A COMBINATION OF ORTHOGONAL POLYNOMIALS SEQUENCES: 2 – 5 TYPE RELATION

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Abstract. In the present paper, a new characterization of the orthogonality of a monic polynomials sequence \( \{Q_n\}_{n \geq 0} \) is obtained. This is defined as a linear combination of another monic orthogonal polynomials sequence \( \{P_n\}_{n \geq 0} \) such as

\[
Q_n(x) + r_n Q_{n-1}(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x) + v_n P_{n-3}(x) + w_n P_{n-4}(x), \quad n \geq 0
\]

where \( w_n r_n \neq 0 \), for every \( n \geq 5 \). Furthermore, the relation between the corresponding linear functionals is showed to be

\[
k(x-c)u = (x^4 + ax^3 + bx^2 + dx + e)v
\]

where \( c, a, b, d, e \in \mathbb{C} \) and \( k \in \mathbb{C} \setminus \{0\} \). Finally, an illustration using special case of the above type relation is given.

1. Introduction

Let \( \mathcal{P} \) be the linear space of polynomials in one variable with complex coefficients and let \( \mathcal{P}' \) be its algebraic dual. \( \langle u, f \rangle \) denotes the action of \( u \) in \( \mathcal{P}' \) on \( f \) in \( \mathcal{P} \) and by \( (u)_n := \langle u, x^n \rangle \), \( n \geq 0 \), the moments of \( u \) with respect to the monomial sequence \( \{x^n\}_{n \geq 0} \). When \( (u)_0 = 1 \), the linear functional \( u \) is said to be normalized.

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In this work we need to recall some operations in $P'$, (see [6], [8]). For any $u$ in $P'$, any $q$ in $P$ and any complex numbers $a, b, c$ with $a \neq 0$, let $Du = u', qu, h_au, \tau_bu$ and $\sigma u$ be respectively the derivative, the left multiplication, the translation, the homothetic and the pair part of the linear functionals defined by duality:

$$
\langle u', f \rangle := -\langle u, f' \rangle,
$$

$$
\langle qu, f \rangle := \langle u, qf \rangle,
$$

$$
\langle h_au, f \rangle := \langle u, h_afx \rangle = \langle u, f(ax) \rangle,
$$

$$
\langle \tau_bu, f \rangle := \langle u, \tau_{-b}f \rangle = \langle u, f(x+b) \rangle,
$$

$$
\langle \sigma u, f \rangle := \langle u, \sigma f \rangle = \langle u, f(x^2) \rangle, \quad f \in P.
$$

The linear functional $u$ is called regular (quasi-definite) if the leading principal submatrices $H_n$ of the Hankel matrix $H = (u_{i+j})_{i,j \geq 0}$ related to the moments $u_n = \langle u, x^n \rangle, n \geq 0$, are nonsingular, for each $n \geq 0$ [6].

**Definition 1.1.** [6] A sequence of monic polynomials $\{P_n\}_{n \geq 0}$ is called orthogonal with respect to the linear functional $u$ if the following orthogonality conditions hold

$$
\langle u, P_n(x) P_m(x) \rangle = 0, \quad n \neq m,
$$

$$
\langle u, P_n^2(x) \rangle \neq 0, \quad n \geq 0,
$$

where $\deg P_n = n$, for every $n \geq 0$.

In this case, $\{P_n\}_{n \geq 0}$ satisfies the following two order recurrence relation:

$$
\begin{cases}
P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_nP_{n-1}(x), & n \geq 1, \\
P_0(x) = 1, \quad P_1(x) = x - \beta_0,
\end{cases}
$$

where $\gamma_n \neq 0$, for each $n \geq 1$.

Let $u$ and $v$ be two regular linear functionals and let $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ be the corresponding sequences of monic orthogonal polynomials. Assume that there exist non-negative integer numbers $M$ and $N$, and sequences of complex numbers $\{r_{i,n}\}_{n \geq 0}$ and $\{s_{k,n}\}_{n \geq 0}$ such that the structure relation

$$
Q_n(x) + \sum_{i=1}^{M} r_{i,n}Q_{n-i}(x) = P_n(x) + \sum_{i=1}^{N} s_{i,n}P_{n-i}(x)
$$

holds for $n \geq 0$. Further, assume that $r_{M,M+N} \neq 0$ and $s_{N,M+N} \neq 0$, det $[\alpha_{ij}]_{i,j=1}^{M+N} \neq 0$ where the entries $\alpha_{ij}$ of the matrix are defined on the basis of $\{r_{i,n}\}_{n \geq 0}$ and
\{s_{k,n}\}_{n \geq 0}$. Then there exist two polynomials $\Phi$ and $\Psi$ with $\deg \Phi = M$ and $\deg \Psi = N$ such that

$$\Phi(z)u = \Psi(z)v.$$  

These polynomials $\Phi$ and $\Psi$ can be constructed in an explicit way \[9\]. On the other hand, the converse result is also analyzed. A characterization theorem for the sequence $\{Q_n\}_{n \geq 0}$ to be orthogonal assuming $\{P_n\}_{n \geq 0}$ is orthogonal is obtained when $M = 0$ and $N = 1$, $M = 1$ and $N = 1$, $M = 0$ and $N = 2$, $M = 1$ and $N = 2$, $M = 0$ and $N = 3$, $M = 0$ and $N = 4$, $M = 1$ and $N = 4$.

In this contribution, we are interested to the case $M = 1$ and $N = 4$.

Let $\{P_n\}_{n \geq 0}$ be the MOPS with respect to the regular functional $u$ and $\{Q_n\}_{n \geq 0}$ be monic polynomials sequence with $\deg Q_n = n$. Suppose that these sequences are related by $2 - 5$ type relation as follows:

\[(1.1) \ P_n(x) + r_nQ_{n-1}(x) = P_{n}(x) + s_nP_{n-1}(x) + t_nP_{n-2}(x) + v_nP_{n-3}(x) + w_nP_{n-4}(x),\]

$n \geq 0$, with the initial conditions $Q_0(x) = P_0(x) = 1$ and $Q_1(x) = P_{-m}(x) = 0$, for $m = 1, 2, 3, 4$ and $\{r_n\}_{n \geq 0} : \{s_n\}_{n \geq 0} : \{t_n\}_{n \geq 0} : \{v_n\}_{n \geq 0} : \{w_n\}_{n \geq 0}$ are sequences of complex numbers with the initial conditions

$$r_0 = s_0 = t_0 = t_1 = v_0 = v_1 = v_2 = w_0 = w_1 = w_2 = w_3 = 0.$$  

The main purpose of this paper is to obtain necessary and sufficient conditions for the orthogonality of the monic polynomials sequence $\{Q_n\}_{n \geq 0}$. In addition, we establish a relation between the linear functionals $u$ and $v$, respectively, corresponding to MOPS's $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ as $k(x - c)u = (x^4 + ax^3 + bx^2 + dx + e)v$ with $c, a, b, d, e \in \mathbb{C}$ and $k \in \mathbb{C} \setminus \{0\}$. This article is organized as follows. In section 2, we develop some basic results and lemmas. Section 3, is devoted to find the characterizations of the orthogonality of the monic polynomials sequence $\{Q_n\}_{n \geq 0}$. Finally, we illustrate a special case of the above type relation.

