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GRAPHIC TOPOLOGY ON FUZZY GRAPHS

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ABSTRACT. In this paper, we study the graphic topology $\mathcal{T}_G$ for a fuzzy graph. We give some properties of this topology, in particular we prove that $\mathcal{T}_G$ is an Alexandroff topology and when two graphs are isomorphic, their graphic topologies will be homeomorphic. We give some properties matching graphs and homeomorphic topology spaces. Finally, we investigate the connectedness of this topology and some relations between the connectedness of the graph and the topology $\mathcal{T}_G$.

1. INTRODUCTION

When we have a topology on a set, we can study some properties of this set (space) that are preserved according to continuous deformations. For a discrete sets, Golomb [9] define a topology for the integers. After that, in [13, 18], the authors define some Alexandroff topologies for connected graphs. Later, Jafarian Amiri et al. [12] introduced an Alexandroff topology for every locally finite graphs called graphic topology.

The Alexandroff spaces were given by P. Alexandroff in 1937 in [2] under the name Diskrete Räume spaces. Recall that an Alexandroff space is a topological

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space (X, T) satisfying every intersection of elements in T is also in T. As consequence, the topology has a unique minimal basis and some interesting results and applications [11, 15–17, 23, 24].

Having a topology for a graph can solve many problems in economy domain, the traffick flow study [3, 13, 18] and many other areas. So, a lot of topologies were defined on graphs and many properties were studied [1, 4, 7, 10, 14, 20, 22, 25, 26].

In this paper, we define the graphic topology on a fuzzy graph. A fuzzy graph, introduced by Rosenfeld [19] in 1975, is a graph with different degrees of vagueness associated with its vertices and edges. This type of graphs has a large application in networks like internet and power grids.

The outline of this paper is as follows. Section 2 is devoted to some preliminaries providing a basic definitions and properties of a topological space and fuzzy graph.

In section 3, we prove elementary results for the graphic topology and we give a nontrivial open set and a nontrivial closed set.

In section 4, some advanced properties of graphic topology for fuzzy graph are given.

Section 5 is added to summarize some relation ship between the connectedness of the graphic topology and the connectedness of the graph.

2. Preliminaries

In this section, we recall some basic definitions and properties of a topological space and fuzzy graphs. For more details, we can see [5, 6, 8, 16, 21, 24].

Let X a non empty set. A topology for the set X is a collection T of subsets of X (i.e $T \subset \mathcal{P}(X)$) satisfying three conditions:

(i) $\emptyset,\ X \in T$;

(ii) For all $A,\ B \in T$, we have $A \cap B \in T$;

(iii) For all $\{A_i\}_{i \in I}$ a family of elements in T, we have $\cup_{i \in I} A_i \in T$.

An element A of the topology T is called an open set. The topology $T = \mathcal{P}(X)$ is called the discrete topology while $T = \{\emptyset, X\}$ is the trivial topology for X.

In this paper, we will introduce and define a topology on fuzzy graph.

Definition 2.1. *Suppose that V is a non empty set and $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$ are two maps. If the following conditions:*

(i) $\mu(a,b) \leq \sigma(a) \wedge \sigma(b)$, *where* $\sigma(a) \wedge \sigma(b) = \min\{\sigma(a), \sigma(b)\}$*;*
(ii) $\mu(a,b) = \mu(b,a)$, *for all* $a, b \in V$*;*
(iii) $\mu(a,a) = 0$, *for all* $a \in V$*,*

we say that $G = (V, \sigma, \mu)$ *is a fuzzy graph and an element* a *of* V *is called a vertex of the graph* G*.*

Example 1. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph given in Figure 1 with* $V = \{a, b, c, d\}$*. The fuzzy subset* σ *of* V *is defined as* $\sigma(a) = 1, \sigma(b) = 0.5, \sigma(c) = 0.4$ *and* $\sigma(d) = 0.25$*. The fuzzy relation* μ *is given by* $\mu(a,b) = 0.2, \mu(b,c) = 0.3, \mu(c,d) = 0.1$*,* $\mu(d,a) = 0.15$ *and* $\mu(a,c) = \mu(b,d) = 0$*.*

