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A COMBINATION OF ORTHOGONAL POLYNOMIALS SEQUENCES: $2-5$ TYPE RELATION

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ABSTRACT. In the present paper, a new characterization of the orthogonality of a monic polynomials sequence $\{Q_n\}_{n\geq 0}$ is obtained. This is defined as a linear combination of another monic orthogonal polynomials sequence $\{P_n\}_{n\geq 0}$ such as

$$Q_n(x)+r_nQ_{n-1}(x)=P_n(x)+s_nP_{n-1}(x)+t_nP_{n-2}\left(x\right)+v_nP_{n-3}\left(x\right)+w_nP_{n-4}(x),\ n\geq 0$$

where $w_nr_n\neq 0$, for every $n\geq 5$. Futhermore, the relation between the corresponding linear functionals is showed to be

$$k\left(x-c\right)u=\left(x^4+ax^3+bx^2+dx+e\right)v$$

where c, a, b, d, $e\in\mathbb{C}$ and $k\in\mathbb{C}\backslash\{0\}$. Finally, an illustration using special case of the above type relation is given.

1. INTRODUCTION

Let $\mathcal{P}$ be the linear space of polynomials in one variable with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. $\langle u,f\rangle$ denotes the action of u in $\mathcal{P}'$ on f in $\mathcal{P}$ and by $(u)_n:=\langle u,x^n\rangle\,,\ n\geq 0$, the moments of u with respect to the monomial sequence $\{x^n\}_{n\geq 0}$. When $(u)_0=1$, the linear functional u is said to be normalized.

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In this work we need to recall some operations in $\mathcal{P}'$, (see [6], [8]). For any u in $\mathcal{P}'$, any q in $\mathcal{P}$ and any complex numbers a, b, c with $a \neq 0$, let $Du = u'$, qu, $h_a u$, $\tau_b u$ and σu be respectively the derivative, the left multiplication, the translation, the homothetic and the pair part of the linear functionals defined by duality:

$$\begin{aligned}
&\langle u', f \rangle := -\langle u, f' \rangle, \\
&\langle qu, f \rangle := \langle u, qf \rangle, \\
&\langle h_a u, f \rangle := \langle u, h_a f \rangle = \langle u, f(ax) \rangle, \\
&\langle \tau_b u, f \rangle := \langle u, \tau_{-b} f \rangle = \langle u, f(x+b) \rangle, \\
&\langle \sigma u, f \rangle := \langle u, \sigma f \rangle = \langle u, f(x^2) \rangle, \ f \in \mathcal{P}.
\end{aligned}$$

The linear functional u is called regular (quasi-definite) if the leading principal submatrices $\mathcal{H}_n$ of the Hankel matrix $\mathcal{H} = (u_{i+j})_{i,j \geq 0}$ related to the moments $u_n = \langle u, x^n \rangle$, $n \geq 0$, are nonsingular, for each $n \geq 0$ [6].

Definition 1.1. *[6] A sequence of monic polynomials $\{P_n\}_{n\geq 0}$ is called orthogonal with respect to the linear functional u if the following orthogonality conditions hold*

$$\begin{aligned}
&\langle u, P_n(x) P_m(x) \rangle = 0, \qquad n \neq m, \\
&\langle u, P_n^2(x) \rangle \neq 0, \qquad n \geq 0,
\end{aligned}$$

where $\deg P_n = n$, for every $n \geq 0$.

In this case, $\{P_n\}_{n\geq 0}$ satisfies the following two order recurrence relation:

$$\begin{cases}
P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x), \quad n \geq 1, \\
P_0(x) = 1, \quad P_1(x) = x - \beta_0,
\end{cases}$$

where $\gamma_n \neq 0$, for each $n \geq 1$.

Let u and v be two regular linear functionals and let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be the corresponding sequences of monic orthogonal polynomials. Assume that there exist non-negative integer numbers M and N, and sequences of complex numbers $\{r_{i,n}\}_{n\geq 0}$ and $\{s_{k,n}\}_{n\geq 0}$ such that the structure relation

$$Q_n(x) + \sum_{i=1}^{M} r_{i,n} Q_{n-i}(x) = P_n(x) + \sum_{i=1}^{N} s_{i,n} P_{n-i}(x)$$

holds for $n \geq 0$. Further, assume that $r_{M,M+N} \neq 0$ and $s_{N,M+N} \neq 0$, $\det [\alpha_{ij}]_{i,j=1}^{M+N} \neq 0$ where the entries α_{ij} of the matrix are defined on the basis of $\{r_{i,n}\}_{n\geq 0}$ and

$\{s_{k,n}\}_{n\geq 0}$. Then there exist two polynomials Φ and Ψ with $\deg \Phi = M$ and $\deg \Psi = N$ such that

$$\Phi(z)u = \Psi(z)v.$$

These polynomials Φ and Ψ can be constructed in an explicit way [9]. On the other hand, the converse result is also analyzed. A characterization theorem for the sequence $\{Q_n\}_{n\geq 0}$ to be orthogonal assuming $\{P_n\}_{n\geq 0}$ is orthogonal is obtained when $M=0$ and $N=1$, $M=1$ and $N=1$, $M=0$ and $N=2$, $M=1$ and $N=2$, $M=0$ and $N=3$, $M=0$ and $N=k$ [1–5].

In this contribution, we are interested to the case $M=1$ and $N=4$.

Let $\{P_n\}_{n\geq 0}$ be the $MOPS$ with respect to the regular functional u and $\{Q_n\}_{n\geq 0}$ be monic polynomials sequence with $\deg Q_n = n$. Suppose that these sequences are related by $2-5$ type relation as follows:

(1.1) $$Q_n(x)+r_nQ_{n-1}(x) = P_n(x)+s_nP_{n-1}(x)+t_nP_{n-2}(x)+v_nP_{n-3}(x)+w_nP_{n-4}(x),$$

$n\geq 0$, with the initial conditions $Q_0(x) = P_0(x) = 1$ and $Q_{-1}(x) = P_{-m}(x) = 0$, for $m = 1, 2, 3, 4$ and $\{r_n\}_{n\geq 0}, \{s_n\}_{n\geq 0}, \{t_n\}_{n\geq 0}, \{v_n\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ are sequences of complex numbers with the initial conditions

$$r_0 = s_0 = t_0 = t_1 = v_0 = v_1 = v_2 = w_0 = w_1 = w_2 = w_3 = 0.$$

The main purpose of this paper is to obtain necessary and sufficient conditions for the orthogonality of the monic polynomials sequence $\{Q_n\}_{n\geq 0}$. In addition, we establish a relation between the linear functionals u and v, respectively, corresponding to $MOPS's$ $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ as $k(x-c)u = (x^4+ax^3+bx^2+dx+e)v$ with $c,\ a, b,\ d,\ e\in\mathbb{C}$ and $k\in\mathbb{C}\backslash\{0\}$. This article is organized as follows. In section 2, we develop some basic results and lemmas. Section 3, is devoted to find the characterizations of the orthogonality of the monic polynomials sequence $\{Q_n\}_{n\geq 0}$. Finally, we illustrate a special case of the above type relation.

2. $2-5$ TYPE RELATION

Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be two sequences of monic orthogonal polynomials with respect to the regular functionals u and v respectively, where $\langle u,1\rangle = \langle v,1\rangle = 1$, let $\{\beta_n\}_{n\geq 0}$, $\{\gamma_n\}_{n\geq 1}$ and $\left\{\widetilde{\beta}_n\right\}_{n\geq 0}$, $\{\widetilde{\gamma}_n\}_{n\geq 1}$ the corresponding sequences characterizing $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively. Suppose that these sequences are related by relation (1.1).