### 2. 2 – 5 Type Relation

Let $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ be two sequences of monic orthogonal polynomials with respect to the regular functionals $u$ and $v$ respectively, where $\langle u, 1 \rangle = \langle v, 1 \rangle = 1$, let $\{\beta_n\}_{n \geq 0} : \{\gamma_n\}_{n \geq 1}$ and $\{\tilde{\beta}_n\}_{n \geq 0} : \{\tilde{\gamma}_n\}_{n \geq 1}$ the corresponding sequences char-
acterizing \( \{P_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \), respectively. Suppose that these sequences are related by relation (1.1).

The initial conditions \( w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) \) and \( w_5 r_5 \neq 0 \) yield a relation between the linear functionals \( u \) and \( v \) such as

\[
\phi u = \psi v
\]

where \( \phi \) and \( \psi \) are polynomials of degree 1 and 4, respectively.

Firstly, if \( w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) \) and \( r_5 \neq 0 \), then there exists a complex number \( c \) such that

\[
\langle (x - c) u, Q_5 (x) \rangle = 0.
\]

Moreover, \( \langle (x - c) u, Q_n (x) \rangle = 0, \quad n \geq 5. \) Indeed,

\[
\begin{align*}
\langle (x - c) u, Q_0 (x) \rangle &= \beta_0 - c, \\
\langle (x - c) u, Q_1 (x) \rangle &= \gamma_1 + (s_1 - r_1) (\beta_0 - c), \\
\langle (x - c) u, Q_2 (x) \rangle &= (s_2 - r_2) \gamma_1 + (t_2 - r_2 (s_1 - r_1)) (\beta_0 - c), \\
\langle (x - c) u, Q_3 (x) \rangle &= (t_3 - r_3 (s_2 - r_2)) \gamma_1 + (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) (\beta_0 - c), \\
\langle (x - c) u, Q_4 (x) \rangle &= [v_4 - r_4 (t_3 - (s_2 - r_2))] \gamma_1 \\
&\quad + [w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) (\beta_0 - c), \\
\langle (x - c) u, Q_5 (x) \rangle &= [w_5 - r_5 (v_4 - r_4 (t_3 - r_3 (s_2 - r_2))) \gamma_1 \\
&\quad - r_5 \left[ w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) (\beta_0 - c) \right] \\
\end{align*}
\]

Then there exists \( c \) such that

\[
\langle (x - c) u, Q_5 (x) \rangle = 0.
\]

This implies

\[
(2.1) \quad c := \beta_0 - \frac{\gamma_1 w_5 - r_5 (v_4 - r_4 (t_3 - r_3 (s_2 - r_2)))}{r_5 w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1)))}.
\]

Thus,

\[
\langle (x - c) u, Q_n (x) \rangle = -r_n \langle (x - c) u, Q_{n-1} (x) \rangle, \quad n \geq 6.
\]

On the other hand [8]

\[
(x - c) u = \sum_{i=0}^{4} \frac{\langle (x - c) u, Q_i (x) \rangle}{\langle v, Q_i^2 (x) \rangle} Q_i (x) v.
\]
Therefore, if \( w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) \) and \( w_5 r_5 \neq 0 \), it can be deduced that the relation between \( u \) and \( v \) is

\[
(x - c) u = q(x) v,
\]

where \( q \) is a polynomial of exact degree 4.

**Lemma 2.1.** Let \( \{P_n\} \) and \( \{Q_n\} \) be two monic orthogonal polynomials sequences (MOPS) with respect to the regular linear functionals \( u \) and \( v \) respectively, where \( \langle u, 1 \rangle = \langle v, 1 \rangle = 1 \). Assuming that there exist sequences of complex numbers

\[
\{r_n\}, \{s_n\}, \{t_n\}, \{v_n\} \quad \text{and} \quad \{w_n\}
\]

with the initial conditions

\[
r_0 = s_0 = t_0 = t_1 = v_0 = v_1 = v_2 = w_0 = w_1 = w_2 = w_3 = 0,
\]

such that the relation (1.1) holds, for every \( n \geq 0 \), then

1. If \( w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) \) and \( r_5 = 0 \), then

\[
w_n \neq r_n (v_{n-1} - r_{n-1} (t_{n-2} - r_{n-2} (s_{n-3} - r_{n-3}))),
\]

for \( n \geq 4 \), and \( r_n = 0 \), for every \( n \geq 5 \). In this case the relation (1.1) reduces to a 1–5 type relation

\[
Q_n(x) = P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x) + c_n P_{n-3}(x) + d_n P_{n-4}(x),
\]

\( n \geq 0 \), with

\[
a_n := s_n - r_n \quad n \geq 1,
\]

\[
b_n := t_n - r_n (s_{n-1} - r_{n-1}) \quad n \geq 2,
\]

\[
c_n := v_n - r_n (t_{n-1} - r_{n-1} (s_{n-2} - r_{n-2})) \quad n \geq 3,
\]

\[
d_n := w_n - r_n (v_{n-1} - r_{n-1} (t_{n-2} - r_{n-2} (s_{n-3} - r_{n-3}))) \quad n \geq 4.
\]

2. If \( w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) \) and \( r_5 \neq 0 \) then \( r_n \neq 0 \), \( n \geq 5 \).

3. If \( w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) \) and \( w_5 r_5 \neq 0 \) then \( w_n r_n \neq 0 \), for \( n \geq 5 \).

Thus, in this case the relation (1.1) is a non-degenerate 2–5 type relation.

**Proof.** We have
\[
\left\{
\begin{array}{l}
\langle u, Q_1 (x) \rangle = s_1 - r_1 , \\
\langle u, Q_2 (x) \rangle = t_2 - r_2 (s_1 - r_1) , \\
\langle u, Q_3 (x) \rangle = v_3 - r_3 (t_2 - r_2 (s_1 - r_1)) , \\
\langle u, Q_4 (x) \rangle = w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) , \\
\langle u, Q_n (x) \rangle = -r_n \langle u, Q_{n-1} \rangle , \quad n \geq 5.
\end{array}
\right.
\tag{2.3}
\]

(1) If \( w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) \) and \( r_5 = 0 \), from \( \text{(2.3)} \), we have

\[\langle u, Q_i (x) \rangle \neq 0, \quad i = 1, 2, 3, 4\]

and

\[\langle u, Q_n (x) \rangle = 0, \quad n \geq 5.\]

So, there exists a polynomial \( q \) of degree 4 such that \( u = q(x)v \). Therefore,

\[Q_n(x) = P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x) + c_n P_{n-3}(x) + d_n P_{n-4}(x),\]

for each \( n \geq 0 \), with \( c_n \neq 0, n \geq 4 \). Again, the relation \( \text{(1.1)} \) leads to

\[s_n = a_n + r_n, \quad n \geq 1,\]

\[t_n = b_n + r_n a_{n-1}, \quad n \geq 2,\]

\[v_n = c_n + r_n b_{n-1}, \quad n \geq 3,\]

\[w_n = d_n + r_n c_{n-1}, \quad n \geq 4,\]

and \( r_n d_{n-1} = 0, n \geq 5 \). So, \( r_n = 0, n \geq 5 \), and \( w_n \neq 0, n \geq 4 \). Then, this case is the degenerate \( 1 - 5 \) type relation.

