S_2 = \{y \in V : gcd(y, m) = 1\}$. Clearly S_1 is the set of all m-multiples in V and $S_2 = V - S_1$.

So $|S_1| = [\frac{n}{m}]$ and $|S_2| = n - [\frac{n}{m}]$.

Then by the definition of $G^M_{m,n}$, the vertices of S_1 are isolated vertices and the vertices of S_2 are adjacent to each other and its degree is $n - [\frac{n}{m}] - 1$. consider a vertex v in S_2. Then v is adjacent to every other vertex in S_2.

So that any one vertex in S_2 along with $[\frac{n}{m}]$ isolated vertices of S_1 in V forms a dominating set for $G^M_{m,n}$ and it is of minimum cardinality. Hence the domination number of $G^M_{m,n}$ is $\gamma(G^M_{m,n}) = [\frac{n}{m}] + 1$. $\square$

Theorem 3.2. *If $1 < m \leq n$ and $m = kp$, where k be any positive integer and p be a prime, then $\gamma(G^M_{m,n}) = [\frac{n}{m}] + 1$.*

Proof. For $1 < m \leq n$, Let $G^M_{m,n}$ be a graph with vertex set $V = \{1, 2, 3, \ldots, n\}$, where k be any positive integer and p be a prime. here two possible cases arise.
case 1: Let $k = p$. then $m = p^2$, p be prime.

Divide the vertex set V into $S_1 = \{x \in V : gcd(x, m) = m\}$, $S_2 = \{y \in V : gcd(y, m) = p\}$ and $S_3 = \{z \in V : gcd(z, m) = 1\}$.

Clearly S_1 is the set of all m-multiples in V and $|S_1| = [\frac{n}{m}]$. And S_2 is the set of all p-multiples in $V - S_1$ and $|S_2| = [\frac{n}{p}] - [\frac{n}{m}]$. Also $S_3 = V - S_2$ and $|S_3| = n - [\frac{n}{p}]$.

Then by the definition of $G^M_{m,n}$, the vertices of S_1 are isolated vertices and the vertices of S_2 are adjacent to every verex of S_3 and not adjacent to a vertex of S_2. The degree of a vertex in S_2 is $n - [\frac{n}{p}]$. and the vertices in S_3 are adjacent to every vertex in S_2 and in S_3 and therefore the degree of a vertex in S_3 is $n - [\frac{n}{m}] - 1$.

Consider a vertex v in S_3. Then v is adjacent to every other vertex in S_2 and in S_3. So that any one vertex in S_3 along with $[\frac{n}{m}]$ isolated vertices of S_1 in V forms a dominating set for $G^M_{m,n}$ and it is of minimum cardinality. Hence the domination number of $G^M_{m,n}$ is, $\gamma(G^M_{m,n}) = [\frac{n}{m}] + 1$.
case 2: Let $k \neq p$. then $m = kp$, where p be prime and $k > 1$ be an integer.

Divide the vertex set V into $S_m = \{x \in V : gcd(x, m) = m\}$, $S_k = \{y \in V : gcd(y, m) = k\}$, $S_p = \{z \in V : gcd(z, m) = p\}$ and $S = \{v \in V : gcd(v, m) = 1\}$.

Clearly S_m is the set of all m-multiples in V and $|S_m| = [\frac{n}{m}]$. And S_k is the set of all k-multiples in $V - S_m$ and $|S_k| = [\frac{n}{k}] - [\frac{n}{m}]$.And S_p is the set of all p-multiples in $V - S_k$ and $|S_k| = [\frac{n}{p}] - [\frac{n}{m}]$. And $S = V - S_p$ and $|S| = n - [\frac{n}{k}] + [\frac{n}{m}] - [\frac{n}{p}]$.

Then by the definition of $G^M_{m,n}$, the vertices of S_m are isolated vertices and the vertices of S_k are adjacent to every vertex of S and not adjacent to the vertices of S_k and the vertices of S_p.The degree of a vertex in S_k is $n - [\frac{n}{p}] - 1$. And the vertices in S_p are adjacent to every vertex in S_p and also in S but not adjacent to the vertices of S_k. and therefore the degree of a vertex in S_p is $n - [\frac{n}{k}] - 1$.

Consider a vertex v in S. Then v is adjacent to every other vertex in V except $[\frac{n}{m}]$ isolated vertices of S_m. So that any one vertex in S along with $[\frac{n}{m}]$ isolated vertices of S_m in v forms a dominating set for $G^M_{m,n}$ and it is of minimum cardinality. Hence the domination number of $G^M_{m,n}$ is $\gamma(G^M_{m,n}) = [\frac{n}{m}] + 1$. □

Theorem 3.3. *If $m > n$ and m is prime, then $\gamma(G^M_{m,n}) = 1$.*

Proof. Let $G^M_{m,n}$ be a graph with vertex set $V = \{1, 2, \ldots, n\}$ where $m > n$ and m be a prime.

Then by the definition, $G^M_{m,n}$ is a complete graph.Hence any one vertex in V can forms a dominating set for $G^M_{m,n}$ and it is of minimum cardinality. Therefore the domination number of $G^M_{m,n}$ is $\gamma(G^M_{m,n}) = 1$. □

Theorem 3.4. *If $m > n$ and m is not a prime, then $\gamma(G^M_{m,n}) = 1$.*

Proof. Let $G^M_{m,n}$ be a graph with vertex set $V = \{1, 2, 3, \ldots, n\}$, where $m > n$ and m be not a prime. If $m > n$, then $m > 1.n$, it is true for any natural number n.

