A COMMON FIXED POINT THEOREM IN MENER SPACE WITH COMPATIBLE MAPPING OF TYPE (K)

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ABSTRACT. The notion of compatible mappings of type (K) in metric space has been introduced by K. Jha, V. Popa, and K.B. Manandhar in 2014. In this paper, we use Meir-Keeler contractive type condition and establish a common fixed point theorem in Menger probabilistic metric space by using compatible mappings of type (K). Our results generalize and improve several similar known results in the literature.

1. INTRODUCTION

The idea of probability in metric space was introduced by K. Menger [13] in 1942, as an extension of the concept of metric space $(X, d)$, by replacing the notion of distance $d(x, y)$ where $(x, y \in X)$ with a distributive function $F_{x,y} : X \times X \rightarrow \mathbb{R}$, where $F(x, y)(t)$ represents the probability that the distance between $x$ and $y$ is less than $t$. This space is called statistical metric space or Menger probabilistic metric space. B. Schweizer and A. Sklar [15] studied this concept and obtained some fundamental results on this Menger space. In 1972, V. M. Sehgal and A. T. Bharucha-Reid [16] initiated the study of contraction mappings in probabilistic

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metric spaces (PM-spaces) which is an important step in the development of fixed point theorems.


The purpose of this paper is to introduce the notion of compatible mappings of type \((K)\) in Menger space with example and establish a common fixed point theorem for pairs of self- mappings in Menger space using this compatible mapping of type \((K)\) under Meir-Keeler contractive condition which generalizes and improves some well-known results in the literature.

2. Preliminaries

Definition 2.1. [15] A function \(F : \mathbb{R} \to \mathbb{R}^+\) is said to be distribution function if it

(i) is non-decreasing,
(ii) is left continuous, and
(iii) \(\inf_{x \in \mathbb{R}} F(x) = 0\) and \(\sup_{x \in \mathbb{R}} F(x) = 1\).

Definition 2.2. [15] Let \(F : X \times X \to L\) (the set of all distribution functions) be a distribution function and \(X\) be a non-empty set. Then, a pair \((X, F)\) is said to be Probabilistic metric space (abbreviated as PM-space) if the distribution function \(F(p, q)\), where \((p, q) \in X \times X\), also denoted by \(F(p, q)\) or \(F_{p,q}\) satisfies following conditions:

(i) \(F_{p,q}(x) = 1\), for every \(x > 0\) if and only if \(p = q\).
(ii) \(F_{p,q}(0) = 0\); for every \(p, q \in X\),
(iii) \(F_{p,q}(x) = F_{q,p}(x)\), for every \(p, q \in X\), and
(iv) For every \( p, q, r \in X \) and for every \( x, y > 0 \), \( F_{p,r}(x) = 1, F_{r,q}(y) = 1 \Rightarrow F_{p,q}(x + y) = 1 \).

Here, \( F(p, q)(x) \) represents the value of \( F(p, q) \) at \( x \in \mathbb{R} \).

**Definition 2.3.** [5] A function \( T : [0, 1] \times [0, 1] \to [0, 1] \) is referred to as **Triangular norm** (shortly t-norm) if it satisfies the following conditions:

(i) \( t(0, 0) = 0 \) and \( t(a, 1) = a \) for every \( a \in [0, 1] \),
(ii) \( t(a, b) = t(b, a) \) for every \( a, b \in [0, 1] \),
(iii) if \( a < c \) and \( b < d \), then \( t(a, b) < t(c, d) \), and
(iv) \( t(a, T(b, c)) = t(t(a, b), c) \) \((a, b, c \in [0, 1])\).

**Definition 2.4.** [18] Menger Space, also known as Menger Probabilistic Metric Space, is a triplet \((X, F, T)\), where \((X, F)\) is a PM space and \( t \) is a \( T \)-norm satisfying following condition:

(v) \( F_{p,q}(x + y) \geq t(F_{p,r}(x), F_{r,q}(y)), \) for all \( p, q, r \in X \) and \( x, y \in \mathbb{R} > 0 \).

