ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **11** (2022), no.10, 869–881
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.10.5

GENERALIZATION OF THE n^{th}-ORDER OPIAL'S INEQUALITY IN (p,q)-CALCULUS

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ABSTRACT. We establish the generalized n^{th}-Order Opial's integral inequality via (p,q)-calculus with some extensions. The other analytical tools used to establish the results were (p,q)-Cauchy repeated integration formula and (p,q)-Cauchy-Schwarz's integral inequality.

1. INTRODUCTION

Opial established an inequality involving integral of a function and its derivative as [14]

$$\int_0^h |f(t)f'(t)|dt \leq \frac{h}{4}\int_0^h (f'(t))^2 dt, \tag{1.1}$$

where $f \in C^1[0,h]$, such that $f(0) = f(h) = 0$, $f'(t) > 0$ and $t \in [0,h]$. The coefficient $h/4$ is the best constant possible.

This inequality, due to its significance, experienced a lot of extensions and generalizations over time in the classical field [5–7, 19], among others.

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2020 *Mathematics Subject Classification.* 05A30, 26A46.
Key words and phrases. Generalization, n^{th}-Order, Opial, Cauchy, Schwarz, inequality, (p,q)- calculus.
Submitted: 24.09.2022; *Accepted:* 10.10.2022; *Published:* 15.10.2022.

The n^{th}-order generalization of the classical Opial's inequality was also established as [19]

$$\int_a^b \left|f(x)f^{(n)}(x)\right| dx \leq \frac{(b-a)^n}{2}\int_a^b \left|f^{(n)}(x)\right|^2 dx, \tag{1.2}$$

where $f \in C^{(n)}[a,b]$ with $f^{(i)}(a) = 0$ for $0 \leq i \leq n-1$, $(n \geq 1)$.

(p,q)-Calculus is a generalization of q-calculus. There has been a lot of development in the study of (p,q)-calculus. Recently, Sadjang [16] investigated on fundamental concepts of (p,q)-calculus. In [9], (p,q)-derivatives and (p,q)-integrals and their properties are also presented.

A (p,q)-analogue of a generalized Opial type inequality was established as [8]

$$\int_0^b |\omega(px)||D_{p,q}\omega(x)|d_{p,q}x \leq \frac{b}{4}\int_0^b |(D_{p,q}\omega(x)|^2 d_{p,q}x. \tag{1.3}$$

where $\omega \in C[0,b]$ with $\omega(0) = \omega(b) = 0$ and $0 < q < p \leq 1$.

See also [1–3, 8, 12] for more analogues of the Opial's type inequalities.

The Opial inequality plays essential role in establishing the existence and uniqueness of initial and boundary values problems for both ordinary and partial differential equations [3, 8].

The main purpose of this work is to further establish a generalization of inequality (1.2) in (p,q)-calculus with some extensions.

2. Preliminaries

The basic concepts of (p,q)-calculus employed in this work are presented in this section. The definitions provided can also be seen in [9, 10, 12, 13, 15, 16] and the references cited therein.

Definition 2.1. *The (p,q)-derivative of a function f is defined as*

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0. \tag{2.1}$$

Definition 2.2. *Let $\alpha > 0$, the (p,q)-bracket is defined as*

$$[\alpha]_{p,q} = p^{\alpha-1} + p^{\alpha-2}q + \cdots + pq^{\alpha-2} + q^{\alpha-1} = \begin{cases} \frac{p^\alpha - q^\alpha}{p-q}, & (p \neq q \neq 1), \\ \frac{1-q^\alpha}{1-q}, & (p = 1), \\ \alpha, & (p = q = 1), \end{cases} \tag{2.2}$$

for $0 < q < p \leq 1$ *and* $\alpha \in \mathbf{R}$.