And $m \nmid 1.u$, for all $u = 2, 3, \ldots, n$. Therefore vertex $u = 1$ is adjacent to every other vertex in V and the set $S = \{u\}$ forms a dominating set and it is of minimum cardinality. Hence the domination number of $G^M_{m,n}$ is $\gamma(G^M_{m,n}) = 1$. □

4. INDEPENDENT DOMINATION IN A GRAPH $G^M_{m,n}$

Cockayne et al. [5, 6] introduced the independent domination number and its notation $i(G)$. Berge [2] has given the relationship between maximum degree and the independent domination number of a graph.

Definition 4.1. *A Dominating set I is said to be an independent dominating set of G if no two vertices of I are adjacent to eachother or the induced subgraph $\langle I \rangle$ has no edges. The independent domination number of G is the minimum cardinality of an independent dominating set of G and is denoted by $\gamma_i(G)$.*

Theorem 4.1. *For a graph $G^M_{m,n}$, if $1 < m \leq n$ and m is prime, then $\gamma_i(G^M_{m,n}) = [\frac{n}{m}] + 1$.*

Proof. Consider a graph $G^M_{m,n}$, where $1 < m \leq n$ and m is prime. From Theorem 3.1, we have seen that the domination number of $G^M_{m,n}$ is $\gamma(G^M_{m,n}) = [\frac{n}{m}] + 1$, if $1 < m \leq n$ and m is prime.

In a graph $G^M_{m,n}$, let I be any dominating set. Then clearly I contains $[\frac{n}{m}] + 1$ isolated vertices. That means $\langle I \rangle$ is a null graph.

Therefore I is independent dominating set with cardinality $[\frac{n}{m}] + 1$. □

Theorem 4.2. *For a graph $G^M_{m,n}$, if $1 < m \leq n$ and $m = kp$, where k is a positive integer and p is a prime, then $\gamma_i(G^M_{m,n}) = [\frac{n}{m}] + 1$.*

Proof. Consider a graph $G^M_{m,n}$, where $1 < m \leq n$. Let $m = kp$, where $k \in Z^+$ and p be a prime.

If $k = p$. then $m = p^2$.

Consider two subsets $I_1 = \{u \in V : gcd(u, m) = m\}$ and $I_2 = \{v \in V : gcd(v, m) = 1\}$. Now set I_1 contains all isolated vertices of $G^M_{m,n}$ and $|I_1| = [\frac{n}{m}]$. and I_2 is a set whose vertices are relatively prime to m and every vertex in I_2 is adjacent to all other vertices of $G^M_{m,n}$. And the cardinality of I_2 is $n - [\frac{n}{p}]$. Denote a set $I = I_1 \cup \{v\}$, where $\{v\} \subset I_2$.

So, the set I is a dominating set of $G^M_{m,n}$, since every vertex in $V - I$ is adjacent to some vertex in I with $\gamma(G^M_{m,n}) = [\frac{n}{m}] + 1$. also the induced subgraph $\langle I \rangle$ of I has no edges. Hence I is an independent dominating set of $G^M_{m,n}$. Therefore $\gamma_i(G^M_{m,n}) = [\frac{n}{m}] + 1$.

Suppose $k \neq p$. Consider two subsets $I_1 = \{u \in V : gcd(u, m) = m\}$ and $I_2 = \{v \in V : gcd(v, m) = 1\}$. their cardinalities, $|I_1| = [\frac{n}{m}]$, $|I_2| = n - [\frac{n}{k}] + [\frac{n}{m}] - [\frac{n}{p}]$. Then by the definition of $G^M_{m,n}$, the vertices of I_1 are isolated vertices and a vertex v in I_2 is adjacent to every other vertex in V. So that any one vertex in I_2 along with $[\frac{n}{m}]$ isolated vertices of S_m in V forms an independent dominating set for $G^M_{m,n}$ and it is minimum. Therefore the independent domination number of $G^M_{m,n}$ is,
$\gamma_i(G^M_{m,n}) = [\frac{n}{m}] + 1$. □

Theorem 4.3. *For a graph $G^M_{m,n}$, if $m > n$ and m is prime, then $\gamma_i(G^M_{m,n}) = 1$.*

Proof. By the definition of $G^M_{m,n}$, if $m > n$ and m is a prime, $G^M_{m,n}$ is a complete graph. Then any v vertex in V can forms a dominating set for $G^M_{m,n}$ and $\{v\}$ is a singleton set. Hence $\{v\}$ forms an independent domination set and therefore the independent domination number of $G^M_{m,n}$ is, $\gamma_i(G^M_{m,n}) = 1$. □

Theorem 4.4. *For a graph $G^M_{m,n}$, if $m > n$ and m is not a prime, then $\gamma_i(G^M_{m,n}) = 1$.*

Proof. In a graph $G^M_{m,n}$, if $m > n$, then $m > 1.n$, it is true for $n \in Z^+$. It is clear that $m \nmid 1.u$, for all $u = 2, 3, \ldots, n$. Then, vertex $u = 1$ is adjacent to every other vertex in $G^M_{m,n}$, and the set $I = \{1\}$ forms a dominating set and it is singleton. Therefore the independent domination number of $G^M_{m,n}$ is, $\gamma_i(G^M_{m,n}) = 1$. □

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