(2) If \( w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) \) and \( r_5 \neq 0 \), according to \( \text{(2.3)} \), we have

\[\langle u, Q_4 (x) \rangle \neq 0 \text{ and } \langle u, Q_5 (x) \rangle \neq 0,\]

and if \( r_n = 0 \), for each \( n \geq 6 \), we get

\[\langle u, Q_n (x) \rangle = 0, \quad n \geq 6.\]

Assuming that there exists \( n \geq 6 \) such that \( r_n = 0 \), putting \( s := \min \{ n \in \mathbb{N} / n \geq 6, r_n = 0 \} \), then

\[\langle u, Q_n (x) \rangle = 0, \quad n \geq s,\]
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and

\[ \langle u, Q_n(x) \rangle \neq 0, \quad 4 \leq n \leq s - 1. \]

So, there exists a polynomial \( q \) of degree \( s - 1 \) such that \( u = q(x)v \) [7]. Therefore,

\[ Q_n(x) = P_n(x) + \sum_{k=1}^{s-1} \alpha_n k P_{n-k}(x), \]

where \( \alpha_n, s - 1 \neq 0, n \geq s - 1 \). Taking into account (1.1), this is not possible. Thus \( r_n \neq 0, n \geq 5 \).

(3) If \( w_4 \neq r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1))) \) and \( w_5 r_5 \neq 0 \), then there exists a constant \( c \) such that

\[ (x - c) u = q(x)v, \]

where \( q \) is a polynomial of degree 4 and by (1.1) we can write

\[ w_n \langle u, P_{n-4}^2(x) \rangle = \langle (x - c) u, Q_n(x) Q_{n-5}(x) \rangle \]

\[ = \langle q(x)v, Q_n(x) + r_n Q_{n-1}(x) \rangle Q_{n-4}(x) \]

\[ = r_n \langle v, q(x)Q_{n-1}(x) Q_{n-4}(x) \rangle \]

\[ = k_1 r_n \langle v, Q_{n-1}^2(x) \rangle, \]

where \( k_1 \) is the leading coefficient of the polynomial \( q \). Now, it is enough to apply (2) to obtain \( r_n \neq 0, n \geq 4 \), and from definition 1.1, we have \( w_n r_n \neq 0, n \geq 5 \).

\[ \square \]

In the following proposition, we show that if \( w_4 \neq r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1))) \) and \( r_5 w_5 \neq 0 \) this is equivalent to assume that the functional \( (x - c) u \) is regular.
Proposition 2.1. Let \( \{P_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \) be two MOPS with respect to the regular functionals \( u \) and \( v \), respectively, with \( \langle u, 1 \rangle = \langle v, 1 \rangle = 1 \). Assume that there exist sequences of complex numbers \( \{r_n\}_{n \geq 0}, \{s_n\}_{n \geq 0}, \{t_n\}_{n \geq 0} \) and \( \{v_n\}_{n \geq 0} \) with initial conditions \( r_0 = s_0 = t_0 = t_1 = v_0 = v_1 = v_2 = 0 \), such that the relation (1.1) holds and the initial conditions \( w_n \neq r_4(v_3 - r_3(t_2 - r_2(s_1 - r_1))) \) and \( r_5w_5 \neq 0 \). Then the following statements are equivalent:

i) The functional \( (x - c)u \) is regular.

ii) \( w_n \neq r_n(v_{n-1} - r_{n-1}(t_{n-2} - r_{n-2}(s_{n-3} - r_{n-3}))) \), \( n \geq 4 \).

Proof. Multiplying the relation (1.1) by \( P_{n-1} \) and applying \( u \), the same way for \( P_{n-2}, P_{n-3} \) and \( P_{n-4} \), we get, respectively,

\[
\langle u, Q_n(x)P_{n-1}(x) \rangle = (s_n - r_n)\langle u, P^2_{n-1}(x) \rangle, \quad n \geq 1,
\]

\[
\langle u, Q_n(x)P_{n-2}(x) \rangle = (t_n - r_n(s_n - r_n))\langle u, P^2_{n-2}(x) \rangle, \quad n \geq 2,
\]

\[
\langle u, Q_n(x)P_{n-3}(x) \rangle = (v_n - r_n(t_n - r_n(s_n - r_n)))\langle u, P^2_{n-3}(x) \rangle, \quad n \geq 3,
\]

\[
\langle u, Q_n(x)P_{n-4}(x) \rangle = [w_n - r_n(v_{n-1} - r_{n-1}(t_{n-2} - r_{n-2}(s_{n-3} - r_{n-3})))\langle u, P^2_{n-4}(x) \rangle, \quad n \geq 4.
\]

Thus

\[
w_n - r_n(v_{n-1} - r_{n-1}(t_{n-2} - r_{n-2}(s_{n-3} - r_{n-3})) \neq 0 \iff \langle u, Q_n(x)P_{n-4}(x) \rangle \neq 0, \quad n \geq 4.
\]

Knowing that \( (x - c)u \) is regular if and only if \( P_n(c) \neq 0 \), for each \( n \geq 0 \).

Moreover, we need to show that \( \langle u, Q_{n+4}(x)P_n(x) \rangle \neq 0 \iff P_n(c) \neq 0 \), for each \( n \geq 0 \). Either for

\[
P_n(x) = \sum_{i=0}^{n} a_{ni}(x - c)^i, \quad n \geq 0,
\]

with \( a_{n0} = P_n(c) \) and \( a_{nn} = 1 \). From Lemma 2.1, we have

\[
(x - c)u = q(x)v.
\]

Hence

\[
\langle u, Q_{n+4}(x)P_n(x) \rangle = \langle (x - c)u, (x - c)^{n-1}Q_{n+4}(x) \rangle
\]

\[
+ \sum_{i=1}^{n-1} a_{ni} \langle (x - c)u, (x - c)^{i-1}Q_{n+4}(x) \rangle
\]

\[
+ P_n(c) \langle u, Q_{n+4}(x) \rangle
\]
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\[ = \langle v, q(x)(x-c)^{n-1}Q_{n+4}(x) \rangle + \sum_{i=1}^{n-1} a_{ni} \langle v, q(x)(x-c)^{i-1}Q_{n+4}(x) \rangle + P_n(c) \langle u, Q_{n+4}(x) \rangle, \quad n \geq 0. \]

Hence
\[ \langle u, Q_{n+4}(x) P_n(x) \rangle = P_n(c) \langle u, Q_{n+4}(x) \rangle, \quad n \geq 0, \]
from Lemma 2.1 and the relation (2.3), we get
\[ \langle u, Q_n(x) \rangle \neq 0, \quad n \geq 4. \]

\[ \square \]

3. CHARACTERIZATION OF ORTHOGONALITY

Let \( \{P_n\}_{n \geq 0} \) be a MOPS with respect to a regular functional \( u \) and let \( \{\beta_n\}_{n \geq 0} \) \( \{\gamma_n\}_{n \geq 1} \) be the corresponding sequences of recurrence coefficients, so that
\[ P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x), \quad n \geq 0, \]
with the initial conditions \( P_0(x) = 1, P_{-1}(x) = 0 \) and the condition \( \gamma_n \neq 0 \), for each \( n \geq 1 \).