The (p,q)-derivative of sum or difference of f and g are defined as

$$D_{p,q}(\alpha f(x) \pm \beta g(x)) = \alpha D_{p,q} f(x) \pm \beta D_{p,q} g(x). \tag{2.3}$$

The (p,q)-derivative of product of f and g is defined as

$$\begin{aligned} D_{p,q}(f(x)g(x)) &= g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x) \\ &= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x). \end{aligned} \tag{2.4}$$

The (p,q)-derivative of a quotient of f and g is defined as

$$\begin{aligned} D_{p,q}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)} \\ &= \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}, \quad g(px)g(qx) \neq 0. \end{aligned} \tag{2.5}$$

Definition 2.3. *Let* $f : [0,b] \to \mathbf{R}$ *be a continuous function and* $0 < q < p \leq 1$*. The definite* (p,q)*-integral of* f *on* $[0,b]$ *is defined as*

$$\int_0^b f(x) d_{p,q}x = (p-q)b \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}} b\right). \tag{2.6}$$

If $a \in (0,b)$*, then the definite* (p,q)*-integral of* f *on* $[a,b]$ *is defined as*

$$\int_a^b f(x) d_{p,q}x = \int_0^b f(x) d_{p,q}x - \int_0^a f(x) d_{p,q}x. \tag{2.7}$$

Remark 2.1. *Letting* $p = 1$ *reduces equation (2.6) to the well known Jackson* q*-integral [11].*

Definition 2.4. *Let* $f \in C[a,b] \to \mathbf{R}$*, if* f *is an antiderivative of* f *and* $x \in [a,b]$*. Then*

$$D_{p,q} \int_a^x f(s) d_{p,q}s = f(x) \tag{2.8}$$

and

$$\int_a^x D_{p,q} f(s) d_{p,q}s = f(x) - f(a), \tag{2.9}$$

$$\int_0^b f(x) d_q x = (1-q)b \sum_{j=0}^{\infty} q^j f(bq^j). \tag{2.10}$$

Definition 2.5. *The function f defined on $[a, b]$ is called (p, q)-increasing or (p, q)-decreasing on $[a, b]$, if $f(qx) \leq f(px)(f(qx) \geq f(px))$, for qx, $px \in [a, b]$.*

It is easily observed that if the function f is increasing (decreasing), then it is also (p, q)-increasing ((p, q)-decreasing).

Definition 2.6. *(Fundamental Theorem of (p, q)-Calculus) If $f \in C[a, b]$ and F is an antiderivative of f defined on $x \in [a, b]$, then*

$$F(x) = \int_a^x f(t) d_{p,q}t. \tag{2.11}$$

Definition 2.7. *((p, q)-Cauchy's Formula) Let $f \in C^{(n)}[a, b]$ be such that $D_{p,q}^{(i)} f(a) = 0$, for $i = 0, 1, 2, ..., n-1$, $(n \geq 1)$ and $0 < q < p \leq 1$, then*

$$\begin{aligned} f(x) &= \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} D_{p,q}^n f(s) d_{p,q}s d_{p,q}x_1 \ldots d_{p,q}x_{n-1} \\ &= \frac{1}{[n-1]!} \int_a^x (px - qs)^{n-1} f(s) d_{p,q}s. \end{aligned} \tag{2.12}$$

The proof of (p, q)-Cauchy's formula is similar to q-Cauchy's formula in [4] by mathematical induction.

Definition 2.8. *($(p, q)-$Hölder's Inequality) Let $\alpha, \beta > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. If f and g are continuous real-valued functions on $I = [0, b]$, then [18]*

$$\int_I |f(x)g(x)| d_{p,q}x \leq \left(\int_I |f(x)|^\alpha d_{p,q}x \right)^{\frac{1}{\alpha}} \left(\int_I |g(x)|^\beta d_{p,q}x \right)^{\frac{1}{\beta}}. \tag{2.13}$$

With equality when $|g(x)| = c|f(x)|^{\alpha-1}$.

If $\alpha = \beta = 2$, the inequality becomes (p, q)-Cauchy-Schwartz's Integral Inequality.

$$\int_I |f(x)g(x)| d_{p,q}x \leq \left(\int_I |f(x)|^2 d_{p,q}x \right)^{\frac{1}{2}} \left(\int_I |g(x)|^2 d_{p,q}x \right)^{\frac{1}{2}}. \tag{2.14}$$

3. Results and Discussion

Lemma 3.1. *Let* $f \in C^{(n)}[a,b]$ *be such that* $D_{p,q}f^{(i)} \in L_2[a,b]$, $0 \le i \le n$ *and* $0 < q < p \le 1$, *then*

$$\left(\int_a^b |D_{p,q}^{(n)} f(x)| d_{p,q}x\right)^2 \le (b-a)\int_a^b |D_{p,q}^{(n)} f(x)|^2 d_{p,q}x. \tag{3.1}$$

