ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **11** (2022), no.10, 925–941
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.10.9

A NEW KERNEL FUNCTION GENERATING THE BEST COMPLEXITY ANALYSIS FOR MONOTONE SDLCP

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ABSTRACT. In this article, we propose a new class of search directions based on new kernel function to solve the monotone semidefinite linear complementarity problem by primal-dual interior point algorithm.

We show that this algorithm based on this function benefits from the best polynomial complexity, namely $O(\sqrt{n}(\log n)^2 \log \frac{n}{\epsilon})$. The implementation of the algorithm showed a great improvement concerning the time and the number of iterations.

1. INTRODUCTION

Let S^n denotes the space of all $n \times n$ real symmetric matrices, S^n_+ and S^n_{++} is the cone of symmetric positive semidefinite, and symmetric positive definite matrices respectively. The semidefinite linear complementarity problem (SDLCP) is defined by:

Find a pair of matrices $(X, Y) \in S^n \times S^n$ that satisfies the following conditions

$$X, Y \in S^n_+, Y = L(X) + Q, \text{ and } X \bullet Y = Tr(XY) = 0, \tag{1.1}$$

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2020 *Mathematics Subject Classification.* 90C30, 90C30.
Key words and phrases. Semidefinite linear complementarity problems, Interior-point methods, Primal dual Newton method,Polynomial complexity.
Submitted: 07.10.2022; *Accepted:* 22.10.2022; *Published:* 26.10.2022.

where $L : S^n \to S^n$ is a linear transformation and $Q \in S^n$. Interior point methods (IPMS) considered the powerful tools to solve linear optimization (LO) and can be extended to more general cases such as complementarity problem (CP), semidefinite optimization(SDO) and semidefinite linear complementarity problem (SDLCP).

The semidefinite linear complementarity problem (SDLCP) which can be viewed as a generalization of the standard linear complementarity problem (LCP) and included the geometric monotone semidefinite linear complementerity introduced by Kojima et al [6], so it became the object of many studies of research these last years due to the importance of applications in mathematical programming and various areas of engineering and scientific fields.

Because their polynomial complexity and their simulation efficiency, primal-dual following path are the most attractive methods among interior point to solve a large wide of optimization problems ([8], [9], [12]). These methods are based on the kernel functions for determining new search directions and new proximity functions for analyzing the complexity of these algorithms, thus we have shown the important role of the kernel function in generating a new design of primal-dual interior point algorithm.

Also these methods are introduced by Bai et al [2] for (LO) and Elghami [4] for (SDO), then it's extended by many authors for different problems in mathematical programming ([1], [3], [7]).

The polynomial complexity of large update primal-dual algorithms is improved in contrast with the classical complexity given by logarithmic barrier functions by using this new form.

A kernel function is an univariate strictly convex function which is defined for all positive real t and is minimal at $t = 1$, whereas the minimal value equals 0. In the other words $\psi(t)$ is a kernel function when it is twice differentiable and satisfies the following conditions

$$\psi(1) = \psi'(1) = 0, \psi''(t) > 0 \; for \; all \; t > 0 \; and \; \lim_{t \to 0} \psi(t) = \lim_{t \to +\infty} \psi(t) = +\infty.$$

We can describe by its second derivative, as follows

$$\psi(t) = \int_1^t \int_1^\zeta \psi''(\xi) d\xi d\zeta.$$

We show that the methods of solving (SDLCP) are similar to the methods of (SDO), but there are difficulties and differences including non-orthogonality of search directions, therefore this will be studied in detail later.

In this paper, we establish the polynomial complexity for (SDLCP) by introducing our new kernel function

$$\psi(t) = \frac{1}{2}(2t^2 + \frac{1}{t^2} - 5) + e^{\frac{1}{t}-1}. \tag{1.2}$$

The idea of this work, is to investigate such a new kernel function and the corresponding barrier function and show that our large-update primal-dual algorithm has best complexity bound in terms of elegant analytic properties of this kernel function.

The paper is organized as follows. In section 3, we present the generic primal-dual algorithm, based on Nestrov-Todd direction, the new kernel function and its growth properties for (SDLCP) are presented in the section 4. In the section 5, we detailed proofs of the complexity of the proposed algorithm (an estimation of the step size and its default value, the worst case iteration complexity). In the section 6, some numerical results are presented. Finally, a conclusion in the section 7.