**Definition 2.5.** [9] A mapping \( A : X \to X \) in Menger space \((X, F, t)\) is said to be **Continuous** at a point \( p \in X \) if for every \( \epsilon > 0 \) and \( \lambda > 0 \), there exists \( \epsilon_1 > 0 \) and \( \lambda_1 > 0 \) such that if \( F_{p,q}(\epsilon_1) > 1 - \lambda_1 \), then \( F_{q,p}(\epsilon_1) > 1 - \lambda \).

**Definition 2.6.** [3] Let \((X, F, T)\) be a Menger Space and \( t \) be a continuous \( T \)-norm. Then,

(1) A sequence \( \{x_n\} \) in \( X \) is said to be **converge** to a point \( x \) in \( X \) (written \( x_n \to x \)) if, and only if, for every \( \epsilon > 0 \) and \( \lambda > 0 \), there exists an integer \( N = (N, \epsilon) > 0 \) such that \( F_{x_n,x}(\epsilon) > 1 - \lambda \) for all \( n \geq N \). In this case, we write \( \lim_{n \to \infty} x_n = x \).

(2) A sequence \( \{x_n\} \) in \( X \) is said to be a **Cauchy sequence** if for every \( \epsilon > 0 \) and \( \lambda > 0 \), there exists an integer \( N = (N, \epsilon) > 0 \) such that \( F_{x_n,x_m}(\epsilon) > 1 - \lambda \) for all \( n, m \geq N \).

(3) A Menger space \((X, F, T)\) is said to be **Complete** if every Cauchy sequence in \( X \) converges to a point in \( X \).

**Definition 2.7.** [8] Let \( X \) be a non-empty set and \( A, S : X \to X \) be arbitrary mappings, then \( x \in X \) is said to be a common fixed point of \( A \) and \( S \) if \( A(x) = S(x) = x \).
Example 1. Let \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) be functions such that \( f(x) = x^2 \) and \( g(x) = x \), then \( x = 0 \) is a common fixed point.


Definition 2.8. [14] Two mappings \( A, S : X \rightarrow X \) are said to be Compatible Mappings in Menger space \( (X, F, t) \) if, and only if,
\[
\lim_{n \to \infty} F_{ASx_n,SAx_n}(t) = 1 \quad \text{for all} \quad t > 0
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z \) for some \( z \) in \( X \).

Definition 2.9. [3] Two mappings \( R, S : X \rightarrow X \) are said to be Compatible Mappings of type A in Menger space \( (X, F, t) \) if, and only if,
\[
\lim_{n \to \infty} F_{SRx_n,RRx_n}(t) = 1 \quad \text{and} \quad \lim_{n \to \infty} F_{RSx_n,SSx_n}(t) = 1 \quad \text{for all} \quad t > 0
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Qx_n = \lim_{n \to \infty} Rx_n = z \) for some \( z \) in \( X \).

In 2021, A.K. Chaudhary, K. Jha, K. B. Manandhar, and P.P. Murthy [6] have introduced the following compatible mapping of type (P) in Menger space as:

Definition 2.10. [6] Two mappings \( A, S : X \rightarrow X \) are said to be Compatible Mappings of type (P) in Menger space \( (X, F, t) \) if, and only if,
\[
\lim_{n \to \infty} F_{AAx_n,SSx_n}(t) = 1 \quad \text{for all} \quad t > 0
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z \) for some \( z \) in \( X \).

The new notion of compatible mappings of type (K) in metric space was introduced by K.B. Manandhar, K. Jha, and V. Popa [11] in 2014.

We introduce the following definition of compatible mappings of type (K) in Menger space.
Definition 2.11. Two mappings \(A, S : X \to X\) are said to be Compatible Mappings of type \((K)\) in Menger space \((X, F, t)\) if, and only if,

\[
\lim_{n \to \infty} F_{AAx_n, Sz}(t) = 1 \quad \text{and} \quad \lim_{n \to \infty} F_{SSx_n, Ax}(t) = 1 \quad \text{for all} \quad t > 0
\]

whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z\) for some \(z\) in \(X\).