Proof. Applying (p,q)-Cauchy-Schwarz's inequality, we have

$$\begin{aligned}\left(\int_a^b |D_{p,q}^{(n)} f(x)| d_{p,q}x\right)^2 &\le \left[\left(\int_a^b 1^2 d_{p,q}x\right)^{\frac{1}{2}} \left(\int_a^b |D_{p,q}^{(n)} f(x)|^2 d_{p,q}x\right)^{\frac{1}{2}}\right]^2 \\ &= (b-a)\int_a^b |D_{p,q}^{(n)} f(x)|^2 d_{p,q}x.\end{aligned}$$

This completes the proof. □

Theorem 3.1. *Let* $f \in C^{(n)}[a,b]$ *be such that* $D_{p,q}^{(i)} f(a) = 0$, *for* $0 \le i \le n-1$, $(n \ge 1)$ *and* $0 < q < p \le 1$, *then*

$$\int_a^b |f(x) D_{p,q}^{(n)} f(x)| d_{p,q}x \le \frac{(b-a)^n}{2}\int_a^b |D_{p,q}^{(n)} f(x)|^2 d_{p,q}x. \tag{3.2}$$

Proof. Let $x \in [a,b]$, $D_{p,q}^{(n)} f(a) = 0$ and

$$\omega(x) = \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} |D_{p,q}^{(n)} f(s)| d_{p,q}s d_{p,q}x_1 ... d_{p,q}x_{n-1}, \tag{3.3}$$

so that
$D_{p,q}^{(n)}\omega(x) = |D_{p,q}^{(n)} f(x)|$, $\omega(x) \ge |f(x)|$ and $D_{p,q}^{(i)}\omega(x) \ge 0$, then

$$\int_a^b |f(x) D_{p,q}^{(n)} f(x)| d_{p,q}x \le \int_a^b \omega(x) D_{p,q}^{(n)}\omega(x) d_{p,q}x. \tag{3.4}$$

Since

$$D_{p,q}^{(i)}\omega(x) \le (x-a) D_{p,q}^{(i+1)}\omega(x), \quad x \in [a,b], \quad 0 \le i \le n-2,$$

it follows that

$$|f(x)| \le \omega(x) \le (x-a) D_{p,q}\omega(x) \le \cdots \le (x-a)^{(n-1)} D_{p,q}^{(n-1)}\omega(x). \tag{3.5}$$

Applying (3.5) to (3.4)) we have

$$
\begin{aligned}
\int_a^b |f(x)D_{p,q}^{(n)}f(x)|d_{p,q}x &\leq \int_a^b (x-a)D_{p,q}\omega(x)D_{p,q}^{(n)}\omega(x)d_{p,q}x \\
&\leq \int_a^b (x-a)^2 D_{p,q}^{(2)}\omega(x)D_{p,q}^{(n)}\omega(x)d_{p,q}x \\
&\ \vdots \\
&\leq \int_a^b (x-a)^{(n-1)} D_{p,q}^{(n-1)}\omega(x)D_{p,q}^{(n)}\omega(x)d_{p,q}x \\
&= (b-a)^{n-1}\int_a^b D_{p,q}^{(n-1)}\omega(x)D_{p,q}^{(n)}\omega(x)d_{p,q}x \\
&= \frac{(b-a)^{n-1}\left[D_{p,q}^{(n)}\omega(b)\right]^2}{2} \\
&= \frac{(b-a)^{n-1}}{2}\left(\int_a^b |D_{p,q}^{(n)}f(x)|d_{p,q}x\right)^2. \qquad (3.6)
\end{aligned}
$$

By Lemma (3.1), we obtain

$$\int_a^b |f(x)D_{p,q}^{(n)}f(x)|d_{p,q}x \leq \frac{(b-a)^{n-1}(b-a)}{2}\int_a^b |D_{p,q}^{(n)}f(x)|^2 d_{p,q}x, \tag{3.7}$$

which yields

$$\int_a^b |f(x)D_{p,q}^{(n)}f(x)|d_{p,q}x \leq \frac{(b-a)^n}{2}\int_a^b |D_{p,q}^{(n)}f(x)|^2 d_{p,q}x.$$

This complete the proof. □

Remark 3.1. *For $n=1$, inequality (3.2) reduces to*

$$\int_a^b |f(x)D_{p,q}f(x)|d_{p,q}x \leq \frac{(b-a)}{2}\int_a^b |D_{p,q}f(x)|^2 d_{p,q}x, \tag{3.8}$$

which is the (p,q)-analogue of the Opial's inequality established in [17].