In this section, we give the characterizations of the orthogonality of a sequence \( \{Q_n\}_{n \geq 0} \) of monic polynomials defined by a non-degenerate type relation (1.1). From Lemma 2.1, the conditions \( w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) \) and \( r_5 w_5 \neq 0 \) must hold, in order to have a non-degenerate 2 – 5 type relation with \( \{P_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \) MOPS and these conditions imply \( w_n r_n \neq 0 \), for each \( n \geq 5 \).

The following is the first characterization of the orthogonality of the sequence \( \{Q_n\}_{n \geq 0} \).

**Proposition 3.1.** Let \( \{P_n\}_{n \geq 0} \) be a MOPS satisfying (3.1), we define recursively a sequence \( \{Q_n\}_{n \geq 0} \) of monic polynomials by the structure relation (1.1) such that \( w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) \) and \( w_n r_n \neq 0 \) for \( n \geq 5 \). Then, \( \{Q_n\}_{n \geq 0} \) is a MOPS with recurrence coefficients \( \{\bar{\beta}_n\}_{n \geq 0} \) and \( \{\bar{\gamma}_n\}_{n \geq 1} \), where
\[ \bar{\beta}_n := \beta_n - s_{n+1} s_n + r_{n+1} r_n, \quad n \geq 0, \]
\[ \tilde{\gamma}_n := \gamma_n + t_n - t_{n+1} + s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1}) - r_n(r_{n+1} - r_n - \tilde{\beta}_n + \tilde{\beta}_{n-1}), \quad n \geq 1, \]  

if and only if \( \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \tilde{\gamma}_4 \neq 0 \) and if the following relations hold:

\begin{align*}
&b_2 - f_2 = a_2(s_1 - r_1), \\
&b_3 - f_3 = a_3(s_2 - r_2), \\
&c_3 - b_3(s_1 - r_1) = a_3(t_2 - s_2(s_1 - r_1)), \\
&b_4 - f_4 = a_4(s_3 - r_3), \\
&c_4 - b_4(s_2 - r_2) = a_4(t_3 - s_3(s_2 - r_2)), \\
&e_4 - b_4(t_2 - s_2(s_1 - r_1)) = a_4[v_3 - s_3(t_2 - s_2(s_1 - r_1)) - t_3(s_1 - r_1)] + c_4(s_1 - r_1), \\
&b_5 - f_5 = a_5(s_4 - r_4), \\
&c_5 - b_5(s_3 - r_3) = a_5(t_4 - s_4(s_3 - r_3)), \\
&e_5 - b_5(t_3 - s_3(s_2 - r_2)) = a_5[v_4 - s_4(t_3 - s_3(s_2 - r_2)) - t_4(s_2 - r_2)] + c_5(s_2 - r_2),
\end{align*}

\begin{align*}
&k_5 - b_5[v_3 - s_3(t_2 - s_2(s_1 - r_1)) - t_3(s_1 - r_1)] = a_5[w_4 - s_4(v_3 - s_3(t_2 - s_2(s_1 - r_1))) - t_3(s_1 - r_1) - t_4(t_2 - s_2(s_1 - r_1)) - v_4(s_1 - r_1)] + c_5(t_2 - s_2(s_1 - r_1)) + e_5(s_1 - r_1).
\end{align*}

As a consequence, for each \( n \geq 6 \), we have

\begin{align*}
&b_n = a_ns_{n-1}, \\
&c_n = a_nt_{n-1},
\end{align*}
(3.16) \[ e_n = a_n v_{n-1}, \]
(3.17) \[ k_n = a_n w_{n-1}, \]
(3.18) \[ f_n = a_n r_{n-1}, \]

where

(3.19) \[ a_n := \gamma_n + t_n - t_{n+1} + s_n (s_{n+1} - s_n - \beta_n + \beta_{n-1}), \quad n \geq 1, \]
(3.20) \[ b_n := s_n \gamma_{n-1} + t_n (s_{n+1} - s_n - \beta_n + \beta_{n-2}) - v_{n+1} + v_n, \quad n \geq 2, \]
(3.21) \[ c_n := t_n \gamma_{n-2} + v_n (s_{n+1} - s_n - \beta_n + \beta_{n-3}) - w_{n+1} + w_n, \quad n \geq 3, \]
(3.22) \[ e_n := v_n \gamma_{n-3} + w_n (s_{n+1} - s_n - \beta_n + \beta_{n-4}), \quad n \geq 4, \]
(3.23) \[ k_n := w_n \gamma_{n-4}, \quad n \geq 5, \]
(3.24) \[ f_n := r_n \gamma_{n-1}, \quad n \geq 2. \]

**Proof.** From the definition of \( Q_n \), we find

\[ Q_{n+1}(x) = P_{n+1}(x) + s_{n+1} P_n(x) + t_{n+1} P_{n-1}(x) + v_{n+1} P_{n-2}(x) \]
\[ + w_{n+1} P_{n-3}(x) - r_{n+1} Q_n(x), \quad n \geq 0, \]

and by replacing (3.1) in (3.23), applying (1.1) to \( x P_n(x) \), and substituting \( x P_{n-1}(x), x P_{n-2}(x), x P_{n-3}(x) \) and \( x P_{n-4}(x) \), using again (3.1), taking into account, polynomials with negative index are zero, we obtain

\[ Q_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x) + s_{n+1} P_n(x) \]
\[ + t_{n+1} P_{n-1}(x) + v_{n+1} P_{n-2}(x) + w_{n+1} P_{n-3}(x) - r_{n+1} Q_n(x) \]
\[ = xQ_n(x) - (\beta_n - s_{n+1} + s_n + r_{n+1} - r_n) Q_n(x) \]
\[ - r_n (\beta_n - s_{n+1} + s_n) Q_{n-1}(x) \]
\[ + [s_n (\beta_n - s_{n+1} + s_n - \beta_{n-1}) + t_{n+1} - t_n - \gamma_n] P_{n-1}(x) \]
\[ + [v_n (\beta_n - s_{n+1} + s_n - \beta_{n-2}) - s_n \gamma_{n-1} - v_n + v_{n+1}] P_{n-2}(x) \]
\[ + [v_n (\beta_n - s_{n+1} + s_n - \beta_{n-3}) - t_n \gamma_{n-2} - w_n + w_{n+1}] P_{n-3}(x) \]
+ [w_n (\beta_n - s_{n+1} + s_n - \beta_{n-4}) - v_n \gamma_{n-3}] P_{n-4}(x) \\
- w_n \gamma_{n-4} P_{n-5}(x) - r_n (Q_n(x) - xQ_{n-1}(x)), \quad n \geq 0.