The initial conditions $w_4 \neq r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1)))$ and $w_5 r_5 \neq 0$ yield a relation between the linear functionals u and v such as

$$\phi u = \psi v$$

where ϕ and ψ are polynomials of degree 1 and 4, respectively.

Firstly, if $w_4 \neq r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1)))$ and $r_5 \neq 0$, then there exists a complex number c such that

$$\langle (x-c)\,u, Q_5(x)\rangle = 0.$$

Moreover, $\langle (x-c)\,u, Q_n(x)\rangle = 0$, $n \geq 5$. Indeed,

$$\begin{aligned}
\langle (x-c)\,u, Q_0(x)\rangle &= \beta_0 - c,\\
\langle (x-c)\,u, Q_1(x)\rangle &= \gamma_1 + (s_1 - r_1)(\beta_0 - c),\\
\langle (x-c)\,u, Q_2(x)\rangle &= (s_2 - r_2)\gamma_1 + (t_2 - r_2(s_1 - r_1))(\beta_0 - c),\\
\langle (x-c)\,u, Q_3(x)\rangle &= (t_3 - r_3(s_2 - r_2))\gamma_1 + (v_3 - r_3(t_2 - r_2(s_1 - r_1)))(\beta_0 - c),\\
\langle (x-c)\,u, Q_4(x)\rangle &= [v_4 - r_4(t_3 - (s_2 - r_2))]\gamma_1\\
&\quad + [w_4 - r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1)))](\beta_0 - c),\\
\langle (x-c)\,u, Q_5(x)\rangle &= [w_5 - r_5(v_4 - r_4(t_3 - r_3(s_2 - r_2)))]\gamma_1\\
&\quad - r_5[w_4 - r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1)))(\beta_0 - c)].
\end{aligned} \tag{2.1}$$

Then there exists c such that

$$\langle (x-c)\,u, Q_5(x)\rangle = 0.$$

This implies

$$c := \beta_0 - \frac{\gamma_1}{r_5}\frac{w_5 - r_5(v_4 - r_4(t_3 - r_3(s_2 - r_2)))}{w_4 - r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1)))}. \tag{2.2}$$

Thus,

$$\langle (x-c)\,u, Q_n(x)\rangle = -r_n \langle (x-c)\,u, Q_{n-1}(x)\rangle, \qquad n \geq 6.$$

On the other hand [8]

$$(x-c)\,u = \sum_{i=0}^{4} \frac{\langle (x-c)\,u, Q_i(x)\rangle}{\langle v, Q_i^2(x)\rangle} Q_i(x)\,v.$$

Therefore, if $w_4 \neq r_4\left(v_3 - r_3\left(t_2 - r_2\left(s_1 - r_1\right)\right)\right)$ and $w_5 r_5 \neq 0$, it can be deduced that the relation between u and v is

$$(x-c)\,u = q(x)v,$$

where q is a polynomial of exact degree 4.

Lemma 2.1. *Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be two monic orthogonal polynomials sequences $(MOPS)$ with respect to the regular linear functionals u and v respectively, where $\langle u,1\rangle = \langle v,1\rangle = 1$. Assuming that there exist sequences of complex numbers*

$$\{r_n\}_{n\geq 0}, \quad \{s_n\}_{n\geq 0}, \quad \{t_n\}_{n\geq 0}, \quad \{v_n\}_{n\geq 0} \quad \textit{and} \quad \{w_n\}_{n\geq 0}$$

with the initial conditions

$$r_0 = s_0 = t_0 = t_1 = v_0 = v_1 = v_2 = w_0 = w_1 = w_2 = w_3 = 0,$$

such that the relation (1.1) holds, for every $n \geq 0$, then

(1) *If $w_4 \neq r_4\left(v_3 - r_3\left(t_2 - r_2\left(s_1 - r_1\right)\right)\right)$ and $r_5 = 0$, then*

$$w_n \neq r_n\left(v_{n-1} - r_{n-1}\left(t_{n-2} - r_{n-2}\left(s_{n-3} - r_{n-3}\right)\right)\right),$$

for $n \geq 4$, and $r_n = 0$, for every $n \geq 5$. In this case the relation (1.1) reduce to a $1-5$ type relation

$$\begin{aligned} Q_n(x) = {} & P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x) \\ & + c_n P_{n-3}(x) + d_n P_{n-4}(x), \end{aligned}$$

$n \geq 0$, with

$$\begin{aligned} a_n &:= s_n - r_n & n &\geq 1, \\ b_n &:= t_n - r_n(s_{n-1} - r_{n-1}) & n &\geq 2, \\ c_n &:= v_n - r_n(t_{n-1} - r_{n-1}(s_{n-2} - r_{n-2})) & n &\geq 3, \\ d_n &:= w_n - r_n(v_{n-1} - r_{n-1}(t_{n-2} - r_{n-2}(s_{n-3} - r_{n-3}))) & n &\geq 4. \end{aligned}$$

(2) *If $w_4 \neq r_4\left(v_3 - r_3\left(t_2 - r_2\left(s_1 - r_1\right)\right)\right)$ and $r_5 \neq 0$ then $r_n \neq 0$, $n \geq 5$.*

(3) *If $w_4 \neq r_4\left(v_3 - r_3\left(t_2 - r_2\left(s_1 - r_1\right)\right)\right)$ and $w_5 r_5 \neq 0$ then $w_n r_n \neq 0$, for $n \geq 5$. Thus, in this case the relation (1.1) is a non-degenerate $2-5$ type relation.*

Proof. We have

$$
(2.3) \qquad \begin{cases}
\langle u, Q_1(x) \rangle = s_1 - r_1, \\
\langle u, Q_2(x) \rangle = t_2 - r_2(s_1 - r_1), \\
\langle u, Q_3(x) \rangle = v_3 - r_3(t_2 - r_2(s_1 - r_1)), \\
\langle u, Q_4(x) \rangle = w_4 - r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1))), \\
\langle u, Q_n(x) \rangle = -r_n \langle u, Q_{n-1} \rangle, \qquad n \geq 5.
\end{cases}
$$

(1) If $w_4 \neq r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1)))$ and $r_5 = 0$, from (2.3), we have

$$\langle u, Q_i(x) \rangle \neq 0, \qquad i = 1,\ 2,\ 3,\ 4$$

and

$$\langle u, Q_n(x) \rangle = 0, \quad n \geq 5.$$

So, there exists a polynomial q of degree 4 such that $u = q(x)v$. Therefore,

$$Q_n(x) = P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x) + c_n P_{n-3}(x) + d_n P_{n-4}(x),$$

for each $n \geq 0$, with $c_n \neq 0$, $n \geq 4$. Again, the relation (1.1) leads to

$$
\begin{array}{ll}
s_n = a_n + r_n & n \geq 1, \\
t_n = b_n + r_n\, a_{n-1} & n \geq 2, \\
v_n = c_n + r_n\, b_{n-1} & n \geq 3, \\
w_n = d_n + r_n\, c_{n-1} & n \geq 4,
\end{array}
$$

and $r_n d_{n-1} = 0$, $n \geq 5$. So, $r_n = 0$, $n \geq 5$, and $w_n \neq 0,\ n \geq 4$. Then, this case is the degenerate $1-5$ type relation.