Lemma 3.2. *Let $f \in C^{(n-1)}[a,b]$ be absolutely continuous such that $D_{p,q}^{(i)}f(a) = 0$, for $0 \leq i \leq n-1$, $(n \geq 1)$, $x \in [a,b]$ and $0 < q < p \leq 1$. Then*

$$\int_a^x (px-qs)^{n-1}|D_{p,q}^{(n)}f(s)|d_{p,q}s = \frac{(px-qa)^{n-\frac{1}{2}}}{[2n-1]_{p,q}^{\frac{1}{2}}}\left(\int_a^x |D_{p,q}^n f(s)|^2 d_{p,q}s\right)^{\frac{1}{2}} \tag{3.9}$$

holds.

Proof. By (p,q)-Cauchy-Schwartz's Inequality we have

$$\int_a^x (px-qs)^{n-1}|D_{p,q}^{(n)}f(s)|d_{p,q}s$$
$$\leq \left(\int_a^x (px-qs)^{2(n-1)}d_{p,q}s\right)^{\frac{1}{2}}\left(\int_a^x |D_{p,q}^n f(s)|^2 d_{p,q}s\right)^{\frac{1}{2}}$$
$$= \left((p-q)(px-qa)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}\left(\frac{q^j}{p^{j+1}}(px-qa)\right)^{2(n-1)}\right)^{\frac{1}{2}}\left(\int_a^x |D_{p,q}^n f(s)|^2 d_{p,q}s\right)^{\frac{1}{2}}$$
$$= \left((p-q)\sum_{j=0}^{\infty}\frac{q^{(2n-1)j}}{p^{(2n-1)j+2n-1}}(px-qa)^{2n-1}\right)^{\frac{1}{2}}\left(\int_a^x |D_{p,q}^n f(s)|^2 d_{p,q}s\right)^{\frac{1}{2}}$$
$$= \left(\frac{(p-q)}{p^{2n-1}-q^{2n-1}}(px-qa)^{2n-1}\right)^{\frac{1}{2}}\left(\int_a^x |D_{p,q}^n f(s)|^2 d_{p,q}s\right)^{\frac{1}{2}}$$
$$= \frac{(px-qa)^{n-\frac{1}{2}}}{[2n-1]_{p,q}^{\frac{1}{2}}}\left(\int_a^x |D_{p,q}^n f(s)|^2 d_{p,q}s\right)^{\frac{1}{2}}.$$

□

Theorem 3.2. *Let $f \in C^{(n-1)}[a,b]$ be such that $D_q f^{(i)}(a) = 0$, for $0 \leq i \leq n-1$, $(n \geq 1)$. Also, let $D_{p,q}^{(n-1)}f(x)$ be absolutely continuous and $\int_a^b |D_{p,q}^n f(x)|^2 d_{p,q}x < \infty$, Then*

$$\int_a^b |f(x)D_{p,q}^n f(x)|d_{p,q}x \leq K(pb-qa)^n \int_a^b |D_{p,q}^n f(x)|^2 d_{p,q}x, \tag{3.10}$$

where $K = \frac{n}{n!\sqrt{2[2n]_q[2n-1]_{p,q}}}$.