2. Preliminaries

Throughout the paper we use the following notation and we review some known facts about matrices and matrix functions which will be used in the analysis of the algorithm. The expression $X \succeq 0$ ($X \succ 0$) means that $X \in S^n_+$ ($X \in S^n_{++}$). The trace of $n \times n$ matrix X is denoted by $Tr(X) = \sum_{i=1}^{n} x_{ii}$. The Frobenius norm of a matrix $X \in \mathbb{R}^{n\times n}$ is defined by $\|X\|_F = \sqrt{X \bullet X} = \sqrt{Tr(X^TX)}$. For any $X \succ 0$, $\lambda_i(X)$, $1 \leq i \leq n$, denote its eigenvalues. $X^{1/2}$ denotes the symmetric square root, for any $X \in S^n_{++}$. The identity matrix of order n is denoted by I. The diagonal matrix with the vector x is denoted by $X = \mathrm{diag}(x)$. we denote by $\lambda(V)$ the vector of eigenvalues of $V \in S^n_{++}$, arranged in non-increasing order, that is $\lambda_1(V) \geq \lambda_2(V) \geq \ldots \geq \lambda_n(V)$.

Theorem 2.1. *(Spectral theorem for symmetric matrices [2]) The real $n \times n$ matrix A is symmetric if and only if there exists a matrix $Q \in \mathbb{R}^{n\times n}$ such that $Q^TQ = I$ and $Q^TAQ = B$, where I is the $n \times n$ identity matrix and B is a diagonal matrix.*

Definition 2.1. *([4], Definition 3.2.1) Let V be a symmetric matrix, and*

$$V = Q^T \operatorname{diag}(\lambda_1(V), \lambda 2(V), \ldots, \lambda_n(V))Q, \tag{2.1}$$

where Q is any orthogonal matrix that diagonalizes V, let $\psi(t)$ be defined as in equation (1.2), the matrix valued function $\psi : S^n \to S^n$ is given by

$$\psi(V) = Q^T \operatorname{diag}(\psi(\lambda_1(V)), \psi(\lambda_2(V)), \ldots, \psi(\lambda_n(V)))Q, \tag{2.2}$$

since $\psi(t)$ is differentiable, and the derivative $\psi'(t)$ is defined by

$$\psi'(V) = Q^T \operatorname{diag}(\psi'(\lambda_1(V)), \psi'(\lambda_2(V)), \ldots, \psi'(\lambda_n(V)))Q, \ for\ t > 0.$$

Using ψ, we define the barrier function (or proximity function) $\Psi(V) : S^n_+ \to \mathbb{R}_+$, as follows

$$\Psi(V) = Tr(\psi(V)) = \sum_{i=1}^{n} \psi(\lambda_i(V)). \tag{2.3}$$

In [4, 5], we can be found some concepts related to matrix functions.

3. Presentation of Problem

The feasible set, the strict feasible set and the solution set of the system (1.1) are subsets of $\mathbb{R}^{n\times n}$ denoted respectively by

$$\begin{aligned}
\mathcal{F} &= \{(X, Y) \in S^n \times S^n, Y = L(X) + Q : X \succeq 0, Y \succeq 0\}, \\
\mathcal{F}^0 &= \{(X, Y) \in \mathcal{F} : X \succ 0, Y \succ 0\}, \\
\mathcal{S} &= \{(X, Y) \in \mathcal{F} : Tr(XY) = 0\}.
\end{aligned}$$

The set $\mathcal{S}$ is nonempty and compact, if $\mathcal{F}^0$ is not empty and L is monotone.

As we know, the basic idea of primal-dual IPMs is to relax the third equation (complementarity condition) in system (1.1) with the following parameterized system

$$\begin{cases} X \succ 0, Y \succ 0, \\ Y = L(X) + Q, \\ XY = \mu I, \end{cases} \tag{3.1}$$

where $\mu > 0$ and I is the identity matrix.

Since L is a linear monotone transformation and (SDLCP) is strictly feasible (i.e, there exists $(X_0, Y_0) \in \mathcal{F}^0$), the system (3.1) has a unique solution for any $\mu > 0$.