(2) If $w_4 \neq r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1)))$ and $r_5 \neq 0$, according to (2.3), we have

$$\langle u, Q_4(x) \rangle \neq 0 \text{ and } \langle u, Q_5(x) \rangle \neq 0,$$

and if $r_n = 0$, for each $n \geq 6$, we get

$$\langle u, Q_n(x) \rangle = 0, \quad n \geq 6.$$

Assuming that there exists $n \geq 6$ such that $r_n = 0$, putting $s := \min\{n \in \mathbb{N} \,/\, n \geq 6,\ r_n = 0\}$, then

$$\langle u, Q_n(x) \rangle = 0, \quad n \geq s,$$

and

$$\langle u, Q_n(x)\rangle \neq 0, \quad 4 \leq n \leq s-1.$$

So, there exists a polynomial q of degree $s-1$ such that $u = q(x)v$ [7]. Therefore,

$$Q_n(x) = P_n(x) + \sum_{k=1}^{s-1} \alpha_{n,k} P_{n-k}(x),$$

where $\alpha_{n,s-1} \neq 0$, $n \geq s-1$. Taking into account (1.1), this is not possible. Thus $r_n \neq 0$, $n \geq 5$.

(3) If $w_4 \neq r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1)))$ and $w_5 r_5 \neq 0$, then there exists a constant c such that

$$(x-c)\,u = q(x)v,$$

where q is a polynomial of degree 4 and by (1.1) we can write

$$\begin{aligned}
& \langle (x-c)\,u, Q_n(x)\,Q_{n-5}(x)\rangle \\
= \; & \langle (x-c)\,u, (P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x) \\
& +v_n P_{n-3}(x) + w_n P_{n-4}(x))\,Q_{n-5}(x)\rangle \\
& -r_n \langle (x-c)\,u, Q_{n-1}(x)\,Q_{n-5}(x)\rangle \\
= \; & w_n \left\langle u, P_{n-4}^2(x)\right\rangle - r_n \langle (x-c)\,u, Q_{n-1}(x)\,Q_{n-5}(x)\rangle, \; n \geq 5.
\end{aligned}$$

Consequently, the following expression is obtained

$$\begin{aligned}
w_n \left\langle u, P_{n-4}^2(x)\right\rangle &= \langle (x-c)\,u, (Q_n(x) + r_n Q_{n-1}(x))\,Q_{n-5}(x)\rangle \\
&= \langle q(x)v, (Q_n(x) + r_n Q_{n-1}(x))\,Q_{n-4}(x)\rangle \\
&= r_n \langle v, q(x) Q_{n-1}(x)\,Q_{n-4}(x)\rangle \\
&= k_1\, r_n \left\langle v, Q_{n-1}^2(x)\right\rangle,
\end{aligned}$$

where k_1 is the leading coefficient of the polynomial q.Now, it is enough to apply (2) to obtain $r_n \neq 0$, $n \geq 4$, and from definition 1.1, we have $w_n r_n \neq 0$, $n \geq 5$.

□

In the following proposition, we show that if $w_4 \neq r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1)))$ and $r_5 w_5 \neq 0$ this is equivalent to assume that the functional $(x-c)\,u$ is regular.

Proposition 2.1. *Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be two $MOPS$ with respect to the regular functionals u and v, respectively, with $\langle u, 1\rangle = \langle v, 1\rangle = 1$. Assume that there exist sequences of complex numbers $\{r_n\}_{n\geq 0}, \{s_n\}_{n\geq 0}, \{t_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ with initial conditions $r_0 = s_0 = t_0 = t_1 = v_0 = v_1 = v_2 = 0$, such that the relation (1.1) holds and the initial conditions $w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1)))$ and $r_5 w_5 \neq 0$. Then the following statements are equivalent:*

$i)$ *The functional $(x - c) u$ is regular.*

$ii)$ $w_n \neq r_n (v_{n-1} - r_{n-1} (t_{n-2} - r_{n-2} (s_{n-3} - r_{n-3})))\,,\ n \geq 4.$

Proof. Multiplying the relation (1.1) by P_{n-1} and applying u, the same way for P_{n-2}, P_{n-3} and P_{n-4}, we get, respectively,

$$\langle u, Q_n (x) P_{n-1} (x)\rangle = (s_n - r_n) \left\langle u, P_{n-1}^2 (x)\right\rangle, \quad n \geq 1,$$

$$\langle u, Q_n (x) P_{n-2} (x)\rangle = (t_n - r_n (s_{n-1} - r_{n-1})) \left\langle u, P_{n-2}^2 (x)\right\rangle, \quad n \geq 2,$$

$$\langle u, Q_n (x) P_{n-3} (x)\rangle = (v_n - r_n (t_{n-1} - r_{n-1} (s_{n-2} - r_{n-2}))) \left\langle u, P_{n-3}^2 (x)\right\rangle, \quad n \geq 3,$$

$$\langle u, Q_n (x) P_{n-4} (x)\rangle = [w_n - r_n(v_{n-1} - r_{n-1}(t_{n-2} - r_{n-2}(s_{n-3} - r_{n-3}))] \left\langle u, P_{n-4}^2\right\rangle,$$

$n \geq 4$. Thus

$$w_n - r_n(v_{n-1} - r_{n-1}(t_{n-2} - r_{n-2}(s_{n-3} - r_{n-3})) \neq 0 \Leftrightarrow \langle u, Q_n (x) P_{n-4} (x)\rangle \neq 0, \ \ n \geq 4.$$

Knowing that $(x - c) u$ is regular if and only if $P_n (c) \neq 0$, for each $n \geq 0$.

Moreover, we need to show that $\langle u, Q_{n+4} (x) P_n (x)\rangle \neq 0 \Leftrightarrow P_n (c) \neq 0$, for each $n \geq 0$. Either for

$$P_n(x) = \sum_{i=0}^{n} a_{ni}(x - c)^i, \qquad n \geq 0,$$

with $a_{n0} = P_n(c)$ and $a_{nn} = 1$. From Lemma 2.1, we have

$$(x - c) u = q(x)v.$$

Hence

$$\begin{aligned} \langle u, Q_{n+4} (x) P_n (x)\rangle &= \left\langle (x - c)u, (x - c)^{n-1} Q_{n+4} (x)\right\rangle \\ &\quad + \sum_{i=1}^{n-1} a_{ni} \left\langle (x - c)u, (x - c)^{i-1} Q_{n+4} (x)\right\rangle \\ &\quad + P_n (c) \langle u, Q_{n+4} (x)\rangle \end{aligned}$$

$$= \left\langle v, q(x)(x-c)^{n-1}Q_{n+4}(x)\right\rangle + \sum_{i=1}^{n-1} a_{ni}\left\langle v, q(x)(x-c)^{i-1}Q_{n+4}(x)\right\rangle$$
$$+P_n(c)\left\langle u, Q_{n+4}(x)\right\rangle, \quad n \geq 0.$$

Hence

$$\left\langle u, Q_{n+4}(x)\, P_n(x)\right\rangle = P_n(c)\left\langle u, Q_{n+4}(x)\right\rangle, \quad n \geq 0,$$

from Lemma 2.1 and the relation (2.3), we get

$$\left\langle u, Q_n(x)\right\rangle \neq 0, \quad n \geq 4.$$

□

3. Characterization of orthogonality

Let $\{P_n\}_{n\geq 0}$ be a $MOPS$ with respect to a regular functional u and let $\{\beta_n\}_{n\geq 0}$ $\{\gamma_n\}_{n\geq 1}$ be the corresponding sequences of recurrence coefficients, so that

(3.1) $$P_{n+1}(x) = (x-\beta_n)\, P_n(x) - \gamma_n P_{n-1}(x), \qquad n \geq 0,$$

with the initial conditions $P_0(x)=1$, $P_{-1}(x)=0$ and the condition $\gamma_n \neq 0$, for each $n \geq 1$.

In this section, we give the characterizations of the orthogonality of a sequence $\{Q_n\}_{n\geq 0}$ of monic polynomials defined by a non-degenerate type relation (1.1).