Proof. Let $x \in [a,b]$. By applying the Cauchy's formula (2.12) we have

$$f(x) = \frac{1}{(n-1)!}\int_a^x (px-qs)^{n-1}D_{p,q}^n f(s)d_{p,q}s. \tag{3.11}$$

This implies

$$|f(x)D_{p,q}^n f(x)| = \frac{|D_{p,q}^n f(x)|}{(n-1)!}\int_a^x (px-qs)^{n-1}|D_{p,q}^n f(s)|d_{p,q}s. \tag{3.12}$$

Applying Lemma (3.2) yields

$$|f(x)D_{p,q}^n f(x)| \leq \frac{|D_{p,q}^n f(x)|}{(n-1)!}\frac{(px-qa)^{n-1/2}}{[2n-1]_{p,q}^{1/2}}\left(\int_a^x |D_{p,q}^n f(s)|^2 d_{p,q}s\right)^{1/2}. \tag{3.13}$$

Integrating (3.13) over $[a,b]$ with respect to x gives

$$\int_a^b |f(x)D_{p,q}^n f(x)|d_{p,q}x \leq \frac{1}{(n-1)![2n-1]_{p,q}^{1/2}} \cdot \int_a^b (px-qa)^{n-1/2}|D_{p,q}^n f(x)| \left(\int_a^x |D_{p,q}^n f(s)|^2 d_{p,q}s\right)^{1/2} d_{p,q}x. \tag{3.14}$$

Applying (p,q)-Cauchy-Schwartz's inequality to the right-hand of (3.14) we obtain

$$\begin{aligned}
\int_a^b |f(x)D_{p,q}^n f(x)|d_{p,q}x &\leq \frac{1}{(n-1)![2n-1]_{p,q}^{1/2}} \left(\int_a^b (px-qa)^{2n-1} d_{p,q}x\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_a^b |D_{p,q}^n f(x)|^2 \left(\int_a^x |D_{p,q}^n f(s)|^2 d_{p,q}s\right) d_{p,q}x\right)^{\frac{1}{2}} \\
&= \frac{1}{(n-1)![2n-1]_{p,q}^{1/2}} \left((p-q)(pb-qa)\sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} \left(\frac{q^j}{p^{j+1}}(pb-qa)\right)^{(2n-1)}\right)^{\frac{1}{2}} \\
&\quad \times \left(\frac{1}{2}\int_a^b D_{p,q}\left(\int_a^x |D_{p,q}^n f(s)|^2 d_{p,q}s\right)^2 d_{p,q}x\right)^{\frac{1}{2}} \\
&= \frac{1}{(n-1)![2n-1]_{p,q}^{1/2}} \left((p-q)\sum_{j=0}^{\infty} \frac{q^{2nj}}{p^{2nj+2n}}(pb-qa)^{(2n)}\right)^{\frac{1}{2}} \sqrt{\frac{1}{2}} \int_a^b |D_{p,q}^n f(x)|^2 d_{p,q}x \\
&= \frac{1}{(n-1)![2n-1]_{p,q}^{1/2}} \sqrt{\frac{1}{2}} \left(\frac{(p-q)}{p^{2n}-q^{2n}}(pb-qa)^{(2n)}\right)^{\frac{1}{2}} \int_a^b |D_{p,q}^n f(x)|^2 d_{p,q}x \\
&= \frac{1}{(n-1)![2n-1]_{p,q}^{1/2}} \sqrt{\frac{1}{2}} \left(\frac{(pb-qa)^{2n}}{[2n]_{p,q}}\right)^{\frac{1}{2}} \int_a^b |D_{p,q}^n f(x)|^2 d_{p,q}x \\
&= \frac{(pb-qa)^n}{(n-1)![2n-1]_{p,q}^{1/2}[2n]_{p,q}^{1/2}} \frac{1}{2^{1/2}} \int_a^b |D_{p,q}^n f(x)|^2 d_{p,q}x \\
&= \frac{n}{n!\sqrt{2[2n]_q[2n-1]_{p,q}}}(pb-qa)^n \int_a^b |D_{p,q}^n f(x)|^2 d_{p,q}x.
\end{aligned}$$

Thus

$$\int_a^b |f(x)D_{p,q}^n f(x)|d_{p,q}x \leq K(pb-qa)^n \int_a^b |D_{p,q}^n f(x)|^2 d_{p,q}x,$$

which completes the proof. □

Remark 3.2. *For $n=1$ in* (3.10)*, we have*

$$\int_a^b |f(x)D_{p,q}f(x)|d_{p,q}x \le \frac{(pb-qa)}{\sqrt{2(p+q)}}\int_a^b |D_{p,q}f(x)|^2 d_{p,q}x, \tag{3.15}$$

which is the (p,q)-analogue of the Opial's inequality established in [17].