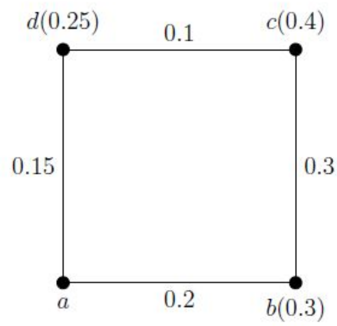

Figure 1. Fuzzy graph in Example 1

A particular type of fuzzy graphs that we will use in section 5 is the fuzzy bipartite graph.

Definition 2.2. *A fuzzy bipartite graph is a fuzzy graph* $G = (V, \sigma, \mu)$ *such that* $V = V_1 \cup V_2$*,* $V_1 \cap V_2 = \emptyset$ *and* $\mu(a,b) = 0$ *for any* $a \in V_1$ *and* $b \in V_2$*.*

For $G = (V, \sigma, \mu)$ be a fuzzy graph. We denote

$$V^\star = \{x \in V;\ \sigma(x) > 0\}.$$

Definition 2.3. *A path of length* n *in a fuzzy graph is a sequence of distinct vertices* $x_0, x_1, \cdots, x_{n-1}, x_n$ *with* $\mu(x_{i-1}, x_i) > 0$ *for* $i = 1, \cdots, n$*.*

Definition 2.4. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph. For any distinct vertices* $x, y \in V^\star$*, we denote* $d(x,y)$ *the length of the shortest path joining* x *and* y*. If there is no path between then, we set* $d(x,y) = +\infty$*.*

Example 2. *In the following fuzzy bipartite graph* $d(a,c) = 2$*.*

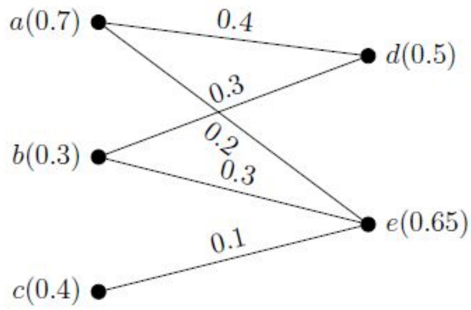

Figure 2. A fuzzy bipartite graph having two paths joining a and d

Now, for all $x \in V^\star$ we set

$$N_x = \{y \in V^\star;\ \mu(x, y) > 0\}, \tag{2.1}$$

the set of the neighbors of x (the neighborhood of x in G). We remark that

$$y \in N_x \text{ if, and only if, } x \in N_y. \tag{2.2}$$

Definition 2.5. *A vertex x of G is called an isolated vertex if $N_x = \emptyset$.*

We remark that any non-isolated vertex is in $V^\star$.

Definition 2.6. *The degree of a vertex x in G is the cardinal of its neighborhood N_x, that is*

$$deg(x) = |N_x|.$$

We denote $\Delta = \max\{deg(x),\ x \in V\}$ and $\delta = \min\{deg(x),\ x \in V\}$.

In what follows, we suppose that the fuzzy graph G is without isolated vertices. Let

$$\mathcal{S}_G = \{N_x;\ x \in V^\star\}. \tag{2.3}$$

Since $V^\star = \bigcup_{x \in V^\star} N_x$, the set $\mathcal{S}_G$ is a subbasis of a topology $\mathcal{T}_G$ for $V^\star$, called the graphic topology of G.

Example 3. *For the fuzzy graph in Example 2, we have*

$$\mathcal{S}_G = \{\{e\}, \{d, e\}, \{a, b\}, \{a, b, c\}\},$$

then the basis is

$$\mathcal{B} = \{\{e\}, \{d, e\}, \{a, b\}, \{a, b, c\}\}$$

and

$$\mathcal{T}_G = \{\emptyset, \{e\}, \{d,e\}, \{a,b\}, \{a,b,c\}, \{a,b,e\}, \{a,b,d,e\}, \{a,b,c,e\}, \{a,b,c,d,e\}\}.$$

In the next section, we will prove that $(V^\star, \mathcal{T}_G)$ is an Alexandroff space under some assumptions. So, recall the following definitions.

