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A NOTE ABOUT THE WAY TO GENERATE SINGULAR MATRIX

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ABSTRACT. In this brief note, we provide, as far as we know, a simple way to demonstrate that when we multiply $k \times n$ matrix with $n \times k$ one, for $k > n$, we always obtain singular $k \times k$ matrix as a result. Some additional results are also provided.

1. INTRODUCTION

Recently [5] have proven than when we multiply $k \times n$ matrix with $n \times k$ one, for $k > n$, we always obtain singular $n \times n$ matrix as result. In this brief note, we prove this assertion more elegantly and simply. The methodology is useful for getting commutator matrices. Finally, some results related to the characteristic polynomial of the resulting matrix obtained by the above product and the trace of this matrix appear in Section 2.

2. MAIN RESULTS

The following result provides, as far as we know, a new and simple way to demonstrate that when we multiply $k \times n$ matrix with $n \times k$ one, for $k > n$, we

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always obtain a singular $n \times n$ matrix as a result. We recall that for a complex matrix $\boldsymbol{Y} = [y_{ij}] \in \mathcal{M}_{n\times n}(\mathbb{C})$, the trace of $\boldsymbol{Y}$ is equal to $\text{tr}(\boldsymbol{Y}) = \sum_{j=1}^{n} y_{jj}$.

Theorem 2.1. *Let $\boldsymbol{A} = [a_{ij}] \in \mathcal{M}_{k\times n}(\mathbb{C})$ and $\boldsymbol{B} = [b_{ij}] \in \mathcal{M}_{n\times k}(\mathbb{C})$ be two complex matrices being $k > n > 0$. Then it is verified that the matrix $\boldsymbol{C} = \boldsymbol{AB}$ is singular.*

Proof. Consider the matrices $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \mathbf{0} \end{pmatrix} \in \mathcal{M}_{k\times k}(\mathbb{C})$ and $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{B} \\ \mathbf{0} \end{pmatrix} \in \mathcal{M}_{k\times k}(\mathbb{C})$, where the last $k - n$ columns of $\boldsymbol{M}$ and the last $k - n$ rows of $\boldsymbol{P}$ consist of zeros. It is simple to see that

$$\boldsymbol{C} = \boldsymbol{AB} = \boldsymbol{MP}.$$

Now, because $\boldsymbol{M}$ and $\boldsymbol{P}$ are singular, we conclude that $\boldsymbol{C}$ is also singular. □

As a consequence of this result we have that $\lambda = 0$ is a characteristic value of $\boldsymbol{C} = [c_{ij}] \in \mathcal{M}_{k\times k}(\mathbb{C})$. Now, if $\text{rank}(\boldsymbol{C}) = r < k$, then the characteristic polynomial of $\boldsymbol{C}$ can be written (see [2, Lemma (i)], [4, Theorem 16.2] and [6]) as

$$p_{\boldsymbol{C}}(\lambda) = |\boldsymbol{C} - \lambda \boldsymbol{I}_k| = |\boldsymbol{MP} - \lambda \boldsymbol{I}_k| = |\boldsymbol{N} - \lambda \boldsymbol{I}_r| = \lambda^{k-r} p_{\boldsymbol{N}}(\lambda), \tag{2.1}$$

where $\boldsymbol{I}_s$ is the identity matrix of order s, $\boldsymbol{N} \in \mathcal{M}_{r\times r}(\mathbb{C})$ is the matrix satisfying $\boldsymbol{PM} = \begin{pmatrix} \boldsymbol{N} & \mathbf{0} \end{pmatrix} \in \mathcal{M}_{k\times k}(\mathbb{C})$ (the matrix with the last $k - r$ columns consist of zeros) and $p_{\boldsymbol{N}}(\lambda)$ is a polynomial of degree r. Then, the characteristic polynomial of $\boldsymbol{C}$ can be found without using this matrix.

As a consequence of this result, we have the following corollary.

Corollary 2.1. *It is verified that*

$$tr\,(\boldsymbol{C}) = tr\,(\boldsymbol{MP}) = tr\,(\boldsymbol{N}) = tr(\boldsymbol{PM}).$$

Proof. It is obtained in straightforward way by having into account (2.1) and the definition of $\boldsymbol{PM} = \begin{pmatrix} \boldsymbol{N} & \mathbf{0} \end{pmatrix}$. □

Recall that a matrix is unitary if and only if its columns form an orthonormal (orthogonal and each of length 1) basis and that the matrix $\boldsymbol{Y} = [y_{ij}] \in \mathcal{M}_{n\times n}(\mathbb{C})$ is a commutator, i.e. $\boldsymbol{Y} = \boldsymbol{RS} - \boldsymbol{SR}$ for some matrices $\boldsymbol{R}, \boldsymbol{S},$ if and only if $\text{tr}(\boldsymbol{Y}) = 0$ (see for instance [7]).

Theorem 2.2. *Given two arbitrary matrices* $\boldsymbol{A} = [a_{ij}] \in \mathcal{M}_{k\times n}(\mathbb{C})$ *and* $\boldsymbol{B} = [b_{ij}] \in \mathcal{M}_{n\times k}(\mathbb{C})$ *with* $k > n$ *and the matrices* $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \end{pmatrix} \in \mathcal{M}_{k\times k}(\mathbb{C})$ *and* $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{pmatrix} \in \mathcal{M}_{k\times k}(\mathbb{C})$*. Then, we have the following:*

(i) *The matrix given by* $\boldsymbol{X} = \boldsymbol{MP} - \boldsymbol{PM} \in \mathcal{M}_{k\times k}(\mathbb{C})$ *is a commutator.*

(ii) *There exists a unitary matrix* $\boldsymbol{W} \in \mathbb{C}^{k\times k}$ *such that* $\boldsymbol{WXW}^{-1}$ *has all entries on its main diagonal equal to zero.*

Proof. $i)$ is direct since $\mathrm{tr}(\boldsymbol{X}) = 0$. (ii) corresponds to a well-known result, for example, in [3]. □

Some other results related to matrices with zero traces can be seen in [1].

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