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PROPERTIES OF THE FORMAL CONTEXT OF ORTHOMODULAR LATTICES

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ABSTRACT. Let L be a finite Orthomodular Lattice and T be the Formal Context of L. Then, considering T as a binary symmetric matrix, we find the determinant of the formal context of the atomic amalgam $B_n + B_m$ of two Boolean algebras $\mathbf{B_n}$ and $\mathbf{B_m}$ consisting of n and m atoms, respectively using the Schur complement formula [8]. We present the proofs of some preliminary results on the determinant of the context table of the Boolean algebra B_n and the characteristic polynomial of B_n. These preliminary results are used in many applications in graph theory.

1. Introduction and Preliminaries

Formal Concept Analysis (FCA) is an analysis technique of connecting object data with attribute data and this knowledge representation has many applications, since its introduction [2]. In the recent years, FCA is broadly utilized in various fields such as syntax, software engineering [10] and information reclamation [3]. If we consider a finite lattice L, there corresponds a formal context. Considering the formal context as a binary matrix, we can study the properties of such matrices. We have used formal context of the corresponding lattice to generate the

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Orthomodular Lattices(OML) [7], [5]. It has been observed that the similar binary matrices arise as adjacency matrices of some graphs.

In graph theory, every graph can be associated with a corresponding adjacency matrix. This adjacency matrix for an undirected simple graph is always a symmetric binary matrix having the principal diagonal entries $'0'$. Many properties of a graph can be revealed by looking at the determinant, eigenvalues and eigenvectors of the adjacency matrix. In fact, the determinant and eigenvalues of adjacency matrices in terms of structural properties of a graph have been studied extensively by Harary [4], [6]. A formula for the coefficients of the characteristic polynomial is derived by A.Mowshowitz [1]. Many determinental formulas involving adjacency matrices of graphs have been given by R.B Bapat and Souvik Roy [8], [9].

Under the Main results section, we find the determinant and the Characteristic polynomial of the Boolean matrix B_n. Also, we find the determinant of the formal context of amalgam of two Boolean algebras using the Schur Complement formula [8], which is an important tool in finding the determinant of certain particular types of matrices. Under the Special cases section, we find the determinant of the adjacency matrix of a complete graph with a pendent vertex, using the formula for the determinant of the Boolean matrix and the determinant of the adjacency matrix of a subgraph induced by the removal of two edges from a vertex of a complete graph K_n with n vertices.

A *formal context* is a triple $K := (G, M, I)$ which consists of the connection I between two sets G and M, where the constituents of G are identified as objects and those of M as attributes. gIm denotes that the object g has an attribute m. This relation is depicted as 1 or $\times$ in the row g and the column m in the matrix form [2].

For a subset $X \subseteq G$ and $Y \subseteq M$, $X' = \{m \in M | xIm for all x \in X\}$ and $Y' = \{g \in G | gIy for all y \in Y\}$. A formal concept of the formal context $K := (G, M, I)$ is an ordered pair (X, Y) such that $X' = Y$ and $Y' = X$. The set of all concepts of the formal context K is denoted by $K(G, M, I)$. The characteristics of a Complete Lattice are satisfied, when all the concepts are ordered by set-inclusion and this Lattice is called the Concept Lattice [2].

Let us denote the Boolean algebra of 'n' atoms by $\mathbf{B_n}$. Let B_n be the formal context associated with Boolean algebra $\mathbf{B_n}$, then $B_n = (b_{ij})$ where

$$b_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}.$$

The Orthomodular lattices (OML) obtained by the atomic amalgam of two Boolean algebras $\mathbf{B_n}$ and $\mathbf{B_m}$ as defined by L.Beran [7] is given below:

Let 0 and 1 be two fixed elements of a set B. For any n of a finite index set $I \neq \phi$, let $\mathbf{B_n}$ denote Boolean algebra or lattice having 0 and 1 as its minimum and the maximum respectively. Suppose

a. Any $\mathbf{B_n}$ has atleast 8 components.
b. If $n \neq m$ in I, $\mathbf{B_n} \cap \mathbf{B_m}$ is either { 0,1 } or { 0, 1, x, x' } for some atom x of $\mathbf{B_n}$ and $\mathbf{B_m}$, where x' is the orthocomplement of x.

   If $B_n \cap B_m$ = { 0, 1, x, x' }, then it is supposed that the complement of x in $\mathbf{B_n}$ say $x^{'(n)}$, is equal to the complement of x in $\mathbf{B_m}$ say $x^{'(m)}$, and it is denoted by $x^{'}$, that is, $x^{'} = x^{'(n)} = x^{'(m)}$ holds good.