From Lemma 2.1, the conditions $w_4 \neq r_4\,(v_3 - r_3\,(t_2 - r_2\,(s_1 - r_1)))$ and $r_5 w_5 \neq 0$ must hold, in order to have a non-degenerate $2-5$ type relation with $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ $MOPS$ and these conditions imply $w_n r_n \neq 0$, for each $n \geq 5$.

The following is the first characterization of the orthogonality of the sequence $\{Q_n\}_{n\geq 0}$.

Proposition 3.1. *Let $\{P_n\}_{n\geq 0}$ be a $MOPS$ satisfying (3.1), we define recursively a sequence $\{Q_n\}_{n\geq 0}$ of monic polynomials by the structure relation (1.1) such that $w_4 \neq r_4\,(v_3 - r_3\,(t_2 - r_2\,(s_1 - r_1)))$ and $w_n r_n \neq 0$ for $n \geq 5$. Then, $\{Q_n\}_{n\geq 0}$ is a $MOPS$ with recurrence coefficients $\left\{\widetilde{\beta}_n\right\}_{n\geq 0}$ and $\{\widetilde{\gamma}_n\}_{n\geq 1}$, where*

(3.2) $$\widetilde{\beta}_n := \beta_n - s_{n+1} + s_n + r_{n+1} - r_n, \quad n \geq 0,$$

$$(3.3)\qquad \begin{aligned}\widetilde{\gamma}_n \;&:= \gamma_n + t_n - t_{n+1} + s_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-1}\right)\\ &\quad -r_n\left(r_{n+1} - r_n - \widetilde{\beta}_n + \widetilde{\beta}_{n-1}\right), \quad n \geq 1,\end{aligned}$$

if and only if $\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3\widetilde{\gamma}_4 \neq 0$ *and if the following relations hold:*

$$(3.4)\qquad b_2 - f_2 = a_2(s_1 - r_1),$$

$$(3.5)\qquad b_3 - f_3 = a_3(s_2 - r_2),$$

$$(3.6)\qquad c_3 - b_3(s_1 - r_1) = a_3\left(t_2 - s_2(s_1 - r_1)\right),$$

$$(3.7)\qquad b_4 - f_4 = a_4(s_3 - r_3),$$

$$(3.8)\qquad c_4 - b_4(s_2 - r_2) = a_4\left(t_3 - s_3(s_2 - r_2)\right),$$

$$(3.9)\qquad \begin{aligned}e_4 - b_4\left(t_2 - s_2(s_1 - r_1)\right) &= a_4\left[v_3 - s_3\left(t_2 - s_2(s_1 - r_1)\right)\right.\\ &- \left. t_3(s_1 - r_1)\right] + c_4(s_1 - r_1),\end{aligned}$$

$$(3.10)\qquad b_5 - f_5 = a_5(s_4 - r_4),$$

$$(3.11)\qquad c_5 - b_5(s_3 - r_3) = a_5(t_4 - s_4(s_3 - r_3)),$$

$$(3.12)\qquad \begin{aligned}e_5 - b_5\left(t_3 - s_3(s_2 - r_2)\right) &= a_5\left[v_4 - s_4\left(t_3 - s_3(s_2 - r_2)\right)\right.\\ &- \left. t_4(s_2 - r_2)\right] + c_5(s_2 - r_2),\end{aligned}$$

$$(3.13)\qquad \begin{aligned}&k_5 - b_5\left[v_3 - s_3\left(t_2 - s_2(s_1 - r_1)\right) - t_3(s_1 - r_1)\right]\\ =\;& a_5\left[w_4 - s_4\left(v_3 - s_3\left(t_2 - s_2(s_1 - r_1)\right)\right) - t_3(s_1 - r_1)\right.\\ &\left. -t_4\left(t_2 - s_2(s_1 - r_1)\right) - v_4(s_1 - r_1)\right]\\ &+c_5\left(t_2 - s_2(s_1 - r_1)\right) + e_5(s_1 - r_1).\end{aligned}$$

As a consequence, for each $n \geq 6$, *we have*

$$(3.14)\qquad b_n = a_n s_{n-1},$$

$$(3.15)\qquad c_n = a_n t_{n-1},$$

$$e_n = a_n v_{n-1}, \quad k_n = a_n w_{n-1}, \tag{3.16}$$

$$f_n = a_n r_{n-1}, \tag{3.17}$$

where

$$a_n := \gamma_n + t_n - t_{n+1} + s_n \left(s_{n+1} - s_n - \beta_n + \beta_{n-1}\right), \quad n \geq 1, \tag{3.18}$$

$$b_n := s_n\gamma_{n-1} + t_n \left(s_{n+1} - s_n - \beta_n + \beta_{n-2}\right) - v_{n+1} + v_n, \quad n \geq 2, \tag{3.19}$$

$$c_n := t_n\gamma_{n-2} + v_n \left(s_{n+1} - s_n - \beta_n + \beta_{n-3}\right) - w_{n+1} + w_n, \quad n \geq 3, \tag{3.20}$$

$$\begin{aligned} e_n &:= v_n\gamma_{n-3} + w_n \left(s_{n+1} - s_n - \beta_n + \beta_{n-4}\right), \quad n \geq 4, \\ k_n &:= w_n\gamma_{n-4}, \quad n \geq 5, \end{aligned} \tag{3.21}$$

$$f_n := r_n\widetilde{\gamma}_{n-1}, \quad n \geq 2. \tag{3.22}$$

Proof. From the definition of Q_n, we find

$$\begin{aligned} Q_{n+1}(x) = {} & P_{n+1}(x) + s_{n+1}P_n(x) + t_{n+1}P_{n-1}(x) + v_{n+1}P_{n-2}(x) \\ & + w_{n+1}P_{n-3}(x) - r_{n+1}Q_n(x), \end{aligned} \tag{3.23}$$

$n \geq 0$, and by replacing (3.1) in (3.23), applying (1.1) to $xP_n(x)$, and substituting $xP_{n-1}(x), xP_{n-2}(x), xP_{n-3}(x)$ and $xP_{n-4}(x)$, using again (3.1), taking into account, polynomials with negative index are zero, we obtain

$$\begin{aligned} Q_{n+1}(x) &= (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x) + s_{n+1}P_n(x) \\ &\quad + t_{n+1}P_{n-1}(x) + v_{n+1}P_{n-2}(x) + w_{n+1}P_{n-3}(x) - r_{n+1}Q_n(x) \\ &= xQ_n(x) - \left(\beta_n - s_{n+1} + s_n + r_{n+1} - r_n\right)Q_n(x) \\ &\quad - r_n\left(\beta_n - s_{n+1} + s_n\right)Q_{n-1}(x) \\ &\quad + \left[s_n\left(\beta_n - s_{n+1} + s_n - \beta_{n-1}\right) + t_{n+1} - t_n - \gamma_n\right]P_{n-1}(x) \\ &\quad + \left[t_n\left(\beta_n - s_{n+1} + s_n - \beta_{n-2}\right) - s_n\gamma_{n-1} - v_n + v_{n+1}\right]P_{n-2}(x) \\ &\quad + \left[v_n\left(\beta_n - s_{n+1} + s_n - \beta_{n-3}\right) - t_n\gamma_{n-2} - w_n + w_{n+1}\right]P_{n-3}(x) \end{aligned}$$

$$+\left[w_n\left(\beta_n - s_{n+1} + s_n - \beta_{n-4}\right) - v_n\gamma_{n-3}\right] P_{n-4}(x)$$
$$-w_n\gamma_{n-4}P_{n-5}(x) - r_n\left(Q_n(x) - xQ_{n-1}(x)\right), \qquad n \geq 0.$$