Example 2. Let \((X, d)\) be a metric space where \(X = [0, 2]\) and let \((X, F, t)\) be Menger space with

\[
F_{x,y}(t) = \begin{cases} 
  t & \text{for } t > 0 \\
  \frac{t}{1+d(x,y)} & \text{for } t = 0
\end{cases}
\]

for all \(x, y \in X\). Let \(A, S : X \to X\) be defined by

\[
A(x) = \begin{cases} 
  2 & \text{for } x \in [0, 1] - \frac{1}{2} \\
  0 & \text{for } x = \frac{1}{2} \\
  \frac{2-x}{2} & \text{for } x \in (1, 2]
\end{cases}
\]

and

\[
S(x) = \begin{cases} 
  0 & \text{for } x \in [0, 1] - \frac{1}{2} \\
  2 & \text{for } x = \frac{1}{2} \\
  \frac{x}{2} & \text{for } x \in (1, 2]
\end{cases}
\]

Taking sequence \(\{x_n\}\) where \(x_n = 1 + \frac{1}{n}, n \in N\). Then, it is neither compatible mappings of type \((A)\) nor compatible mappings of type \((P)\) but \((A, S)\) is compatible mappings of type \((K)\).

3. Main Result

The following lemma helps us to prove our main theorem:

Lemma 3.1. \([17]\) Let \(\{x_n\}\) be a sequence in Menger space \((X, F, t)\), where \(t\) is continuous \(T\)-norm and \(t(x, x) \geq x\) for all \(x \in [0, 1]\). If there exists a constant \(k \in (0, 1)\) such that \(\lim_{n \to \infty} F_{x_n, x_{n+1}}(kt) \geq F_{x_n, x_n}(x)\), for all \(t > 0\) and \(n \in N\), then \(\{x_n\}\) is a Cauchy sequence in \(X\).

Lemma 3.2. \([17]\) Let \((X, F, t)\) be a Menger space. If there exists \(k \in (0, 1)\) such that for all \(p, q \in X\), \(F_{p, q}(kt) \geq F_{p, q}(t)\) then \(p = q\).
Now, we prove our main theorem by using compatible mappings of type (K) in Complete Menger Space:

**Theorem 3.1.** Let \((X, F, t)\) be a complete Menger space with \(t(x, y) = \min(x, y)\) for all \(x, y \in [0, 1]\) and \(A, B, S, T : X \to X\) be four self mappings such that

\[
\begin{align*}
(3.1) & \quad A(X) \subset T(X) \text{ and } B(X) \subset S(X), \\
(3.2) & \quad \text{the pairs } (A, S) \text{ and } (B, T) \text{ are compatible mappings of type } (K), \\
(3.3) & \quad S \text{ and } T \text{ be continuous, and} \\
(3.4) & \quad \text{there exists a constant } k \in (0, 1) \text{ and for every } \epsilon \in (0, 1), \text{ there exists } \delta \in (0, \epsilon] \text{ such that } \\
& \quad \epsilon - \delta < M_{x,y} \leq \epsilon \implies F_{Ax,By}(kt) \geq \epsilon \text{ and } F_{Ax,By}(kt) \geq M_{x,y}(t) \\
& \quad \text{where } M_{x,y}(t) = \min\{(F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Ax,Ty}(t))\} \\
& \quad \text{for all } x, y \in X \text{ and } t \geq 0.
\end{align*}
\]

Then, \(A, B, S,\) and \(T\) have a unique common fixed point in \(X\).

**Proof.** Consider \(x_0 \in X\). From (i), we have \(A(X) \subset T(X)\) and \(B(X) \subset S(X)\). So if there exists a point \(x_1 \in X\) such that \(Ax_0 = Tx_1\). And for \(x_1 \in X\) there exist \(x_2 \in X\) such that \(Bx_1 = Sx_2\) and so on. And inductively, we may construct sequence \(\{y_n\}\) in \(X\) such that \(y_{2n-1} = Ax_{2n-2} = Tx_{2n-1}\), and \(y_{2n} = Bx_{2n-1} = Sx_{2n}\), for \(n = 1, 2, 3 \ldots\) Putting \(x = x_{2n}\) and \(y = x_{2n+1}\) in (3.4), we get