Then $\mathbf{B} = \bigcup(\mathbf{B}_n; n \in I)$ is defined as an atomic amalgam of the Boolean algebras $\mathbf{B_n} = (\mathbf{B_n}, \wedge, \vee, 0, 1)$. $(\mathbf{B_n}; n \in I)$ are called the initial blocks of the atomic amalgam.

If we consider the formal context of an amalgam of Boolean algebras as a matrix, then that matrix turns out to be a symmetric binary matrix of n^{th} order having principal diagonal entries $'0'$. The Lattice $\mathbf{B_3}$ and the corresponding formal context B_3 are shown in figure 1 and figure 2. In this paper, we call a matrix associated with Boolean algebra of n atoms as Boolean matrix, denoted by B_n. Also, the formal context of the atomic amalgam of Boolean algebras $\mathbf{B_n}$ and $\mathbf{B_m}$ is denoted by $B_n + B_m$. The amalgam of Boolean algebra $\mathbf{B_4}$ and $\mathbf{B_5}$ and its formal context is given in the figure 3 and 4.

2. DETERMINANT AND EIGENVALUES OF B_n

The following theorem gives the formula to calculate the determinant of formal context of the Boolean algebras:

	a_1'	a_2'	a_3'
a_1	0	1	1
a_2	1	0	1
a_3	1	1	0

FIGURE 1. formal formalB_3

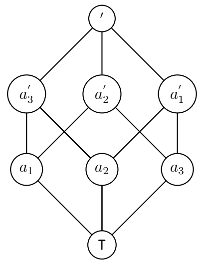

FIGURE 2. Lattice $\mathbf{B_3}$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

FIGURE 3. formal formal $B_4 + B_5$

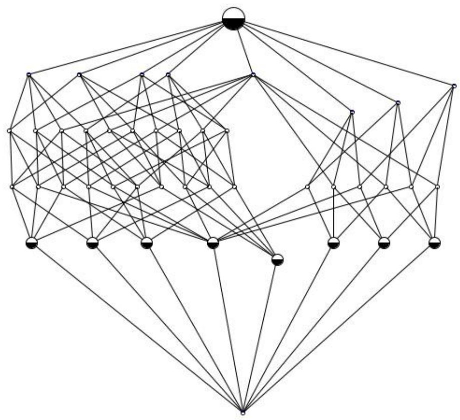

FIGURE 4. $B_4 + B_5$

Theorem 2.1. *Let B_n be the formal context of the Boolean algebra with n atoms, $\mathbf{B_n}$. Then* $\det(B_n) = (-1)^{n-1}(n-1)$.

Proof. The theorem is proved by mathematical induction on n. While expanding the matrix B_n across the first row to find the determinant, we come across a matrix A_n shown below:

$$A_n = (a_{ij}), \quad \text{where} \quad a_{ij} = \begin{cases} 0 & \text{if } (i = j), 2 \leq (i, j) \leq n, \\ 1 & \text{otherwise.} \end{cases} \tag{2.1}$$

We simultaneously prove that determinant of A_n is 1, when n is odd and (-1), when n is even, by mathematical induction on n.

We observe that the result is true for $n = 2$ $\det(B_2) = 0 - 1 = -1$ and $\det(A_2) = 0 - 1 = -1$. The result is true for $n = 2$.

The result is assumed to be true for $n = k - 1$.

When $n = k$:

Case 1: When k is even; $(k-1)$ is odd. Expanding the determinant across the first row we get:

$$\begin{aligned}\det(B_k) &= [0 \times \det(B_{k-1})] + \det(A_{k-1}) \sum_{2}^{k} \left[(-1)^{j-1}(-1)^{j-2}\right] \\ &= \det(A_{k-1}) \left[(-1)^{k-1}(k-1)\right].\end{aligned}$$

Noting that $k-1$ is odd , $\det(A_{k-1}) = 1$. Therefore $\det(B_k) = (-1)^{k-1}(k-1)$.

Case 2: When k is odd: Note that $(k-1)$ is even. Expanding the determinant across the first row we get:

$$\det(B_k) = \det(A_{k-1}) \left[(-1)^{k-2}(k-1)\right].$$

Noting that $k-1$ is even , $\det(A_{k-1}) = -1$, we get $\det(B_k) = (-1)^{k-1}(k-1)$.

Therefore by mathematical induction, the theorem is true for all n.