Putting

$$\widetilde{\beta}_n := \beta_n - s_{n+1} + s_n + r_{n+1} - r_n, \qquad n \geq 0,$$

we get, for $n \geq 0$,

$$\begin{aligned} Q_{n+1}(x) &= \left(x - \widetilde{\beta}_n\right) Q_n(x) + r_n\left(r_{n+1} - r_n - \widetilde{\beta}_n\right) Q_{n-1}(x) \\ &\quad - \left[\gamma_n + t_n - t_{n+1} + s_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-1}\right)\right] P_{n-1}(x) \\ &\quad - \left[s_n\gamma_{n-1} + t_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-2}\right) + v_n - v_{n+1}\right] P_{n-2}(x) \\ &\quad - \left[t_n\gamma_{n-2} + v_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-3}\right) + w_n - w_{n+1}\right] P_{n-3}(x) \\ &\quad - \left[v_n\gamma_{n-3} + w_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-4}\right)\right] P_{n-4}(x) - w_n\gamma_{n-4}P_{n-5}(x) \\ &\quad - r_n\left(Q_n(x) - xQ_{n-1}(x)\right). \end{aligned}$$

So, $\{Q_n\}_{n\geq 0}$ is a $MOPS$ if and only if $\widetilde{\gamma}_n \neq 0$, for each $n \geq 1$, and

$$\begin{aligned} & r_n\left(r_{n+1} - r_n - \widetilde{\beta}_n\right) Q_{n-1}(x) \\ & - \left[\gamma_n + t_n - t_{n+1} + s_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-1}\right)\right] P_{n-1}(x) \\ & - \left[s_n\gamma_{n-1} + t_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-2}\right) + v_n - v_{n+1}\right] P_{n-2}(x) \\ & - \left[t_n\gamma_{n-2} + v_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-3}\right) + w_n - w_{n+1}\right] P_{n-3}(x) \\ & - \left[v_n\gamma_{n-3} + w_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-4}\right)\right] P_{n-4}(x) - w_n\gamma_{n-4}P_{n-5}(x) \\ & - r_n\left(Q_n(x) - xQ_{n-1}(x)\right) \end{aligned}$$

$$= -\widetilde{\gamma}_n Q_{n-1}(x), \qquad n \geq 0. \tag{3.24}$$

On the other hand, $\{Q_n\}_{n\geq 0}$ will be a $MOPS$ if and only if $\widetilde{\gamma}_n \neq 0$, for each $n \geq 1$, and

$$Q_n(x) - xQ_{n-1}(x) = -\widetilde{\beta}_{n-1}Q_{n-1}(x) - \widetilde{\gamma}_{n-1}Q_{n-2}(x), \qquad n \geq 1. \tag{3.25}$$

Replacing (3.25) in (3.24), we get the following expression

$$\begin{aligned}(3.26)\qquad & \left[\widetilde{\gamma}_n + r_n\left(r_{n+1} - r_n - \widetilde{\beta}_n + \widetilde{\beta}_{n-1}\right)\right] Q_{n-1}(x) + r_n\widetilde{\gamma}_{n-1}Q_{n-2}(x) \\ = \; & \left[\gamma_n + t_n - t_{n+1} + s_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-1}\right)\right] P_{n-1}(x) \\ & + \left[s_n\gamma_{n-1} + t_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-2}\right) + v_n - v_{n+1}\right] P_{n-2}(x) \\ & + \left[t_n\gamma_{n-2} + v_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-3}\right) + w_n - w_{n+1}\right] P_{n-3}(x) \\ & + \left[v_n\gamma_{n-3} + w_n\left(s_{n+1} - s_n - \beta_n + \beta_{n-4}\right)\right] P_{n-4}(x) + w_n\gamma_{n-4}P_{n-5}(x),\end{aligned}$$

which holds, for each $n \geq 1$.

First, suppose that $\{Q_n\}_{n\geq 0}$ is a $MOPS$ with recurrence coefficients $\left\{\widetilde{\beta}_n\right\}_{n\geq 0}$ and $\{\widetilde{\gamma}_n\}_{n\geq 1}$. Then,

$$Q_{n+1}(x) = \left(x - \widetilde{\beta}_n\right) Q_n(x) - \widetilde{\gamma}_n Q_{n-1}(x), \quad n \geq 0, \tag{3.27}$$

with the initial conditions $Q_0(x) = 1$, $Q_{-1}(x) = 0$ and the condition $\widetilde{\gamma}_n \neq 0$ for every $n \geq 1$.

To obtain (3.26), it is enough to replace $Q_n(x) - xQ_{n-1}(x)$ in (3.24).

Conversely, if (3.26) is satisfied and $\widetilde{\gamma}_n \neq 0$, for every $n \geq 1$, we show that $\{Q_n\}_{n\geq 0}$ satisfies (3.27). Notice that, for every $n \geq 1$,

$$\begin{aligned} & r_n\left(\widetilde{\beta}_{n-1}Q_{n-1}(x) + \widetilde{\gamma}_{n-1}Q_{n-2}(x)\right) = -\widetilde{\gamma}_n Q_{n-1}(x) \\ & + \left(r_{n+1} - r_n - \widetilde{\beta}_n\right)\left(s_nP_{n-1}(x) + t_nP_{n-2}(x) + v_nP_{n-3}(x) + w_nP_{n-4}(x) - r_nQ_{n-1}(x)\right) \\ & + s_n\left(\beta_{n-1}P_{n-1}(x) + \gamma_{n-1}P_{n-2}(x)\right) + t_n\left(\beta_{n-2}P_{n-2}(x) + \gamma_{n-2}P_{n-3}(x)\right) \\ & + v_n\left(\beta_{n-3}P_{n-3}(x) + \gamma_{n-3}P_{n-4}(x)\right) + w_n\left(\beta_{n-4}P_{n-4}(x) + \gamma_{n-4}P_{n-5}(x)\right) \\ & + \left(\gamma_n + t_n - t_{n+1}\right) P_{n-1}(x) + \left(v_n - v_{n+1}\right) P_{n-2}(x) \\ & + \left(w_n - w_{n+1}\right) P_{n-3}(x) - v_{n+1}P_{n-4}(x).\end{aligned}$$

Applying (1.1) in $s_nP_{n-1}(x) + t_nP_{n-2}(x) + v_nP_{n-3}(x) + w_nP_{n-4}(x)$ and using (3.1), we get

$$\begin{aligned} r_n\left[\widetilde{\beta}_{n-1}Q_{n-1}(x) + \widetilde{\gamma}_{n-1}Q_{n-2}(x)\right] &= r_n\left(xQ_{n-1}(x) - Q_n(x)\right) + \left(x - \widetilde{\beta}_n\right) Q_n(x) \\ &- Q_{n+1}(x) - \widetilde{\gamma}_n Q_{n-1}(x), \quad n \geq 1.\end{aligned}$$

Consequently,

$$\begin{aligned}
&Q_{n+1}(x) - \left(x - \widetilde{\beta}_n\right) Q_n(x) + \widetilde{\gamma}_n Q_{n-1}(x) \\
&= -r_n \left[Q_n(x) - \left(x - \widetilde{\beta}_{n-1}\right) Q_{n-1}(x) + \widetilde{\gamma}_{n-1} Q_{n-2}(x)\right], \; n \geq 1.
\end{aligned}$$

Moreover, from (1.1)

$$Q_1(x) = P_1(x) + s_1 - r_1 = x - \beta_0 + s_1 - r_1 = x - \widetilde{\beta}_0,$$

we deduce recursively

$$Q_{n+1}(x) = \left(x - \widetilde{\beta}_n\right) Q_n(x) - \widetilde{\gamma}_n Q_{n-1}(x), \quad n \geq 1.$$

Thus, $\{Q_n\}_{n\geq 0}$ is a $MOPS$ with recurrence coefficients $\left\{\widetilde{\beta}_n\right\}_{n\geq 0}$ and $\{\widetilde{\gamma}_n\}_{n\geq 1}$.