Definition 2.7. *A topological space (X, T) is called an Alexandroff space if any intersection of elements in T is an element in T. We say, also, the topology is an Alexandroff topology.*

Definition 2.8. *A fuzzy graph $G = (V, \sigma, \mu)$ is called locally finite fuzzy graph if N_x is a finite set, for all $x \in V^\star$.*

Definition 2.9. *Let (X, T) be a topological space and $A \subset X$.*

(i) *We denote A^c the complement of A in X, that is*

$$A^c = \{x \in X;\ x \notin A\}.$$

(ii) *A is called a closed set of X if A^c is an open set of X.*
(iii) *We denote $\overline{A}$ the smallest closed set of X containing A.*

We end this section by the following definition that we will use in section 4.

Definition 2.10. *A fuzzy graph $G = (V, \sigma, \mu)$ is called complete if*

$$\mu(a,b) = \sigma(a) \wedge \sigma(b), \text{ for all } a, b \in V.$$

Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The complement of G is the fuzzy graph defined by $\overline{G} = (V, \sigma, \overline{\mu})$, where

$$\overline{\mu}(a,b) = \sigma(a) \wedge \sigma(b) - \mu(a,b) = \min(\sigma(a), \sigma(b)) - \mu(a,b),$$

for all $a, b \in V$.

In this paper, by a fuzzy graph we will mean a simple fuzzy graph without isolated vertex.

3. Elementary Results

In the sequel, a fuzzy graph G will be locally finite without isolated vertices.

Theorem 3.1. *Suppose that $G = (V, \sigma, \mu)$ is a fuzzy graph. Then, $\mathcal{T}_G$ is an Alexandroff topology on $V^\star$.*

Proof. We have to prove that any intersection of open sets is an a open set. Since this topology $\mathcal{T}_G$ is defined by a subbasis, it is sufficient to prove that any intersection of elements in $\mathcal{S}_G$ is an open set.
Let $S \subset V^\star$ and consider $\cap_{x\in S} N_x$.

(i) If $\cap_{x\in S} N_x = \emptyset$, then $\cap_{x\in S} N_x$ is an open set.
(ii) If $\cap_{x\in S} N_x \neq \emptyset$, then let $z \in \cap_{x\in S} N_x$. We have $z \in N_x$, for all $x \in S$. From (2.2), we have $x \in N_z$, for all $x \in S$. So, $S \subseteq N_z$. But N_z is a finite set and so S is also finite. Hence, $\cap_{x\in S} N_x$ is an open set.

So, $(V^\star, \mathcal{T}_G)$ is an Alexandroff space. □

As consequence of the Theorem 3.1, the topology $\mathcal{T}_G$ has minimal basis $\mathcal{U} = \{U_x,\ x \in V^\star\}$, where $U_x = \cap A$, for all $A \in \mathcal{T}_G$ such that $x \in A$. That is, U_x is the smallest open set satisfying $x \in U_x$, [2, 5, 8, 16, 23, 24, 26]. We have the following characterisation of U_x.

Theorem 3.2. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then, $U_x = \cap_{y\in N_x} N_y$ and U_x is finite, for all x a vertex in $V^\star$.*

Proof. Let $x \in V^\star$. We have $N_x \neq \emptyset$ since x is not isolated. If we consider

$$\bigcap_{y\in N_x} N_y,$$

we have N_y is an open set and so by Theorem 3.1, it is an open set. For all $y \in N_x$, $x \in N_y$. So, $x \in N_y$, for all $y \in N_x$. Then, $x \in \bigcap_{y\in N_x} N_y$. Therefore

$$U_x \subseteq \bigcap_{y\in N_x} N_y.$$

Conversely, U_x is the minimal element in $\mathcal{T}_G$ containing x. Since $\mathcal{S}_G$ is a subbasis for the topology so, there exists $S \subseteq V^\star$ such that

$$U_x = \bigcap_{y\in S} N_y.$$

We have for all $y \in S$, $x \in N_y$. That is, for all $y \in S$, $y \in N_x$. Then, $S \subseteq N_x$

$$\bigcap_{y \in N_x} N_y \subseteq \bigcap_{y \in S} N_y.$$

But $\bigcap_{y \in S} N_y = U_x$ and so the result follows. □

Corollary 3.1. *Let G be a fuzzy graph and let $x, y \in V^\star$ two distinct vertices. Then, we have*