To prove that $\det(A_k) = \begin{cases} 1 & \text{if k is odd} \\ -1 & \text{if k is even} \end{cases}$.

For $n = k$.

Case 1: When k is even. Expanding the determinant across the first row:

$$\begin{aligned}\det(A_k) &= [1 \times \det(B_{k-1})] + \det(A_{k-1}) \sum_{2}^{k} \left[(-1)^{j-1}(-1)^{j-2}\right] \\ &= \det(B_{k-1}) - \det(A_{k-1})[(k-1)].\end{aligned}$$

By induction assumption, substituting for $\det(B_{k-1})$ and $\det(A_{k-1})$ we get:

$$\begin{aligned}\det(A_k) &= (-1)^{k-2}(k-2) - (k-1) \times 1 \\ &= k - 2 - k + 1 = -1.\end{aligned}$$

So, $\det(A_k) = -1$ when k is even.

Case 2: When k is odd; $(k-2)$ is odd. Note that $\det(A_k) = 1$ when k is odd. With a similar discussion as above, we have $det(A_k) = 1$. Therefore, the result is true for all n. □

Remark 2.2. *The matrix A_n defined in 2.1 is used in the following theorems. The notation A_n is preserved for future use. Its determinant is 1 when n is odd and (-1) when n is even. Also, $B_n = (b_{ij})$ is preserved for the formal context of Boolean algebras.*

The following theorem gives the characteristic equation of the formal context of Boolean algebras.

Theorem 2.3. *Let $\mathbf{B_n}$ be the Boolean algebra with n atoms and B_n be the matrix of its formal context, then the Characteristic Equation of B_n is $(-1)^n(\lambda+1)^{n-1}(\lambda-n+1)$.*

Proof. The theorem is proved by mathematical induction on n.

For $n = 2$, the above equation holds.

The result is assumed to be true for $n = k$.

Now proving for $n = k + 1$. On expanding the determinant $\det(B_{k+1} - \lambda I)$ across the first row, we come across a minor whose entries are same as the matrix denoted by C_k defined as follows: $C_k = (c_{ij})$,

$$c_{ij} = \begin{cases} -\lambda & \text{if } i = j;\ 2 \le i \le k;\ 2 \le j \le k \\ 1 & \text{otherwise} \end{cases}$$

and

$$\det(B_{k+1} - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 & 1 & . & . & . & 1 \\ 1 & -\lambda & 1 & 1 & . & . & . & 1 \\ 1 & 1 & -\lambda & 1 & . & . & . & \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & . & . & . & -\lambda \end{vmatrix} \tag{2.2}$$

$$= -\lambda \times \det(B_k - \lambda I) + \det(C_k) \sum_{2}^{k} \left[(-1)^{1+j}(-1)^{j-2}\right]$$

$$= -\lambda \times \det(B_k - \lambda I) - \det(C_k) \times k.$$

We prove that $\det(C_n) = (-1)^{n-1}(\lambda + 1)^{n-1}$.

The result is true for $n = 2$, i.e, $\det(C_2) = (-1)(\lambda + 1)$. The result is assumed to be true for $n = k$. For $n = k + 1$:

$$\begin{aligned}\det(C_{k+1}) &= [1 \times \det(B_k - \lambda I)] + \det(C_k) \sum_{2}^{k+1} \left[(-1)^{1+j}(-1)^{j-2}\right] \\ &= \det(B_k - \lambda I) - \det(C_k) \times k.\end{aligned}$$

By induction assumption for B_k and C_k, we have $\det(C_{k+1}) = (-1)^k(\lambda + 1)^k$.

Hence, by mathematical induction, we have proved that the result holds good for all n.

Using this result in 2.2,

$$\begin{aligned}\det(B_{k+1} - \lambda I) &= -\lambda \times \det(B_k - \lambda I) - k \times \det(C_k) \\ &= -\lambda(-1)^k(\lambda + 1)^{k-1}(\lambda - k + 1) - k(-1)^{k-1}(\lambda + 1)^{k-1} \\ &= (-1)^{k+1}(\lambda + 1)^k(\lambda - k).\end{aligned}$$

Hence, by mathematical induction, we have proved that the result holds good for all n. □

3. Determinant of the Formal Context of the Amalgam of two Boolean Algebras

In this section, we present the main result of this paper. Let **L** be the amalgam of $\mathbf{B_n}$ and $\mathbf{B_m}$. In the following theorem, we find the determinant of formal context $B_n + B_m$ of **L**.