From (1.1) and (3.3) we have (3.26), that is equivalent to

$$\begin{aligned}
(f_n - r_{n-1}a_n)\, Q_{n-2}(x) &= (b_n - s_{n-1}a_n)\, P_{n-2}(x) + (c_n - t_{n-1}a_n)\, P_{n-3}(x) \\
&\quad + (e_n - v_{n-1}a_n)\, P_{n-4}(x) + (k_n - w_{n-1}a_n)\, P_{n-5}(x), \quad n \geq 2,
\end{aligned}$$

where a_n, b_n, c_n, f_n, e_n and k_n are defined by $(3.18) - (3.22)$.

Now, we show that (3.28) holds with $\widetilde{\gamma}_n \neq 0$, $n \geq 1$ if and only if $\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3\widetilde{\gamma}_4 \neq 0$ and the relations $(3.4) - (3.17)$ hold. Replacing in (3.28) by $n = 2$, $n = 3$, $n = 4$ and $n = 5$, we get the relations $(3.4) - (3.13)$.Therefore it is easy to check that $(3.14) - (3.17)$ hold for $n \geq 6$. Indeed, since $w_4 \neq r_4\,(v_3 - r_3\,(t_2 - r_2\,(s_1 - r_1)))$ and $r_n \neq 0$, for $n \geq 5$. Then by (2.3), we deduce

$$\langle u, Q_n \rangle \neq 0, \quad n \geq 4.$$

Applying u to both sides of (3.28), we get

$$(f_n - r_{n-1}a_n)\, \langle u, Q_{n-2} \rangle = 0, \quad n \geq 6.$$

This leads to

$$f_n = r_{n-1}a_n, \quad n \geq 6,$$

and this proves (3.17). Multiplying (3.28) by P_{n-2}, P_{n-3}, P_{n-4} and P_{n-5} and applying u, we obtain, respectively $(3.14) - (3.16)$.

Conversely, from (3.16) and (3.17) we have

$$\widetilde{\gamma}_{n-1} = \frac{r_{n-1}}{r_n}\frac{w_n}{w_{n-1}}\gamma_{n-4}, \quad n \geq 6,$$

this implies

$$\widetilde{\gamma}_n = \frac{r_n}{r_{n+1}}\frac{w_{n+1}}{w_n}\gamma_{n-3}, \qquad n \geq 5.$$

Thus

$$\widetilde{\gamma}_n \neq 0, \quad n \geq 5.$$

Hence, the conditions $(3.4) - (3.17)$ with $\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3\widetilde{\gamma}_4 \neq 0$ imply that (3.28) holds and $\widetilde{\gamma}_n \neq 0$, for every $n \geq 1$. □

Now, we will prove that the orthogonality of the sequence $\{Q_n\}_{n\geq 0}$ can be also characterized by the fact that there are five sequences depending on the parameters r_n, s_n, t_n, v_n, $w_n$and the recurrence coefficients which remain constants.

Theorem 3.1. *Let $\{P_n\}_{n\geq 0}$ be a $MOPS$ with respect to a regular linear functional u and the sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ is given by the relation (1.1). If $\{Q_n\}_{n\geq 0}$ is a $MOPS$ with respect to a regular linear functional v, then*

$$k(x-c)u = \left(x^4 + ax^3 + bx^2 + dx + e\right)v \tag{3.28}$$

with c, a, b, d, $e \in \mathbb{C}$ and $k \in \mathbb{C}\backslash\{0\}$ and the normalizations for these linear functionals $\langle u, 1\rangle = \langle v, 1\rangle = 1$.

Proof. Applying the regular linear functional u corresponding to the $MOPS$ $\{P_n\}_{n\geq 0}$ in (1.1), we obtain, for each $n \geq 4$

$$\langle (x-c)u, Q_n(x)\rangle = 0.$$

Then, according to [8] taking into account (1.1), we expand the linear functional u in terms of the dual basis $\left\{\frac{Q_i v}{\langle v, Q_i^2\rangle}\right\}_{i\geq 0}$ of the $MOPS$ $\{Q_n\}_{n\geq 0}$ as follows:

$$(x-c)u = \sum_{i=0}^{4}\frac{\langle (x-c)u, Q_i\rangle}{\langle v, Q_i^2\rangle}Q_i v.$$

Since $\{Q_n\}_{n\geq 0}$ is a $MOPS$ with respect to v, the recurrence coefficients $\left\{\widetilde{\beta}_n\right\}_{n\geq 0}$ and $\{\widetilde{\gamma}_n\}_{n\geq 1}$ are given by (3.2) and (3.3), furthermore

$$\widetilde{\gamma}_n = \frac{\langle v, Q_n^2\rangle}{\left\langle v, Q_{n-1}^2\right\rangle} \neq 0, \quad n \geq 1. \tag{3.29}$$

Indeed, making both sides of (3.28) acting on the polynomials Q_0, Q_1, Q_2, Q_3 and Q_4, and taking into account (2.1) , we get

$$
\begin{aligned}
(3.30) \quad & k\left[\left(t_3-r_3\left(s_2-r_2\right)\right)\gamma_1+\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)\left(\beta_0-c\right)\right] \\
= & \left(\widetilde{\beta}_3+\widetilde{\beta}_2+\widetilde{\beta}_1+\widetilde{\beta}_0\right)\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3+a\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3
\end{aligned}
$$

$$
\begin{aligned}
(3.31) \quad & k\left[\left(s_2-r_2\right)\gamma_1+\left(t_2-r_2\left(s_1-r_1\right)\right)\left(\beta_0-c\right)\right] \\
= & \left({}_1\widetilde{\gamma}_2\widetilde{\gamma}_3+\widetilde{\gamma}_1+\widetilde{\gamma}_2+\widetilde{\beta}_0^2+\widetilde{\beta}_1^2+\widetilde{\beta}_2^2+\widetilde{\beta}_1\widetilde{\beta}_0+\widetilde{\beta}_1\widetilde{\beta}_2+\widetilde{\beta}_2\widetilde{\beta}_0\right)\widetilde{\gamma}_1\widetilde{\gamma}_2 \\
& +a\left(\widetilde{\beta}_2+\widetilde{\beta}_1+\widetilde{\beta}_0\right)\widetilde{\gamma}_1\widetilde{\gamma}_2+b\widetilde{\gamma}_1\widetilde{\gamma}_2
\end{aligned}
$$

$$
\begin{aligned}
(3.32) \quad & k\left[\gamma_1+\left(s_1-r_1\right)\left(\beta_0-c\right)\right] \\
= & 2\left(\widetilde{\beta}_1+\widetilde{\beta}_0\right)\widetilde{\gamma}_1^2+\left(\widetilde{\beta}_2+2\widetilde{\beta}_1+\widetilde{\beta}_0\right)\widetilde{\gamma}_1\widetilde{\gamma}_2+\left(\widetilde{\beta}_0^3+\widetilde{\beta}_1^3+\widetilde{\beta}_1^2\widetilde{\beta}_0+\widetilde{\beta}_1\widetilde{\beta}_0^2\right)\widetilde{\gamma}_1 \\
& +a\left(\widetilde{\gamma}_1+\widetilde{\gamma}_2+\widetilde{\beta}_1^2+\widetilde{\beta}_0^2+\widetilde{\beta}_1\widetilde{\beta}_0\right)\widetilde{\gamma}_1+b\left(\widetilde{\beta}_1+\widetilde{\beta}_0\right)\widetilde{\gamma}_1+d\,\widetilde{\gamma}_1
\end{aligned}
$$