(i) *If $N_x = \{y\}$, then $U_x = N_y$.*
(ii) *If $y \in N_x$, then $U_x \subset N_y$.*
(iii) *If $U_y \subset N_x$, then $U_x \subset N_y$.*

Proof.

(i) If $N_x = \{y\}$, then $U_x = \cap_{z \in N_x} N_z = N_y$.
(ii) If $y \in N_x$, then $U_x = \cap_{z \in N_x} N_z \subset N_y$.
(iii) If $U_y \subset N_x$, then $y \in U_y \subset N_x$. From (ii), $U_x \subset N_y$.

□

Proposition 3.1. *Suppose that G is a fuzzy graph. Then we have: For any x and $y \in V^\star$, $y \in U_x$ equivalent to $N_x \subset N_y$. That is, $U_x = \{y \in V^\star;\ N_x \subset N_y\}$.*

Proof. From the Theorem 3.2, $U_x = \cap_{z \in N_x} N_z$. Then, we have

$$\begin{aligned} y \in U_x &\Leftrightarrow y \in N_z, \forall z \in N_x \\ &\Leftrightarrow \forall z \in N_x, y \in N_z \\ &\Leftrightarrow \forall z \in N_x, z \in N_y \\ &\Leftrightarrow N_x \subset N_y. \end{aligned}$$

□

Corollary 3.2. *Suppose that G is a fuzzy graph and $x, y \in V^\star$. If $y \in U_x$, then $N_x \subset N_y$ and so $deg(x) \leq deg(y)$.*

Proposition 3.2. *Let G be a fuzzy graph and $x \in V^\star$. Then, $U_x \cap N_x = \emptyset$ and $U_x \cap U_y = \emptyset$, for all $y \in N_x$.*

Proof. If $y \in U_x \setminus \{x\}$, then $N_x \subset N_y$ and $y \neq x$. Since $N_x \neq \emptyset$, there exists $z \in N_x$ and so $z \in N_y$. Therefore, $d(x, y) = 2$ and we get

$$U_x \subset \{x\} \cup \{y \in V^\star;\ d(x, y) = 2\}. \tag{3.1}$$

Then, $U_x \cap N_x = \emptyset$ and also $N_x \subset U_x^c$. If $y \in N_x$, then $U_y \subset N_x$. So, $U_y \subset U_x^c$. Hence $U_x \cap U_y = \emptyset$. □

In the Example 2, $d(a, c) = d(a, b) = 2$. But $U_a = \{a, b\}$ and so the two sets in (3.1) are not equal.

From the Proposition 3.2, we have: $U_x \subset N_x^c$, so $\{x\} \subset U_x \subset N_x^c$. Then, $\overline{\{x\}} \subset \overline{U_x} \subset N_x^c$, and $N_x \subset U_x^c$, so $\overline{N_x} \subset U_x^c$.

Proposition 3.3. *Let G be a fuzzy graph and $x \in V^\star$. Then, $y \in \overline{\{x\}}$ if and only if $N_y \subset N_x$. This means, $\overline{\{x\}} = \{y \in V^\star;\ N_y \subset N_x\}$.*

Proof. $y \in \overline{\{x\}}$ if and only if, for all open set O containing y, $O \cap \{x\} \neq \emptyset$. But, this is equivalent to $U_y \cap \{x\} \neq \emptyset$. So, $y \in \overline{\{x\}}$ if and only if $x \in U_y$ and the result follows by Proposition 3.1. □