Putting

\bar{\beta}_n := \beta_n - s_{n+1} + s_n + r_{n+1} - r_n, \quad n \geq 0,

we get, for \( n \geq 0 \),

\begin{align*}
Q_{n+1}(x) &= \left( x - \bar{\beta}_n \right) Q_n(x) + r_n \left( r_{n+1} - r_n - \bar{\beta}_n \right) Q_{n-1}(x) \\
&\quad - [\gamma_n + t_n - t_{n+1} + s_n (s_{n+1} - s_n - \beta_n + \beta_{n-1})] P_{n-1}(x) \\
&\quad - [s_n \gamma_{n-1} + t_n (s_{n+1} - s_n - \beta_n + \beta_{n-2}) + v_n - v_{n+1}] P_{n-2}(x) \\
&\quad - [t_n \gamma_{n-2} + v_n (s_{n+1} - s_n - \beta_n + \beta_{n-3}) + w_n - w_{n+1}] P_{n-3}(x) \\
&\quad - [v_n \gamma_{n-3} + w_n (s_{n+1} - s_n - \beta_n + \beta_{n-4})] P_{n-4}(x) - w_n \gamma_{n-4} P_{n-5}(x) \\
&\quad - r_n (Q_n(x) - xQ_{n-1}(x)).
\end{align*}

So, \( \{Q_n\}_{n \geq 0} \) is a MOPS if and only if \( \bar{\gamma}_n \neq 0 \), for each \( n \geq 1 \), and

\begin{align*}
&\quad r_n \left( r_{n+1} - r_n - \bar{\beta}_n \right) Q_{n-1}(x) \\
&\quad - [\gamma_n + t_n - t_{n+1} + s_n (s_{n+1} - s_n - \beta_n + \beta_{n-1})] P_{n-1}(x) \\
&\quad - [s_n \gamma_{n-1} + t_n (s_{n+1} - s_n - \beta_n + \beta_{n-2}) + v_n - v_{n+1}] P_{n-2}(x) \\
&\quad - [t_n \gamma_{n-2} + v_n (s_{n+1} - s_n - \beta_n + \beta_{n-3}) + w_n - w_{n+1}] P_{n-3}(x) \\
&\quad - [v_n \gamma_{n-3} + w_n (s_{n+1} - s_n - \beta_n + \beta_{n-4})] P_{n-4}(x) - w_n \gamma_{n-4} P_{n-5}(x) \\
&\quad - r_n (Q_n(x) - xQ_{n-1}(x)) \\
&= -\bar{\gamma}_n Q_{n-1}(x), \quad n \geq 0.
\end{align*}

(3.24)

On the other hand, \( \{Q_n\}_{n \geq 0} \) will be a MOPS if and only if \( \bar{\gamma}_n \neq 0 \), for each \( n \geq 1 \), and

\begin{align*}
Q_n(x) - xQ_{n-1}(x) &= -\bar{\beta}_{n-1} Q_{n-1}(x) - \bar{\gamma}_{n-1} Q_{n-2}(x), \quad n \geq 1.
\end{align*}

(3.25)
Replacing \(3.25\) in \(3.24\), we get the following expression

\[
(3.26) \quad [\tilde{\gamma}_n + r_n \left( r_{n+1} - r_n - \tilde{\beta}_n + \tilde{\beta}_{n-1} \right)] Q_{n-1}(x) + r_n \tilde{\gamma}_{n-1} Q_{n-2}(x) = \left[ \gamma_n + t_n - t_{n+1} + s_n (s_{n+1} - s_n - \beta_n + \beta_{n-1}) \right] P_{n-1}(x) + [s_n \gamma_{n-1} + t_n (s_{n+1} - s_n - \beta_n + \beta_{n-2}) + v_n - v_{n+1}] P_{n-2}(x) + [t_n \gamma_{n-2} + v_n (s_{n+1} - s_n + \beta_n - \beta_{n-3}) + w_n - w_{n+1}] P_{n-3}(x) + [v_n \gamma_{n-3} + w_n (s_{n+1} - s_n - \beta_n + \beta_{n-4})] P_{n-4}(x) + w_n \gamma_{n-4} P_{n-5}(x),
\]

which holds, for each \(n \geq 1\).

First, suppose that \(\{Q_n\}_{n \geq 0}\) is a MOPS with recurrence coefficients \(\{\tilde{\beta}_n\}_{n \geq 0}\) and \(\{\tilde{\gamma}_n\}_{n \geq 1}\). Then,

\[
(3.27) \quad Q_{n+1}(x) = \left( x - \tilde{\beta}_n \right) Q_n(x) - \tilde{\gamma}_n Q_{n-1}(x), \quad n \geq 0,
\]

with the initial conditions \(Q_0(x) = 1, Q_1(x) = 0\) and the condition \(\tilde{\gamma}_n \neq 0\) for every \(n \geq 1\).

To obtain \(3.26\), it is enough to replace \(Q_n(x) - x Q_{n-1}(x)\) in \(3.24\).

Conversely, if \(3.26\) is satisfied and \(\tilde{\gamma}_n \neq 0\), for every \(n \geq 1\), we show that \(\{Q_n\}_{n \geq 0}\) satisfies \(3.27\). Notice that, for every \(n \geq 1\),

\[
\begin{align*}
& r_n \left( \tilde{\beta}_n Q_{n-1}(x) + \tilde{\gamma}_n Q_{n-2}(x) \right) = -\tilde{\gamma}_n Q_{n-1}(x) \\
& + \left( r_{n+1} - r_n - \tilde{\beta}_n \right) \left( s_n P_{n-1}(x) + t_n P_{n-2}(x) + v_n P_{n-3}(x) + w_n P_{n-4}(x) - r_n Q_{n-1}(x) \right) \\
& + s_n \left( \beta_n - 1 \right) P_{n-1}(x) + \gamma_n - \gamma_{n-1} P_{n-2}(x) + t_n \left( \beta_n - 2 \right) P_{n-2}(x) + \gamma_n - 2 P_{n-3}(x) \\
& + v_n \left( \beta_n - 3 P_{n-3}(x) + \gamma_n - 3 P_{n-4}(x) + w_n \left( \beta_n - 4 \right) P_{n-4}(x) + \gamma_n - 4 P_{n-5}(x) \right) \\
& + \left( \gamma_n + t_n - t_{n+1} \right) P_{n-1}(x) + \left( v_n - v_{n+1} \right) P_{n-2}(x) \\
& + \left( w_n - w_{n+1} \right) P_{n-3}(x) - v_{n+1} P_{n-4}(x). \\
\end{align*}
\]