$$
\begin{aligned}
(3.33) \quad & k\left(\beta_0-c\right) \\
= & \widetilde{\beta}_0^4+\left(\widetilde{\gamma}_1+\widetilde{\gamma}_2+3\widetilde{\beta}_0^2+\widetilde{\beta}_1^2+2\widetilde{\beta}_1\widetilde{\beta}_0\right)\widetilde{\gamma}_1+a\left(\left(\widetilde{\beta}_1+2\widetilde{\beta}_0\right)\widetilde{\gamma}_1+\widetilde{\beta}_0^3\right) \\
& +b\left(\widetilde{\gamma}_1+\widetilde{\beta}_0^2\right)+d\,\widetilde{\beta}_0+e
\end{aligned}
$$

$$
\begin{aligned}
(3.34) \quad & k\left\{\left[v_4-r_4\left(t_3-\left(s_2-r_2\right)\right)\right]\gamma_1\right. \\
& \left.+\left[w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)\right]\left(\beta_0-c\right)\right\} \\
= & \widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3\widetilde{\gamma}_4
\end{aligned}
$$

where, $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, $\widetilde{\gamma}_3$ and $\widetilde{\gamma}_4$ are given by (3.2) and(3.3) .

Using the relations $(3.30)-(3.34)$ and taking into account (2.2) , thus, the values of c, a, b, d, e and k are given as follows

$$
\begin{aligned}
k &= \frac{r_5}{w_5}\frac{\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3\widetilde{\gamma}_4}{\gamma_1}, \\
c &= \beta_0-\frac{\gamma_1}{r_5}\frac{w_5-r_5\left(v_4-r_4\left(t_3-r_3(s_2-r_2)\right)\right)}{w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)},
\end{aligned}
$$

$$
\begin{aligned}
a &= -\widetilde{\beta}_3-\widetilde{\beta}_2-\widetilde{\beta}_1-\widetilde{\beta}_0+\frac{v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)}{w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)} \\
&\quad +\frac{r_5}{w_5}\frac{\left(t_3-r_3\left(s_2-r_2\right)\right)w_4-\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)v_4}{w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)} \\
b &= \widetilde{\beta}_3\widetilde{\beta}_0-\widetilde{\beta}_1\widetilde{\beta}_2+\widetilde{\beta}_2^2\widetilde{\beta}_1+\widetilde{\beta}_1^2\widetilde{\beta}_2+\widetilde{\beta}_0\widetilde{\beta}_2\widetilde{\beta}_1+\widetilde{\beta}_3\widetilde{\beta}_2\widetilde{\beta}_1-\widetilde{\beta}_1^2-\widetilde{\beta}_2^2-\widetilde{\gamma}_1-\widetilde{\gamma}_2-\widetilde{\gamma}_3 \\
&\quad +\left[\frac{\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)}{w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)}\right. \\
&\quad \left.+\frac{r_5}{w_5}\frac{\left(t_3-r_3\left(s_2-r_2\right)\right)w_4-\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)v_4}{w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)}\right]\left(\widetilde{\beta}_2\widetilde{\beta}_1+\widetilde{\beta}_0\right) \\
&\quad +\frac{r_5}{w_5}\left(s_2-r_2\right)\widetilde{\gamma}_3+\frac{\widetilde{\gamma}_3}{w_5}\frac{w_5-r_5\left(v_4-r_4\left(t_3-\left(s_2-r_2\right)\right)\right)}{w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)}\left(t_2-r_2\left(s_1-r_1\right)\right), \\
d &= -\left(\widetilde{\beta}_0^3+\widetilde{\beta}_1^3+\widetilde{\beta}_1^2\widetilde{\beta}_0+\widetilde{\beta}_1\widetilde{\beta}_0^2\right)-\left(2\widetilde{\beta}_1+2\widetilde{\beta}_0\right)\widetilde{\gamma}_1-\left(\widetilde{\beta}_2+2\widetilde{\beta}_1+\widetilde{\beta}_0\right)\widetilde{\gamma}_2 \\
&\quad +\left[\frac{\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)}{w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)}\right. \\
&\quad \left.+\frac{r_5}{w_5}\frac{\left(t_3-r_3\left(s_2-r_2\right)\right)w_4-\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)v_4}{w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)}\right]\left(\widetilde{\beta}_2\widetilde{\beta}_1+\widetilde{\beta}_0\right) \\
&\quad +\frac{r_5}{w_5}\left(s_2-r_2\right)\widetilde{\gamma}_3+\frac{\widetilde{\gamma}_3}{w_5}\frac{w_5-r_5\left(v_4-r_4\left(t_3-\left(s_2-r_2\right)\right)\right)}{w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)}\left(t_2-r_2\left(s_1-r_1\right)\right), \\
d &= -\left(\widetilde{\beta}_0^3+\widetilde{\beta}_1^3+\widetilde{\beta}_1^2\widetilde{\beta}_0+\widetilde{\beta}_1\widetilde{\beta}_0^2\right)-\left(2\widetilde{\beta}_1+2\widetilde{\beta}_0\right)\widetilde{\gamma}_1-\left(\widetilde{\beta}_2+2\widetilde{\beta}_1+\widetilde{\beta}_0\right)\widetilde{\gamma}_2 \\
&\quad +\left[r_5+\frac{w_5-r_5\left(v_4-r_4\left(t_3-\left(s_2-r_2\right)\right)\right)}{w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)}\left(s_1-r_1\right)\right]\frac{\widetilde{\gamma}_2\widetilde{\gamma}_3}{w_5} \\
&\quad -a\left[\widetilde{\gamma}_1+\widetilde{\gamma}_2+\widetilde{\beta}_1^2+\widetilde{\beta}_0^2+\widetilde{\beta}_1\widetilde{\beta}_0\right]-b\left(\widetilde{\beta}_1+\widetilde{\beta}_0\right), \\
e &= \frac{\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3}{w_5}\frac{w_5-r_5\left(v_4-r_4\left(t_3-\left(s_2-r_2\right)\right)\right)}{w_4-r_4\left(v_3-r_3\left(t_2-r_2\left(s_1-r_1\right)\right)\right)} \\
&\quad -\left(\widetilde{\beta}_0^4+\left(\widetilde{\gamma}_1+\widetilde{\gamma}_2+3\widetilde{\beta}_0^2+\widetilde{\beta}_1^2+2\widetilde{\beta}_1\widetilde{\beta}_0\right)\widetilde{\gamma}_1\right) \\
&\quad -a\left(\left(\widetilde{\beta}_1+2\widetilde{\beta}_0\right)\widetilde{\gamma}_1+\widetilde{\beta}_0^3\right)-b\left(\widetilde{\gamma}_1+\widetilde{\beta}_0^2\right)-d\widetilde{\beta}_0.
\end{aligned}
$$

$\square$

4. Particular case

In this section, a special case of relation (1.1) is obtained. Let us consider the symmetric $MOPS$ $\{P_n\}_{n\geq 1}$, this means that $\beta_n = 0$, for each $n \geq 0$. From Proposition 3.1, the equations (3.14), (3.15), (3.16) and (3.17) become, for each $n \geq 6$,

$$s_{n+1} = s_n + \frac{v_{n-1}}{w_{n-1}}\gamma_{n-4} - \frac{v_n}{w_n}\gamma_{n-3}, \tag{4.1}$$

$$t_{n+1} = t_n + \gamma_n - \frac{w_n}{w_{n-1}}\gamma_{n-4} + s_n\left(s_{n+1} - s_n\right), \tag{4.2}$$

$$\begin{aligned} v_{n+1} &= v_n + s_n\gamma_{n-1} + t_n\left(s_{n+1} - s_n\right) \\ &\quad - s_{n-1}\left[\gamma_n + t_n - t_{n+1} + s_n\left(s_{n+1} - s_n\right)\right], \end{aligned} \tag{4.3}$$

$$\begin{aligned} w_{n+1} &= w_n + t_n\gamma_{n-2} + v_n\left(s_{n+1} - s_n\right) \\ &\quad - t_{n-1}\left[\gamma_n + t_n - t_{n+1} + s_n\left(s_{n+1} - s_n\right)\right], \end{aligned} \tag{4.4}$$

$$r_{n+1} = r_n + \frac{w_n}{w_{n-1}}\frac{\gamma_{n-4}}{r_n} - \frac{w_{n+1}}{w_n}\frac{\gamma_{n-3}}{r_{n+1}}, \qquad n \geq 5. \tag{4.5}$$