Proposition 3.4. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then, $U = \{x \in V^\star,\ deg(x) = \Delta\}$ is an open set for $(V^\star, \mathcal{T}_G)$.*

Proof. Suppose that $x \in U$. We will prove that $x \in U_x \subset U$. Let $y \in U_x$, from corollary 3.2, $deg(x) \leq deg(y)$ and so $deg(y) = \Delta$. Hence, $y \in U$, that is $U_x \subset U$ and so $x \in U_x \subset U$. □

Proposition 3.5. *Suppose that $G = (V, \sigma, \mu)$ is a fuzzy graph. Then, $V = \{a \in V^\star,\ deg(a) = \delta\}$ is a closed set for $(V^\star, \mathcal{T}_G)$.*

Proof. We will prove that $\overline{V} \subset V$. Since $(V^\star, \mathcal{T}_G)$ is an Alexandroff space, we have any union of closed sets is a closed set and so,

$$\overline{V} = \bigcup_{a \in V} \overline{\{a\}}.$$

Let $y \in \overline{V}$, there exists $a \in V$ such that $y \in \overline{\{a\}}$. From proposition 3.3, $N_y \subset N_a$, then $deg(y) \leq deg(a)$ and so, $deg(y) = \delta$. Hence, $y \in V$ and the proof follows. □

4. Advanced Properties of Graphic Topology

Definition 4.1. *Consider $(V_1, \mathcal{T}_1)$ and $(V_2, \mathcal{T}_2)$ two topological spaces and let $\psi : V_1 \to V_2$ be a function. ψ is called a continuous or a map if for all $O \in \mathcal{T}_2$, $\psi^{-1}(O) \in \mathcal{T}_1$.*

When the function ψ is bijective and, ψ and ψ^{-1} are continuous, we say that ψ is an homeomorphism and the two spaces V_1 and V_1 are homeomorphic.

Definition 4.2. *Consider $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ two fuzzy graphs. We say that G_1 and G_2 are isomorphic if there exists a bijection $\psi : V_1 \to V_2$ satisfying*

$$\mu_2(\psi(x), \psi(y)) = \mu_1(x, y), \text{ for all } x, y \in V_1 \tag{4.1}$$

and

$$\sigma_2(\psi(x)) = \sigma_1(x), \text{ for all } x \in V_1. \tag{4.2}$$

Theorem 4.1. *Suppose that $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ are two fuzzy graphs. If G_1 and G_2 are isomorphic, then the topological spaces $(V_1^\star, \mathcal{T}_{G_1})$ and $(V_2^\star, \mathcal{T}_{G_2})$ are homeomorphic.*

Proof. Since the fuzzy graphs are isomorphic, there exists $\psi : V_1 \to V_2$ a bijection map satisfying (4.1) and (4.2). Let B an open set of $(V_2^\star, \mathcal{T}_{G_2})$ in the subbasis $\mathcal{S}_{G_2}$. So, there exists $y \in V_2^\star$ such that $B = N_y$. We set $x = \psi^{-1}(y)$, we get

$$\begin{aligned}
\psi^{-1}(B) &= \{z \in V_1,\ \psi(z) \in B\} \\
&= \{z \in V_1,\ \psi(z) \in N_y\} \\
&= \{z \in V_1,\ \mu_2(y, \psi(z)) > 0\} \\
&= \{z \in V_1,\ \mu_2(\psi(x), \psi(z)) > 0\} \\
&= \{z \in V_1,\ \mu_1(x, z) > 0\} \\
&= \{z \in V_1^\star,\ \mu_1(x, z) > 0\} \\
&= \{z \in V_1^\star,\ z \in N_x\} \\
&= N_x.
\end{aligned}$$

So, $\psi^{-1}(B) \in \mathcal{S}_{G_1}$, that is $\psi^{-1}(B)$ is an open set of $V_1^\star$. Therefore $\psi^{-1}(B)$ is an open set of $V_1^\star$, for all B an open set of $V_2^\star$. Hence, the function ψ is continuous. In a similar way, we prove that ψ^{-1} is continuous. So, the two spaces $(V_1^\star, \mathcal{T}_{G_1})$ and $(V_2^\star, \mathcal{T}_{G_2})$ are homeomorphic. ☐

Remark 4.1. *If we consider a fuzzy cycle graph C_n and a complete graph K_n of order $n > 4$, then the graphic topology $\mathcal{T}_{C_n}$ is the discrete topology as $\mathcal{T}_{K_n}$ but the two spaces are not homeomorphic. So, the converse of the Theorem 4.1 is not true.*

In what follows, without loss of generality, we suppose that $V^\star = V$.