Applying \(1.1\) in \(s_n P_{n-1}(x) + t_n P_{n-2}(x) + v_n P_{n-3}(x) + w_n P_{n-4}(x)\) and using \(3.1\), we get

\[
\begin{align*}
& r_n \left( \tilde{\beta}_n Q_{n-1}(x) + \tilde{\gamma}_n Q_{n-2}(x) \right) = r_n \left( x Q_{n-1}(x) - Q_n(x) \right) + \left( x - \tilde{\beta}_n \right) Q_n(x) \\
& - Q_{n+1}(x) - \tilde{\gamma}_n Q_{n-1}(x), \quad n \geq 1.
\end{align*}
\]
Consequently,
\[ Q_{n+1}(x) - \left( x - \tilde{\beta}_n \right) Q_n(x) + \tilde{\gamma}_n Q_{n-1}(x) \]
\[ = -r_n \left[ Q_n(x) - \left( x - \tilde{\beta}_{n-1} \right) Q_{n-1}(x) + \tilde{\gamma}_{n-1} Q_{n-2}(x) \right], \quad n \geq 1. \]

Moreover, from (1.1)
\[ Q_1(x) = P_1(x) + s_1 - r_1 = x - \beta_0 + s_1 - r_1 = x - \tilde{\beta}_0, \]
we deduce recursively
\[ Q_{n+1}(x) = \left( x - \tilde{\beta}_n \right) Q_n(x) - \tilde{\gamma}_n Q_{n-1}(x), \quad n \geq 1. \]

Thus, \( \{ Q_n \}_{n \geq 0} \) is a MOPS with recurrence coefficients \( \{ \tilde{\beta}_n \}_{n \geq 0} \) and \( \{ \tilde{\gamma}_n \}_{n \geq 1} \).

From (1.1) and (3.3) we have (3.26), that is equivalent to
\[ (f_n - r_{n-1} a_n) Q_{n-2}(x) = (b_n - s_{n-1} a_n) P_{n-2}(x) + (e_n - r_{n-1} a_n) P_{n-3}(x) \]
\[ + (e_n - v_{n-1} a_n) P_{n-4}(x) + (k_n - w_{n-1} a_n) P_{n-5}(x), \quad n \geq 2, \]
where \( a_n, b_n, c_n, f_n, e_n \) and \( k_n \) are defined by (3.18) - (3.22).

Now, we show that (3.28) holds with \( \tilde{\gamma}_n \neq 0, n \geq 1 \) if and only if \( \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \tilde{\gamma}_4 \neq 0 \) and the relations (3.4) - (3.17) hold. Replacing in (3.28) by \( n = 2, n = 3, n = 4 \) and \( n = 5 \), we get the relations (3.4) - (3.13). Therefore it is easy to check that (3.14) - (3.17) hold for \( n \geq 6 \). Indeed, since \( w_4 \neq r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) \) and \( r_n \neq 0, \) for \( n \geq 5 \). Then by (2.3), we deduce
\[ \langle u, Q_n \rangle \neq 0, \quad n \geq 4. \]

Applying \( u \) to both sides of (3.28), we get
\[ (f_n - r_{n-1} a_n) \langle u, Q_{n-2} \rangle = 0, \quad n \geq 6. \]

This leads to
\[ f_n = r_{n-1} a_n, \quad n \geq 6, \]
and this proves (3.17). Multiplying (3.28) by \( P_{n-2}, P_{n-3}, P_{n-4} \) and \( P_{n-5} \) and applying \( u \), we obtain, respectively (3.14) - (3.16).

Conversely, from (3.16) and (3.17) we have
\[ \tilde{\gamma}_{n-1} = \frac{r_{n-1}}{r_n} \frac{w_n}{w_{n-1}} \tilde{\gamma}_{n-4}, \quad n \geq 6, \]
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this implies

\[ \tilde{\gamma}_n = \frac{r_n}{r_{n+1}} \frac{w_{n+1}}{w_n} \gamma_{n-3}, \quad n \geq 5. \]

Thus

\[ \tilde{\gamma}_n \neq 0, \quad n \geq 5. \]

Hence, the conditions \((3.4)-(3.17)\) with \(\tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \tilde{\gamma}_4 \neq 0\) imply that \((3.28)\) holds and \(\tilde{\gamma}_n \neq 0, \) for every \(n \geq 1.\) \(\square\)

Now, we will prove that the orthogonality of the sequence \(\{Q_n\}_{n \geq 0}\) can be also characterized by the fact that there are five sequences depending on the parameters \(r_n, s_n, t_n, v_n, w_n\) and the recurrence coefficients which remain constants.

**Theorem 3.1.** Let \(\{P_n\}_{n \geq 0}\) be a MOPS with respect to a regular linear functional \(u\) and the sequence of monic polynomials \(\{Q_n\}_{n \geq 0}\) is given by the relation \((1.1)\). If \(\{Q_n\}_{n \geq 0}\) is a MOPS with respect to a regular linear functional \(v,\) then

\[ (3.28) \quad k(x - c)u = (x^4 + ax^3 + bx^2 + dx + e)\ v \]

with \(c, a, b, d, e \in \mathbb{C}\) and \(k \in \mathbb{C}\setminus\{0\}\) and the normalizations for these linear functionals \(\langle u, 1 \rangle = \langle v, 1 \rangle = 1.\)

**Proof.** Applying the regular linear functional \(u\) corresponding to the MOPS \(\{P_n\}_{n \geq 0}\) in \((1.1)\), we obtain, for each \(n \geq 4\)

\[ \langle (x - c)u, Q_n(x) \rangle = 0. \]

Then, according to \([8]\) taking into account \((1.1)\), we expand the linear functional \(u\) in terms of the dual basis \(\{\frac{Q_i u}{\langle v, Q_i^2 \rangle}\}_{i \geq 0}\) of the MOPS \(\{Q_n\}_{n \geq 0}\) as follows:

\[ (x - c)u = \sum_{i=0}^{4} \frac{\langle (x - c)u, Q_i \rangle}{\langle v, Q_i^2 \rangle} Q_i v. \]

Since \(\{Q_n\}_{n \geq 0}\) is a MOPS with respect to \(v,\) the recurrence coefficients \(\{\tilde{\beta}_n\}_{n \geq 0}\) and \(\{\tilde{\gamma}_n\}_{n \geq 1}\) are given by \((3.2)\) and \((3.3)\), furthermore

\[ (3.29) \quad \tilde{\gamma}_n = \frac{\langle v, Q_n^2 \rangle}{\langle v, Q_{n-1}^2 \rangle} \neq 0, \quad n \geq 1. \]
Indeed, making both sides of (3.28) acting on the polynomials $Q_0$, $Q_1$, $Q_2$, $Q_3$ and $Q_4$, and taking into account (2.1), we get

(C.30) \[ k \left[ (t_3 - r_3 (s_2 - r_2)) \gamma_1 + (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) (\beta_0 - c) \right] = \left( \beta_3 + \beta_2 + \beta_1 + \beta_0 \right) \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 + a \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \]

(C.31) \[ k \left[ (s_2 - r_2) \gamma_1 + (t_2 - r_2 (s_1 - r_1)) (\beta_0 - c) \right] = \left( 1 \tilde{\gamma}_2 \tilde{\gamma}_3 + \tilde{\gamma}_1 + \tilde{\gamma}_2 + \tilde{\beta}_0^2 + \tilde{\beta}_1^2 + \tilde{\beta}_2 + \tilde{\beta}_1 \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{\beta}_2 + \tilde{\beta}_2 \tilde{\beta}_0 \right) \tilde{\gamma}_1 \tilde{\gamma}_2 \]

\[ + a \left( \tilde{\beta}_2 + \tilde{\beta}_1 + \tilde{\beta}_0 \right) \tilde{\gamma}_1 \tilde{\gamma}_2 + b \tilde{\gamma}_1 \tilde{\gamma}_2 \]