Remark 3.3. *Inequality* (3.10) *is sharper than (3.2) when $n \ge 2$ as $q \to 1$.*

Theorem 3.3. *Let $r,s > 0$ satisfying $\beta = s+r > 1$. Also, let $f \in C^n[a,b]$ such that $D_{p,q}^{(i)}f(a) = 0$, $0 \le i \le n-1$ $(n \ge 1)$, $D_{p,q}^{n-1}f$ is absolutely continuous with $\int_a^b |D_{p,q}^n f(x)|^\beta d_{p,q}x < \infty$. Then*

$$\int_a^b |f(x)|^s |D_{p,q}^n f(x)|^r d_{p,q}x \le M(pb-qa)^{ns}\int_a^b |D_{p,q}^n f(x)|^\beta d_{p,q}x, \tag{3.16}$$

where

$$M = \phi r^{r\phi}\left(\frac{n\,[\beta n]_{p,q}^{\phi}}{\left[\frac{\beta n-1}{\beta-1}\right]_{p,q}^{1-\phi}}\right)^s (n!)^{-s}, \quad \phi = (\beta)^{-1}. \tag{3.17}$$

Proof. Let $x \in [a,b]$. By (2.12) we have

$$f(x) = \frac{1}{(n-1)!}\int_a^x (px-qs)^{n-1}D_{p,q}^n f(s)d_{p,q}s. \tag{3.18}$$

Applying Hölder's inequality with $\alpha = \frac{s+r}{s+r-1}$ and $\beta = s+r$ on (3.18) we obtain

$$\begin{aligned}|f(x)| &= \frac{1}{(n-1)!}\left(\int_a^x (px-qs)^{\alpha(n-1)}d_{p,q}s\right)^{\frac{1}{\alpha}}\left(\int_a^x |D_{p,q}^n f(s)|^\beta d_{p,q}s\right)^{\frac{1}{\beta}}\\ &= \frac{1}{(n-1)!}\left((p-q)(px-qa)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}\left(\frac{q^j}{p^{j+1}}(px-qa)\right)^{\alpha(n-1)}\right)^{\frac{1}{\alpha}}\\ &\quad\cdot\left(\int_a^x |D_{p,q}^n f(s)|^\beta d_{p,q}s\right)^{\frac{1}{\beta}}\end{aligned}$$

$$= \frac{1}{(n-1)!}\left((p-q)\sum_{j=0}^{\infty}\frac{q^{(\alpha(n-1)+1)j}}{p^{(\alpha(n-1)+1)j+(\alpha(n-1)+1)}}(px-qa)^{\alpha(n-1)+1}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left(\int_a^x |D^n_{p,q}f(s)|^{\beta}d_{p,q}s\right)^{\frac{1}{\beta}}$$

$$= \frac{1}{(n-1)!}\left(\frac{(p-q)}{p^{(\alpha(n-1)+1)}-q^{(\alpha(n-1)+1)}}(px-qa)^{(\alpha(n-1)+1)}\right)^{\frac{1}{\alpha}}\left(\int_a^x |D^n_{p,q}f(s)|^{\beta}d_{p,q}s\right)^{\frac{1}{\beta}}$$

$$= \frac{1}{(n-1)!}\left(\frac{p-q}{p^{(\alpha(n-1)+1)}-q^{(\alpha(n-1)+1)}}\right)^{\frac{1}{\alpha}}(px-qa)^{(n-1)+\alpha^{-1}}\left(\int_a^x |D^n_{p,q}f(s)|^{\beta}d_{p,q}s\right)^{\frac{1}{\beta}}$$

$$= \frac{n}{n!\,[\alpha(n-1)+1]_{p,q}^{\frac{1}{\alpha}}}(px-qa)^{(n-1)+\alpha^{-1}}\left(\int_a^x |D^n_{p,q}f(s)|^{\beta}d_{p,q}s\right)^{\frac{1}{\beta}}.$$