Proposition 4.1. *Suppose that $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ are two fuzzy graphs and $\psi : V_1 \to V_2$ a function. The following two assertions are equivalent.*

(i) *ψ is a continuous function from $(V_1, \mathcal{T}_{G_1})$ to $(V_2, \mathcal{T}_{G_2})$.*
(ii) *$N_y \subset N_x \Longrightarrow N_{\psi(y)} \subset N_{\psi(x)}$, for all $x, y \in V_1$.*

Proof. Suppose that ψ is continuous. If $N_y \subset N_x$, then by Proposition 3.1, $x \in U_y$. Consider the open set $U_{\psi(y)}$, we have $y \in \psi^{-1}\Big(U_{\psi(y)}\Big)$ and so $U_y \subset \psi^{-1}\Big(U_{\psi(y)}\Big)$. We get $x \in \psi^{-1}\Big(U_{\psi(y)}\Big)$, that is, $\psi(x) \in U_{\psi(y)}$. From the Proposition 3.1, we obtain $N_{\psi(y)} \subset N_{\psi(x)}$.

Conversely, suppose that we have (ii). Let B an open set for $\mathcal{T}_{G_2}$, we will prove that $\psi^{-1}(B)$ is an open set for $\mathcal{T}_{G_1}$. Let $x \in \psi^{-1}(B)$, we have $\psi(x) \in B$ and so $U_{\psi(x)} \subset B$, since B is an open set. Now, for all $y \in U_x$, $N_x \subset N_y$ (Proposition 3.1). Then, $N_{\psi(x)} \subset N_{\psi(y)}$ and so $\psi(y) \in U_{\psi(x)}$. Therefore $\psi(y) \in U_{\psi(x)} \subset B$ and so, $y \in \psi^{-1}(B)$. We get $U_x \subset \psi^{-1}(B)$, for all $x \in \psi^{-1}(B)$ and the proof of the Proposition 4.1 follows. □

We have the following characterisation of homeomorphic graphic topology spaces.

Theorem 4.2. *Suppose that $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ are two fuzzy graphs and $\psi : V_1 \to V_2$ be a bijective function. Then, ψ is an homeomorphism between the topological spaces $(V_1, \mathcal{T}_{G_1})$ and $(V_2, \mathcal{T}_{G_2})$ if, and only if,*

$$N_y \subset N_x \Longleftrightarrow N_{\psi(y)} \subset N_{\psi(x)}, \ \textit{for all } x, y \in V_1. \tag{4.3}$$

Proof. Suppose that ψ is an homeomorphism. Let $x, y \in V_1$, from Proposition 4.1 and the fact that ψ is continuous, we have

$$N_y \subset N_x \Longrightarrow N_{\psi(y)} \subset N_{\psi(x)}.$$

Now, if we apply the same result to ψ^{-1}, we get

$$N_{\psi(y)} \subset N_{\psi(x)} \Longrightarrow N_y \subset N_x,$$

and then (4.3) follows.