Theorem 3.1. $\det(B_n + B_m) = (-1)^{m+n-3}[(n-1)(m-2) + (n-2)(m-1)]$

Proof. Let $A = B_n + B_m$. Then A can be partitioned as $A = \begin{bmatrix} B_n & C \\ C' & B_m \end{bmatrix}$. Using Schur Complement formula ([8]), we have, $A = B_{n-1} \times [B_m - C'B_{n-1}^{-1}C]$. We note that the matrix $C'B_{n-1}^{-1}C$ is a matrix of order m with the first entry given by $\dfrac{(-1)^n(n-1)}{detB_{n-1}}$ which is $\dfrac{n-1}{n-2}$ and rest of the entries 0. Expanding this $det[B_m - C'B_{n-1}^{-1}C]$ across the first row, we observe two cases.

Case 1: When m is odd: (m-1) is even. We know that the $det(A_{m-1}) = -1$.

$$\begin{aligned} det(A) &= det(B_{n-1})\left[-\frac{n-1}{n-2}\det(B_{m-1}) - \det(A_{m-1})\sum_{2}^{m}[(-1)^{1+j}(-1)^{j-2}\right] \\ &= (-1)^{n-2}(n-2)\left[(-1)^{m-1}\frac{n-1}{n-2}(m-2) + (-1)^{m-1}(m-1)\right] \\ &= (-1)^{m+n-3}\left[(m-2)(n-1) + (m-1)(n-2)\right] \end{aligned}$$

Case 2: When m is even, (m-1) is odd. We know that $\det(A_{m-1}) = 1$:

$$\begin{aligned} detA &= det(B_{n-1})\left[-\frac{n-1}{n-2}\det(B_{m-1}) + \det(A_{m-1})\sum_{2}^{m}[(-1)^{1+j}(-1)^{j-2}\right] \\ &= (-1)^{n-2}(n-2)\left[(-1)^{m-1}\frac{n-1}{n-2}(m-2) + (-1)^{m-1}(m-1)\right] \\ &= (-1)^{m+n-3}\left[(n-1)(m-2) + (m-1)(n-2)\right]. \end{aligned}$$

□

Remark 3.2. *The above theorem can be reduced to:* $det(B_n+B_m) = (-1)^{m+n-3}[2nm-3(n+m)+4]$.

Corollary 3.3. *When* $m = n$ *the theorem becomes:* $det(B_n+B_n) = -2(n-1)(n-2)$.

Special Cases

We consider the n^{th} order adjacency matrix P_n of a complete graph K_{n-1} with one pendent vertex. We find the determinant of such matrices.
$P_n = (p_{ij})$ where

$$p_{ij} = \begin{cases} b_{ij} & \text{if } 1 \le i,j \le (n-1) \\ 1 & \text{if } i = n, j = n-1 \\ 1 & \text{if } i = n-1, j = n \\ 0 & \text{otherwise} \end{cases}.$$

For example, we consider a complete graph K_4 with one pendent vertex whose adjacency matrix is denoted by P_5 shown in figure 6.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

FIGURE 5. P_5

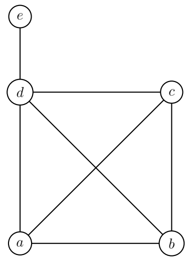

FIGURE 6. Graph K_4 with one pendent vertex

Theorem 3.4. $det(P_n) = (-1)^{n-2}(n-3)$.

Proof. Expanding the determinant along the last row and then, expanding again along the last column, we get:

$$\det(P_n) = (-1)^{2n-1}(-1)^{2n-2}\left[det(B_{n-2})\right] = (-1)^{n-2}(n-3).$$

□

Remark 3.5. *Using the help of the* $\det(A_n)$*, we find the determinant of adjacency matrix of a graph with n vertices, where (n-1) vertices form a complete graph and* n^{th} *vertex is not connected to 2 vertices,* $n \geq 4$*. The adjacency matrix of such graphs is given as* $S = (s_{ij})$*, where*

$$s_{ij} = \begin{cases} s_{ij} = b_{ij} & 1 \leq i, j \leq n-1 \\ s_{ij} = 0 & \textit{i = (n-2), (n-1), (n); j = (n)} \\ s_{ij} = 0 & \textit{i = (n); j =(n-2), (n-1), (n)} \\ s_{ij} = 1 & \textit{otherwise} \end{cases} . \tag{3.1}$$

Determinant of the matrix S is $(-1)^{n-2}(n-3)$.

4. CONCLUSION

Similar to algebraic theory of graphs, one can develop the algebraic theory for formal context. The determinant of the amalgam of Orthomodular lattices in general may be found.

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