Letting

$$A = \frac{n}{n!\,[\alpha(n-1)+1]_{p,q}^{\frac{1}{\alpha}}}$$

Thus, we have

$$\int_a^b |f(x)|^s|D^n_{p,q}f(x)|^r d_{p,q}x \leq$$

$$A^s\int_a^b (px-qa)^{s((n-1)+\alpha^{-1})}|D^n_{p,q}f(x)|^r\left(\int_a^x |D^n_{p,q}f(s)|^{\beta}d_{p,q}s\right)^{\frac{s}{\beta}}d_{p,q}x. \tag{3.19}$$

Applying (p,q)-Hölder's inequality with indices $\frac{\beta}{s}$ and $\frac{\beta}{r}$ on the right-hand side of (3.19) we obtain

$$\int_a^b |f(x)|^s|D^n_{p,q}f(x)|^r d_{p,q}x \leq A^s\left(\int_a^b (px-qa)^{\beta((n-1)+\alpha^{-1})}d_{p,q}x\right)^{\frac{s}{\beta}}$$

$$\cdot\left(\int_a^b |D^n_{p,q}f(x)|^{\beta}\left(\int_a^x |D^n_{p,q}f(s)|^{\beta}d_{p,q}s\right)^{s/r}d_{p,q}x\right)^{\frac{r}{\beta}}$$

$$= A^s\left((p-q)(pb-qa)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}\left(\frac{q^j}{p^{j+1}}(pb-qa)\right)^{\beta((n-1)+\alpha^{-1})}\right)^{\frac{s}{\beta}}$$

$$
\begin{aligned}
&\cdot\left(\frac{r}{\beta}\int_a^b D_{p,q}\left(\int_a^x |D_{p,q}^n f(s)|^\beta d_{p,q}s\right)^{\frac{s}{r}+1} d_{p,q}x\right)^{\frac{r}{\beta}}\\
&= A^s\left((p-q)\sum_{j=0}^{\infty}\frac{q^{(\beta((n-1)+\alpha^{-1})+1)j}}{p^{(\beta((n-1)+\alpha^{-1})+1)j+(\beta((n-1)+\alpha^{-1})+1)}}(pb-qa)^{(\beta((n-1)+\alpha^{-1})+1)}\right)^{\frac{s}{\beta}}\\
&\cdot\left(\frac{r}{\beta}\int_a^b D_{p,q}\left(\int_a^x |D_{p,q}^n f(s)|^\beta d_{p,q}s\right)^{\frac{\beta}{r}} d_{p,q}x\right)^{\frac{r}{\beta}}\\
&= A^s\left(\frac{(p-q)}{p^{\beta n}-q^{\beta n}}(pb-qa)^{\beta n}\right)^{\frac{s}{\beta}}\phi r^{r\phi}\int_a^b |D_{p,q}^n f(x)|^\beta d_{p,q}x\\
&= A^s\left(\frac{(p-q)}{p^{\beta n}-q^{\beta n}}\right)^{\frac{s}{\beta}}(pb-qa)^{ns}\phi r^{r\phi}\int_a^b |D_{p,q}^n f(x)|^\beta d_{p,q}x\\
&= \phi r^{r\phi}\frac{A^s}{[\beta n]_{p,q}^{s\phi}}(pb-qa)^{ns}\int_a^b |D_{p,q}^n f(x)|^\beta d_{p,q}x\\
&= \phi r^{r\phi}\left(\frac{n\,[\beta n]_{p,q}^{\phi}}{\left[\frac{\beta n-1}{\beta-1}\right]_{p,q}^{1-\phi}}\right)^s (n!)^{-s}(pb-qa)^{ns}\int_a^b |D_{p,q}^n f(x)|^\beta d_{p,q}x.
\end{aligned}
$$

This completes the proof. ☐

Hence,

$$\int_a^b |f(x)|^s |D_{p,q}^n f(x)|^r d_{p,q}x \le M(pb-aq)^{ns}\int_a^b |D_{p,q}^n f(x)|^\beta d_{p,q}x. \tag{3.20}$$

Remark 3.4. *Taking $n=1$ in (3.16) yields*

$$\int_a^b |f(x)D_{p,q}f(x)|d_{p,q}x \le \sqrt{\frac{p+q}{2}}(pb-qa)\int_a^b |D_{p,q}f(x)|^2 d_{p,q}x. \tag{3.21}$$

ACKNOWLEDGMENT

The Authors are grateful to the anonymous reviewers.

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