The equations (3.2) and (3.3) become

$$\begin{aligned} \widetilde{\beta}_n &= s_n - s_{n+1} + r_{n+1} - r_n, \quad n \geq 0, \\ \widetilde{\gamma}_n &= \gamma_n + t_n - t_{n+1} + s_n\left(s_{n+1} - s_n\right) - r_n\left(s_{n+1} - s_n + \widetilde{\beta}_{n-1}\right), \quad n \geq 1, \end{aligned}$$

for each $n \geq 6$, we have

$$\begin{aligned} \widetilde{\beta}_n &= \frac{v_n}{w_n}\gamma_{n-3} - \frac{v_{n-1}}{w_{n-1}}\gamma_{n-4} + \frac{w_n}{w_{n-1}}\frac{\gamma_{n-4}}{r_n} - \frac{w_{n+1}}{w_n}\frac{\gamma_{n-3}}{r_{n+1}} \\ &= \frac{\gamma_{n-3}}{w_n}\left(v_n - \frac{w_{n+1}}{r_{n+1}}\right) - \frac{\gamma_{n-4}}{w_{n-1}}\left(v_{n-1} - \frac{w_n}{r_n}\right). \end{aligned} \tag{4.6}$$

$$\widetilde{\gamma}_n = \frac{w_n}{w_{n-1}}\gamma_{n-4} - r_n\left(\frac{v_{n-1}}{w_{n-1}}\gamma_{n-4} - \frac{v_n}{w_n}\gamma_{n-3}\right) - r_n\widetilde{\beta}_{n-1}. \tag{4.7}$$

In this case, we treat the following three subcases.

i) If $s_n = s_1$ and $r_{n-1} = r_1$, for each $n \geq 6$, from (4.1) and (4.5), we obtain

$$\begin{aligned} \frac{v_n}{w_n}\gamma_{n-3} &= \frac{v_{n-1}}{w_{n-1}}\gamma_{n-4} = \dots = \frac{v_4}{w_4}\gamma_1, \\ \frac{w_{n+1}}{w_n}\gamma_{n-3} &= \frac{w_n}{w_{n-1}}\gamma_{n-4} = \dots = \frac{w_5}{w_4}\gamma_1, \end{aligned}$$

the relation (4.7) yields

(4.8) $$\widetilde{\gamma}_n = \frac{w_5}{w_4}\gamma_1, \quad n \geq 6.$$

We conclude that $\widetilde{\beta}_n = 0$ and $\widetilde{\gamma}_n$ are constants, for each $n \geq 6$. From (4.2), we have

(4.9) $$t_{n+1} = t_n + \gamma_n - \frac{w_5}{w_4}\gamma_1.$$

ii) If $s_n = s_1$, $r_{n-1} = r_1$ and $t_n = t_2$, for each $n \geq 6$, from (4.9) and (4.8), we get

$$\widetilde{\gamma}_n = \gamma_n, \qquad n \geq 6.$$

The coefficients γ_n are constants, for each $n \geq 6$, then $\{P_n\}_{n\geq 0}$ is the sequence of anti-associated polynomials of order 6 for the Chebyshev polynomials of the second kind [10].

iii) If $r_{n-1} = r_1$, $s_n = s_1$, $t_n = t_2$ and $v_n = v_3$, for each $n \geq 6$, from (4.3), we have

$$v_{n+1} = v_n + s_n\gamma_{n-1} - s_{n-1}\gamma_n, \qquad n \geq 6,$$

hence

$$v_{n+1} = v_n + s_1\left(\gamma_{n-1} - \gamma_n\right), \qquad n \geq 7,$$

it is clear that $s_1\left(\gamma_{n-1} - \gamma_n\right) = 0$, for all $n \geq 7$.

iv) If $r_{n-1} = r_1$, $s_n = s_1$, $t_n = t_2$, $v_n = v_3$ and $w_n = w_4$, for each $n \geq 6$, from (4.3), we have

$$w_{n+1} = w_n + t_n\gamma_{n-2} - t_{n-1}\gamma_n, \qquad n \geq 6,$$

hence

$$w_{n+1} = w_n + t_2\left(\gamma_{n-2} - \gamma_n\right), \qquad n \geq 8,$$

it is clear that $t_2\left(\gamma_{n-2} - \gamma_n\right) = 0$, for all $n \geq 8$.

Remark 4.1. *If $r_{n-1} = r_1$, $s_n = s_1$ and $t_n = t_2$ or $r_{n-1} = r_1$, $s_n = s_1$, $t_n = t_2$ and $v_n = v_3$ or $r_{n-1} = r_1$, $s_n = s_1$, $t_n = t_2$, $v_n = v_3$ and $w_n = w_4$ for each $n \geq 6$, then*

$$\begin{aligned} \widetilde{\beta}_n &= 0, & n \geq 6, \\ \widetilde{\gamma}_n &= \gamma_n = \gamma_6, & n \geq 6. \end{aligned}$$

Example 1. *Let $\{P_n\}_{n\geq 0}$ be the sequence of monic Chebyshev polynomials of the second kind orthogonal with respect to the weight function $\mathcal{W}(x) = (1-x^2)^{\frac{1}{2}}$ on $(-1,1)$. Then $\beta_n = 0$, $\gamma_n = \dfrac{1}{4}$, $n \geq 1$, and the relations (4.1), (4.2), (4.3) and (4.5), for each $n \geq 6$, become*

$$\begin{aligned}
s_{n+1} &= s_n + \frac{1}{4}\left(\frac{v_{n-1}}{w_{n-1}} - \frac{v_n}{w_n}\right), \\
t_{n+1} &= t_n + \frac{1}{4}\left(1 - \frac{w_n}{w_{n-1}}\right) + s_n\left(s_{n+1} - s_n\right), \\
v_{n+1} &= v_n + \frac{1}{4}s_n + t_n\left(s_{n+1} - s_n\right) - s_{n-1}\left[\frac{1}{4} + t_n - t_{n+1} + s_n\left(s_{n+1} - s_n\right)\right], \\
w_{n+1} &= w_n + \frac{1}{4}t_n + v_n\left(s_{n+1} - s_n\right) - t_{n-1}\left[\frac{1}{4} + t_n - t_{n+1} + s_n\left(s_{n+1} - s_n\right)\right], \\
r_n &= r_{n-1} + \frac{1}{4}\left(\frac{w_{n-1}}{r_{n-1}w_{n-2}} - \frac{w_n}{r_n w_{n-1}}\right).
\end{aligned}$$

Assume that $r_{n-1} = r_1$, $s_n = s_1$ and $t_n = t_2$, for each $n \geq 6$, we obtain

$$w_{n+1} = w_n + \frac{1}{4}\left(t_n - t_{n-1}\right), \qquad n \geq 6,$$

in particular,

$$w_{n+1} = w_n, \qquad n \geq 7.$$

In this situation, we deduce constant connection coefficients, for $n \geq 7$.

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