Conversely, Suppose (4.3) is true. From Proposition 4.1, the function ψ is continuous. Let $a, b \in V_2$, if $N_a \subset N_b$, then $N_{\psi(\psi^{-1}(a))} \subset N_{\psi(\psi^{-1}(b))}$. But

$$N_{\psi(\psi^{-1}(a))} \subset N_{\psi(\psi^{-1}(b))} \Longrightarrow N_{\psi^{-1}(a)} \subset N_{\psi^{-1}(b)}.$$

We get

$$N_a \subset N_b \Longrightarrow N_{\psi^{-1}(a)} \subset N_{\psi^{-1}(b)}, \quad \text{for all } a, b \in V_2. \tag{4.4}$$

From Proposition 4.1, the function ψ^{-1} is continuous. Therefore ψ is an homeomorphism. $\square$

Proposition 4.2. *Suppose that $G = (V, \sigma, \mu)$ is a fuzzy graph and $V^\star = V$. If $\mu(a, b) = \frac{1}{2}(\sigma(a) \wedge \sigma(b))$, then $\overline{G} = G$ and $\mathcal{T}_{\overline{G}} = \mathcal{T}_G$, where $\overline{G}$ is the complement of G.*

Proof.

$$\begin{aligned}
\overline{\mu}(a, b) &= \sigma(a) \wedge \sigma(b) - \mu(a, b) \\
&= \sigma(a) \wedge \sigma(b) - \frac{1}{2}(\sigma(a) \wedge \sigma(b)) \\
&= \frac{1}{2}(\sigma(a) \wedge \sigma(b)) \\
&= \mu(a, b).
\end{aligned}$$

So, G and $\overline{G}$ have the same graphic topology. $\square$

Next, we end this section by giving a necessary and sufficient condition for compactness of graphic topology such that a topological space V is called compact if each open cover of V has a finite subcover. We have the result.

Theorem 4.3. *Suppose that $G = (V, \sigma, \mu)$ is a fuzzy graph. The topological space $(V^\star, \mathcal{T}_G)$ is compact iff $V^\star$ is finite.*

Proof. First, when $V^\star$ is finite, it is clear that from any open cover we have a finite subcover.

Conversely, suppose that $(V^\star, \mathcal{T}_G)$ is a compact topological space. If we consider the open cover given by the minimal basis $\mathcal{U}$ it has a finite subcover But it is minimal as basis, Therefore $\mathcal{U}$ is finite. Hence, $V^\star$ is finite. $\square$

5. Graphic Topology and Connectedness

Let $(V, \mathcal{T})$ be a topological space. The empty set is called trivial open set. An open set for V is called proper if it is not equal to V.

Definition 5.1. *A topological space $(V, \mathcal{T})$ is called connected if V can not be written as union of two disjoin proper open sets. Sometimes, we say also the topology $\mathcal{T}$ is connected.*

Example 4. *Consider $V = \{1, 2, 3\}$, $\tau_1 = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, V\}$ and $\tau_2 = \{\emptyset, \{1\}, \{2, 3\}, V\}$. It is clear that τ_1 is connected but the topology τ_2 is not connected.*

Definition 5.2. *A fuzzy graph $G = (V, \sigma, \mu)$ is called connected if for all $x, y \in V^\star$ there exists a path joining x and y.*

If a fuzzy graph is not connected, we have what we call connected components.

Definition 5.3. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Let $V_1, V_2, \cdots$ be subsets of V such that*

(i) *$V = \cup_i V_i$;*

(ii) *$V_i \cap V_j = \emptyset$, for all $i \neq j$;*

(iii) *For $i = 1, 2, \cdots$, for all $x, y \in V_i$, there exists a path joining x and y.*

(iv) *for all $x \in V_i$, $y \in V_j$ and $i \neq j$, there is no path joining x and y.*

Then, each subset V_i is called connected component of the graph G.

We observe that if the fuzzy graph is connected, then it has one connected component. Also, if the fuzzy graph is finite, then it has a finite connected components.

In what follows, we suppose that $V = V^\star$. The first elementary result is the following.