(C.32) \[ k \left[ \gamma_1 + (s_1 - r_1) (\beta_0 - c) \right] = 2 \left( \beta_1 + \tilde{\beta}_0 \right) \tilde{\gamma}_1^2 + \left( \tilde{\beta}_2 + 2 \tilde{\beta}_1 + \tilde{\beta}_0 \right) \tilde{\gamma}_1 \tilde{\gamma}_2 + \tilde{\beta}_3 \tilde{\beta}_0^2 + \tilde{\beta}_1 \tilde{\beta}_2 + \tilde{\beta}_2 \tilde{\beta}_0 \right) \tilde{\gamma}_1 \]

\[ + a \left( \tilde{\gamma}_1 + \tilde{\gamma}_2 + \tilde{\beta}_2 + \tilde{\beta}_0 \right) \tilde{\gamma}_1 + b \left( \tilde{\beta}_1 + \tilde{\beta}_0 \right) \tilde{\gamma}_1 + d \tilde{\gamma}_1 \]

(C.33) \[ k (\beta_0 - c) = \tilde{\beta}_0^2 + \left( \beta_1 + 3 \beta_0 \right) \tilde{\gamma}_1^3 + a \left( \beta_1 + 2 \beta_0 \right) \tilde{\gamma}_1 + b \beta_0^2 + d \tilde{\beta}_0 + e \]

(C.34) \[ k \left[ [v_4 - r_4 (t_3 - (s_2 - r_2))] \gamma_1 + [w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1)))) (\beta_0 - c) \right] = \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \tilde{\gamma}_4 \]

where, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, $\tilde{\gamma}_3$ and $\tilde{\gamma}_4$ are given by $(3.2)$ and $(3.3)$.

Using the relations (C.30) – (C.34) and taking into account (2.2), thus, the values of $c$, $a$, $b$, $d$, $e$ and $k$ are given as follows

\[ k = \frac{r_5 \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \tilde{\gamma}_4}{w_5} \tilde{\gamma}_1, \]

\[ c = \beta_0 - \gamma_4 \frac{w_5 - r_5 (v_4 - r_4 (t_3 - r_3 (s_2 - r_2)))}{r_5 w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1)))} \]
\[ a = -\tilde{\beta}_3 - \tilde{\beta}_2 - \tilde{\beta}_1 - \tilde{\beta}_0 + \frac{v_3 - r_3 (t_2 - r_2 (s_1 - r_1))}{w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1)))} + \frac{r_5 (t_3 - r_3 (s_2 - r_2)) w_4 - (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) v_4}{w_5 (w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))))} \]

\[ b = \tilde{\beta}_3 \tilde{\beta}_0 - \tilde{\beta}_1 \tilde{\beta}_2 + \tilde{\beta}_2^3 \tilde{\beta}_1 + \tilde{\beta}_1^2 \tilde{\beta}_2 + \tilde{\beta}_0 \tilde{\beta}_2 \tilde{\beta}_1 + \tilde{\beta}_3 \tilde{\beta}_2 \tilde{\beta}_1 - \tilde{\beta}_1^3 - \tilde{\beta}_2^2 - \tilde{\gamma}_1 - \tilde{\gamma}_2 - \tilde{\gamma}_3 \]

\[ + \frac{(v_3 - r_3 (t_2 - r_2 (s_1 - r_1)))}{w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))))} \left( \frac{r_5 (t_3 - r_3 (s_2 - r_2)) w_4 - (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) v_4}{w_5 (w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))))} \right) \left( \tilde{\beta}_2 \tilde{\beta}_1 + \tilde{\beta}_0 \right) \]

\[ d = - \left( \tilde{\beta}_0^3 + \tilde{\beta}_1^3 + \tilde{\beta}_1^2 \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{\beta}_0^2 \right) - \left( 2\tilde{\beta}_1 + \tilde{\beta}_0 \right) \tilde{\gamma}_1 - \left( \tilde{\beta}_2 + 2\tilde{\beta}_1 + \tilde{\beta}_0 \right) \tilde{\gamma}_2 \]

\[ + \frac{(v_3 - r_3 (t_2 - r_2 (s_1 - r_1)))}{w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))))} \left( \frac{r_5 (t_3 - r_3 (s_2 - r_2)) w_4 - (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) v_4}{w_5 (w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))))} \right) \left( \tilde{\beta}_2 \tilde{\beta}_1 + \tilde{\beta}_0 \right) \]

\[ + \frac{(v_3 - r_3 (t_2 - r_2 (s_1 - r_1)))}{w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))))} \left( \frac{r_5 (t_3 - r_3 (s_2 - r_2)) w_4 - (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))) v_4}{w_5 (w_4 - r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1))))} \right) \left( \tilde{\beta}_2 \tilde{\beta}_1 + \tilde{\beta}_0 \right) \]

\[ e = \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \frac{w_5 - r_5 (v_4 - r_4 (t_3 - (s_2 - r_2)))}{w_4 r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1)))} \left( \tilde{\beta}_1 + \tilde{\beta}_0 \right) - \frac{w_5 - r_5 (v_4 - r_4 (t_3 - (s_2 - r_2)))}{w_4 r_4 (v_3 - r_3 (t_2 - r_2 (s_1 - r_1)))} \left( \tilde{\beta}_1 + \tilde{\beta}_0 \right) \left( \tilde{\beta}_1 + \tilde{\beta}_0 \right) - d \tilde{\beta}_0. \]
In this section, a special case of relation (1.1) is obtained. Let us consider the symmetric MOPS \( \{P_n\}_{n \geq 1} \), this means that \( \beta_n = 0 \), for each \( n \geq 0 \). From Proposition 3.1, the equations (3.14), (3.15), (3.16) and (3.17) become, for each \( n \geq 6 \),

\[
\begin{align*}
(4.1) \quad & s_{n+1} = s_n + \frac{v_{n-1}}{w_{n-1}} \gamma_{n-4} - \frac{v_n}{w_n} \gamma_{n-3}, \\
(4.2) \quad & t_{n+1} = t_n + \gamma_n - \frac{w_n}{w_{n-1}} \gamma_{n-4} + s_n (s_{n+1} - s_n), \\
(4.3) \quad & v_{n+1} = v_n + s_n \gamma_{n-1} + t_n (s_{n+1} - s_n) \\
& \quad - s_{n-1} [\gamma_n + t_n - t_{n+1} + s_n (s_{n+1} - s_n)], \\
(4.4) \quad & w_{n+1} = w_n + t_n \gamma_{n-2} + v_n (s_{n+1} - s_n) \\
& \quad - t_{n-1} [\gamma_n + t_n - t_{n+1} + s_n (s_{n+1} - s_n)], \\
(4.5) \quad & r_{n+1} = r_n + \frac{w_n \gamma_{n-4}}{w_n r_n} - \frac{w_{n+1} \gamma_{n-3}}{w_{n+1} r_{n+1}}, \quad n \geq 5.
\end{align*}
\]