\[
\begin{align*}
F_{y_{2n+1},y_{2n+2}}(kt) & = F_{Ax_{2n},Bx_{2n+1}}(kt) \\
& \geq \min\{(F_{Sx_{2n},Tx_{2n+1}}(t), F_{Ax_{2n},Sx_{2n}}(t), F_{Bx_{2n},Tx_{2n+1}}(t), F_{Ax_{2n},Tx_{2n+1}}(t))\} \\
& \geq \min\{(F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n+2}}(t), F_{y_{2n+2},y_{2n+1}}(t), F_{y_{2n+1},y_{2n+1}}(t))\} \\
& \geq \min\{(F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n+2}}(t))\},
\end{align*}
\]

so, \(F_{y_{2n+1},y_{2n+2}}(kt) \geq F_{y_{2n},y_{2n+1}}(kt)\). Therefore, for every \(n \in N\), \(F_{y_n,y_{n+1}}(kt) \geq F_{y_{n-1},y_n}\).

So, by Lemma 3.1, \(\{y_n\}\) is Cauchy sequence in \(X\).

Since the Menger space \((X, F, t)\) is complete, so \(\{y_n\}\) converges to a point \(z\) in \(X\) and consequently the sub sequences \(\{Ax_{2n-2}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}\) and \(\{Tx_{2n-1}\}\), also converges to \(z\).

Since \(S\) and \(T\) are continuous and \((A, S)\) and \((B, T)\) are compatible mappings of type (K). So, we have

\[
\begin{align*}
(3.1) & \quad AAx_{2n-2} \to Sz \text{ and } SSx_{2n} \to Az \text{ and } BBx_{2n-1} \to Tz \text{ and } TTx_{2n-1} \to Bz.
\end{align*}
\]
Also, from (3.4), we get
\[ F_{Ax,2n-2,Bz2n-1}(kt) \geq \min \{ (F_{Ax,2n-2,Tz2n-1}(t), F_{Ax,2n-2,Sz2n-2}(t), F_{Bz2n-1,Tz2n-1}(t), F_{Ax,2n-2,Tz2n-1}(t)) \} \]

As \( n \to \infty \) and by using equation (3.1), we have
\[ F_{Sz,Tz}(kt) \geq \min \{ (F_{Sz,Tz}(t), F_{Sz,Sz}(t), F_{Tz,Tz}(t), F_{Sz,Tz}(t)) \} \]
or, \( F_{Sz,Tz}(kt) \geq F_{Sz,Tz}(t) \). From lemma (3.2), we get
(3.2) \( Sz = Tz \)

Again, from (3.4), we have
\[ F_{Az,Bz2n-1}(kt) \geq \min \{ (F_{Sz,Tz2n-1}(t), F_{Az,Sz}(t), F_{Bz,Tz2n-1}(t), F_{Az,Tz2n-1}(t)) \} \]
Again, taking \( n \to \infty \) and using equation (3.1) and (3.2), we get
\[ F_{Az,Tz}(t) \geq \min \{ (F_{Sz,Sz}(t), F_{Az,Tz}(t), F_{Tz,Tz}(t), F_{Az,Tz}(t)) \} \]
or, \( F_{Az,Tz}(kt) \geq F_{Az,Tz}(t) \). From Lemma 3.2, we get
(3.3) \( Az = Tz \).

Then, from equations (3.2) and (3.3), we obtain
\[ F_{Az,Bz}(kt) \geq \min \{ (F_{Sz,Tz}(t), F_{Az,Sz}(t), F_{Bz,Tz}(t), F_{Az,Tz}(t)) \} \]
or,
\[ F_{Az,Bz}(kt) \geq \min \{ (F_{Az,Az}(t), F_{Az,Az}(t), F_{Bz,Az}(t), F_{Az,Az}(t)) \} \]
or, \( F_{Az,Bz}(kt) \geq F_{Az,Bz}(t) \). So, we get
(3.4) \( Az = Bz \).

From equation relations (3.2), (3.3) and (3.4), we get
(3.5) \( Az = Bz = Tz = Sz \).