Proposition 5.1. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. If G is disconnected, then the graphic topology $\mathcal{T}_G$ is disconnected.*

Proof. Suppose that G is disconnected. So, we have

$$V = \bigcup_{i=1} V_i,$$

where $\{V_i\}_i$ the connected components of G. Since $V_i = \bigcup_{x \in V_i} N_x$, we have V_i is an open set for $(V, \mathcal{T}_G)$ and $W = \cup_{i=2} V_i$ is also an open set. We have $V_i \cap V_j = \emptyset$, for all $i \neq j$, hence $V_1 \cap W = \emptyset$. Therefore, $\mathcal{T}_G$ is disconnected. □

Proposition 5.2. *Let $G = (V, \sigma, \mu)$ be a bipartite fuzzy graph, then $\mathcal{T}_G$ is disconnected.*

Proof. Set $V = A \cup B$, where $A \cap B = \emptyset$ and

$$\mu(x, y) = 0 \text{ for all } (x, y) \in A^2 \text{ and for all } (x, y) \in B^2.$$

Let $O_1 = \bigcup_{x \in A} N_x$ and $O_2 = \bigcup_{x \in B} N_x$. We have $O_1 \neq \emptyset$, $O_2 \neq \emptyset$ and $V = O_1 \bigcup O_2$. Also, $O_1 \subset B$ and $O_2 \subset A$ and therefore $O_1 \cap O_2 = \emptyset$. So, $\mathcal{T}_G$ is disconnected. □

Corollary 5.1. *If $G = (V, \sigma, \mu)$ is a connected bipartite fuzzy graph, then $\mathcal{T}_G$ is disconnected.*

Proposition 5.3. *Let $G = (V, \sigma, \mu)$ be a fuzzy cycle of order $n > 4$ (connected), then $\mathcal{T}_G$ is disconnected.*

Proof. Since G is a cycle $x_1, \cdots, x_n$. We have $N_{x_i} = \{x_i\}$ and so $\mathcal{T}_G$ is the discrete topology and so disconnected. □

Proposition 5.4. *Let $G = (V, \sigma, \mu)$ be a connected fuzzy graph of order $n > 4$ such that $\{x_1, \cdots, x_{n-1}\}$ is an $n-1$ cycle, $deg(x_{n-1}) = 3$ and $deg(x_n) = 1$. Then $\mathcal{T}_G$ is disconnected.*

Proof. For any $i = 1, \cdots, n-1$, $U_{x_i} = \{x_i\}$ and $U_{x_n} = N_{x_{n-1}} = \{x_n, x_{n-2}, x_1\}$. So, $\mathcal{T}_G$ is disconnected. □

Note that a fuzzy graph can be connected and also its graphic topology $\mathcal{T}_G$ is connected as the fuzzy graph given in Figure 3.

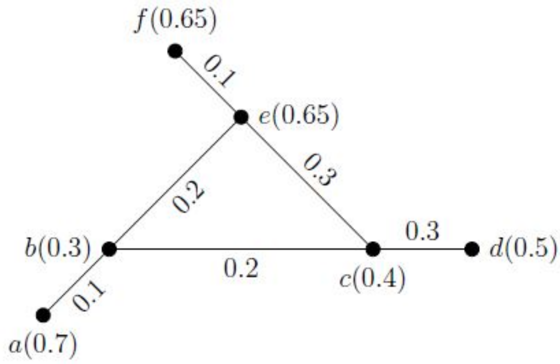

Figure 3. A connected fuzzy graph having connected graphic topology

Question: For a connected fuzzy graph G, when $\mathcal{T}_G$ is disconnected?

6. Conclusions

Let $G = (V, \sigma, \mu)$ be a fuzzy graph. In this paper, we introduced the graphic topology $\mathcal{T}_G$ for a fuzzy graph $G = (V, \sigma, \mu)$, where $V^\star$ the vertex set satisfying $\sigma(x) \neq 0$. We proved some properties of this topology in particular we prove that $\mathcal{T}_G$ is an Alexandroff topology and so, give most of the topological properties by using minimal basis. As an example, we give a necessary and sufficient condition for two graphs to be homeomorphic. We also investigate the connectedness of the graphic topology and prove some relations between the connectedness of the graph and the topology $\mathcal{T}_G$.

As future work, we can think about the open problem: are there some necessary and sufficient conditions for connectivity of $\mathcal{T}_G$?

Conflicts of Interest

The authors declare no conflict of interest.

Authors' contributions

The authors contributed to each part of this work equally and read and approved the final version of the paper.

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