The equations (3.2) and (3.3) become

\[
\begin{align*}
\tilde{\beta}_n &= s_n - s_{n+1} + t_{n+1} - r_n, \quad n \geq 0, \\
\tilde{\gamma}_n &= \gamma_n + t_n - t_{n+1} + s_n (s_{n+1} - s_n) - r_n (s_{n+1} - s_n + \tilde{\beta}_{n-1}), \quad n \geq 1,
\end{align*}
\]

for each \( n \geq 6 \), we have

\[
\begin{align*}
(4.6) \quad & \tilde{\beta}_n = \frac{v_n}{w_n} \gamma_{n-3} - \frac{v_{n-1}}{w_{n-1}} \gamma_{n-4} + \frac{w_n}{w_{n-1}} \gamma_{n-4} - \frac{w_{n+1}}{w_{n+1}} \gamma_{n-3} \\
& \quad \quad = \frac{\gamma_{n-3}}{w_n} \left( v_n - \frac{w_{n+1}}{r_{n+1}} \right) - \frac{\gamma_{n-4}}{w_{n-1}} \left( v_{n-1} - \frac{w_n}{r_n} \right), \\
(4.7) \quad & \tilde{\gamma}_n = \frac{w_n}{w_{n-1}} \gamma_{n-4} - r_n \left( \frac{v_{n-1}}{w_{n-1}} \gamma_{n-4} - \frac{v_n}{w_n} \gamma_{n-3} \right) - r_n \tilde{\beta}_{n-1}.
\end{align*}
\]

In this case, we treat the following three subcases.

i) If \( s_n = s_1 \) and \( r_{n-1} = r_1 \), for each \( n \geq 6 \), from (4.1) and (4.5), we obtain

\[
\begin{align*}
\frac{v_n}{w_n} \gamma_{n-3} &= \frac{v_{n-1}}{w_{n-1}} \gamma_{n-4} = \ldots = \frac{v_1}{w_1} \gamma_1, \\
\frac{w_{n+1}}{w_n} \gamma_{n-3} &= \frac{w_n}{w_{n-1}} \gamma_{n-4} = \ldots = \frac{w_0}{w_0} \gamma_1,
\end{align*}
\]

the relation (4.7) yields
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(4.8) \[ \tilde{\gamma}_n = \frac{w_5}{w_4} \gamma_1, \quad n \geq 6. \]

We conclude that \( \tilde{\beta}_n = 0 \) and \( \tilde{\gamma}_n \) are constants, for each \( n \geq 6 \). From (4.2), we have

(4.9) \[ t_{n+1} = t_n + \gamma_n - \frac{w_5}{w_4} \gamma_1. \]

ii) If \( s_n = s_1, r_{n-1} = r_1 \) and \( t_n = t_2 \), for each \( n \geq 6 \), from (4.9) and (4.8), we get

\[ \tilde{\gamma}_n = \gamma_n, \quad n \geq 6. \]

The coefficients \( \gamma_n \) are constants, for each \( n \geq 6 \), then \( \{ P_n \}_{n \geq 0} \) is the sequence of anti-associated polynomials of order 6 for the Chebyshev polynomials of the second kind \[10\].

iii) If \( r_{n-1} = r_1, s_n = s_1, t_n = t_2 \) and \( v_n = v_3 \), for each \( n \geq 6 \), from (4.3), we have

\[ v_{n+1} = v_n + s_n \gamma_{n-1} - s_{n-1} \gamma_n, \quad n \geq 6, \]

hence

\[ v_{n+1} = v_n + s_1 \left( \gamma_{n-1} - \gamma_n \right), \quad n \geq 7, \]

it is clear that \( s_1 (\gamma_{n-1} - \gamma_n) = 0 \), for all \( n \geq 7 \).

iv) If \( r_{n-1} = r_1, s_n = s_1, t_n = t_2 \) and \( w_n = w_4 \), for each \( n \geq 6 \), from (4.3), we have

\[ w_{n+1} = w_n + t_n \gamma_{n-2} - t_{n-1} \gamma_n, \quad n \geq 6, \]

hence

\[ w_{n+1} = w_n + t_2 \left( \gamma_{n-2} - \gamma_n \right), \quad n \geq 8, \]

it is clear that \( t_2 (\gamma_{n-2} - \gamma_n) = 0 \), for all \( n \geq 8 \).

Remark 4.1. If \( r_{n-1} = r_1, s_n = s_1 \) and \( t_n = t_2 \) or \( r_{n-1} = r_1, s_n = s_1, t_n = t_2 \) and \( v_n = v_3 \) or \( r_{n-1} = r_1, s_n = s_1, t_n = t_2, v_n = v_3 \) and \( w_n = w_4 \) for each \( n \geq 6 \), then

\[ \tilde{\beta}_n = 0, \quad n \geq 6, \]

\[ \tilde{\gamma}_n = \gamma_n = \gamma_6, \quad n \geq 6. \]
Example 1. Let \( \{P_n\}_{n \geq 0} \) be the sequence of monic Chebyshev polynomials of the second kind orthogonal with respect to the weight function \( W(x) = (1 - x^2)^{\frac{1}{2}} \) on \((-1, 1)\). Then \( \beta_n = 0, \gamma_n = \frac{1}{4}, n \geq 1, \) and the relations \( (4.1), (4.2), (4.3) \) and \( (4.5) \), for each \( n \geq 6 \), become

\[
\begin{align*}
    s_{n+1} &= s_n + \frac{1}{4} \left( \frac{v_{n-1}}{w_{n-1}} - \frac{v_n}{w_n} \right), \\
    t_{n+1} &= t_n + \frac{1}{4} \left( 1 - \frac{w_n}{w_{n-1}} \right) + s_n (s_{n+1} - s_n), \\
    v_{n+1} &= v_n + \frac{1}{4} s_n + t_n (s_{n+1} - s_n) - s_{n-1} \left[ \frac{1}{4} + t_n - t_{n+1} + s_n (s_{n+1} - s_n) \right], \\
    w_{n+1} &= w_n + \frac{1}{4} t_n + v_n (s_{n+1} - s_n) - t_{n-1} \left[ \frac{1}{4} + t_n - t_{n+1} + s_n (s_{n+1} - s_n) \right], \\
    r_n &= r_{n-1} + \frac{1}{4} \left( \frac{w_{n-1}}{r_{n-1} w_{n-2}} - \frac{w_n}{r_n w_{n-1}} \right),
\end{align*}
\]

Assume that \( r_{n-1} = r_1, s_n = s_1 \) and \( t_n = t_2, \) for each \( n \geq 6, \) we obtain

\[
w_{n+1} = w_n + \frac{1}{4} (t_n - t_{n-1}), \quad n \geq 6,
\]

in particular,

\[
w_{n+1} = w_n, \quad n \geq 7.
\]

In this situation, we deduce constant connection coefficients, for \( n \geq 7. \)

References


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