Now, we have to show that \( Az = z \).

From condition (3.4), we have
\[ F_{Ax,Bz2n-1}(kt) \geq \min \{ (F_{Sz,Tz2n-1}(t), F_{Az,Sz}(t), F_{Bz2n-1,Tz2n-1}(t), F_{Az,Tz2n-1}(t)) \} \].
Taking $n \to \infty$ and using equations (3.2) and (3.3), we get
\[ F_{Az,z}(kt) \geq \min\{(F_{Sz,z}(t), F_{Az,Sz}(t), F_{Az,z}(t)) \} \]
\[ \geq \min\{(F_{Az,z}(t), F_{Az,Az}(t), F_{Az,z}(t)) \} \]
or, $F_{Az,z}(kt) \geq F_{Az,z}(t)$, or, by Lemma 3.2 $Az = z$.

Hence, from equation (3.5), we get $z = Az = Bz = Tz = Sz$ and $z$ is a common fixed point of $A, B, S$, and $T$.

**Uniqueness:** Suppose $w \neq z$ is another common fixed point of $A, B, S$, and $T$. Then $Aw = Bw = Sw = Tw = w$. Therefore, from condition (3.4), we have
\[ F_{z,w}(kt) = F_{Az,Bw}(kt) \geq \min\{(F_{Sz,Tw}(t), F_{Az,Sz}(t), F_{Bw,Tw}(t), F_{Az,Tw}(t)) \}, \]
or, $F_{z,w}(kt) \geq \min\{(F_{z,w}(t), F_{z,z}(t), F_{w,w}(t), F_{z,w}(t)) \}$, or, $F_{z,w}(kt) \geq F_{z,w}(t)$.

By Lemma 3.2, we get $z = w$. Hence, $z = Az = Bz = Tz = Sz$ and $z$ is unique in $X$. This completes the proof. \qed

### 4. Verification and Application

We verify our main result through following example:

**Example 3.** Let $(X, F, t)$ be a complete Menger Space with $t(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$ where $X = [1, 10]$ with metric $d$ defined by $d(x, y) = |x - y|$ and $F$ is defined by
\[ F_{x,y}(t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases} \]
for all $x, y \in X$. We define $A, B, S, T : X \to X$ as:
\[ A(x) = \begin{cases} 1 & \text{for } x \leq 4 \\ 2 & \text{for } x > 4, \end{cases} \]
\[ B(x) = \begin{cases} 1 & \text{for } x \leq 5 \\ 2 & \text{for } x > 5, \end{cases} \]
and $S(x) = T(x) = x$ for all $x \in X$. Taking sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{n}, n \in \mathbb{N}$. Then, the mappings $A, B, S$ and $T$ satisfy all the conditions of the above Theorem 3.1 and have a unique common fixed point $x = 1$. 
Our main result help to prove following corollary. In the theorem 3.1, if we take
\( A = B, T = S \), then we have following result:

**Corollary 4.1.** Let \( A \) and \( S \) be self-mappings in complete Menger space \( (X,F,t) \) with continuous \( t(x,y) = \min(x,y) \) for all \( x, y \in [0, 1] \) satisfying the following conditions:

1. \( A(X) \subset S(X) \),
2. the pairs \( (A,S) \) be compatible mappings of type \((K)\),
3. \( S \) be continuous, and
4. there exists a constant \( k \in (0, 1) \) and for every \( \epsilon \in (0, 1) \), there exists \( \delta \in (0, \epsilon] \) such that \( \epsilon - \delta < M_{x,y} < \epsilon \) implies \( F_{Ax,Ay}(kt) \geq \epsilon \) and \( F_{Ax, Ay}(kt) \geq M_{x,y}(t) \)
where \( M_{x,y}(t) = \min \{ (F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(t), F_{Ax,Sy}(t)) \} \),
for all \( x, y \in X \) and \( t > 0 \). Then, \( A \) and \( S \) have a unique common fixed point in \( X \).

**Remark 4.1.** Our results generalize the results of Jungck et.al [7], Meir Keeler [12] and Jha et al [11]. Also, this result improves other similar results in the literature.

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