As $\mu \to 0$ the sequence $(X(\mu), Y(\mu))$ approaches the solution (X, Y) of problem (SDLCP). The system (3.1) is equivalent to

$$\left\{\begin{array}{l} L(\Delta X) = \Delta Y, \\ \Delta X + P\Delta Y P^T = \mu Y^{-1} - X, \end{array}\right. \tag{3.2}$$

where P is defined in [11]

$$\begin{aligned} P &= X^{\frac{1}{2}}(X^{\frac{1}{2}}YX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}} \\ &= Y^{-\frac{1}{2}}(Y^{\frac{1}{2}}XY^{\frac{1}{2}})^{\frac{1}{2}}Y^{-\frac{1}{2}}. \end{aligned}$$

Let $D = P^{\frac{1}{2}}$, where $P^{\frac{1}{2}}$ denotes the symmetric square root of P. The matrix D can be used to scale X and Y to the same matrix V, defined by

$$V = \frac{1}{\sqrt{\mu}}D^{-1}XD^{-1} = \frac{1}{\sqrt{\mu}}DYD, \tag{3.3}$$

thus we have

$$V^2 = \frac{1}{\mu}D^{-1}XYD. \tag{3.4}$$

Note that the matrix V and D are symmetric and positive definite. Using (3.3), the system (3.1) becomes

$$\left\{\begin{array}{l} \tilde{L}(D_X) = D_Y, \\ D_X + D_Y = V^{-1} - V, \end{array}\right. \tag{3.5}$$

with

$$D_X = \frac{1}{\sqrt{\mu}}D^{-1}\Delta XD^{-1},\ D_Y = \frac{1}{\sqrt{\mu}}D\Delta YD,\ and\ \tilde{L}(D_X) = DL(DD_XD)D. \tag{3.6}$$

The linear transformation $\tilde{L}$ is also monotone on S^n. Under our hypothesis the new linear system (3.5) has a unique symmetric solution (D_X, D_Y). These directions are not orthogonal, because

$$\begin{aligned} D_X \bullet D_Y &= Tr(D_Y D_X) \\ &= \tfrac{1}{\mu}\Delta X \bullet L(\Delta X) \geq 0. \end{aligned}$$

Thus, this property makes the analysis more difficult from SDO problem.

So far, we have described the schema that defines classical NT-direction. According to [4,12], we replace the right hand side of the second equation in system (3.5) by $-\psi'(V)$. Thus, we will use the following system to define new search directions

$$\left\{ \begin{array}{l} \tilde{L}(D_X) = D_Y, \\ D_X + D_Y = -\psi'(V). \end{array} \right. \tag{3.7}$$

The new search directions D_X and D_Y are obtained by solving system (3.7), so that ΔX and ΔY are computed via (3.6). By taking along the search directions with a step size α defined by some line search rules, we can construct a new couple (X_+, Y_+) according to $X_+ = X + \alpha\Delta X$ and $Y_+ = Y + \alpha\Delta Y$.

The generic form of the large-update primal-dual interior point algorithm for solving SDLCP is stated as follows.

Algorithm Generic interior point algorithm for SDLCP

Input: A threshold parameter $\tau \geq 1$; an accuracy parameter $\epsilon \geq 0$;
barrier update parameter θ, $0 < \theta < 1$; $X^0 \succ 0$, $Y^0 \succ 0$ and
$\mu^0 = Tr(X^0Y^0)/n$ such that $\Psi(X^0, Y^0, \mu^0) \leq \tau$
begin
$X := X^0; Y := Y^0; \mu := \mu^0;$
while $n\mu \geq \epsilon$ **do**
begin
$\mu = (1-\theta)\mu;$
while $\Psi(X, Y, \mu) > \tau$ **do**
begin
Solve system (3.7) and use (3.6) to obtain $(\Delta X, \Delta Y)$;
determine a suitable step size α;
update $(X, Y) := (X, Y) + \alpha(\Delta X, \Delta Y)$
end
end
end

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4. Properties of New Kernel Function

In this part, we present the new kernel function defined as follows

$$\psi(t) = \frac{1}{2}(2t^2 + \frac{1}{t^2} - 5) + e^{\frac{1}{t}-1}. \tag{4.1}$$

the properties of this function are crucial in our complexity analysis. We list the first three derivatives of ψ, as follows

$$\psi'(t) = 2t - \frac{1}{t^3} - \frac{1}{t^2}e^{\frac{1}{t}-1}, \tag{4.2}$$

$$\psi''(t) = 2 + \frac{3}{t^4} + (\frac{2}{t^3} + \frac{1}{t^4})e^{\frac{1}{t}-1}, \tag{4.3}$$

$$\psi'''(t) = -\frac{12}{t^5} - (\frac{6}{t^5} + \frac{1}{t^6} + \frac{6}{t^4})e^{\frac{1}{t}-1}. \tag{4.4}$$

It is easy to verify that $\lim_{t\to 0}\psi(t) = \lim_{t\to+\infty},\psi(t) = +\infty$ and $\psi(1) = \psi'(1) = 0$. Then $\psi(t)$ is a barrier kernel function, and from (4.3), (4.4) we see that $\psi''(t) > 1, \psi'''(t) < 0$.

Lemma 4.1. *Let $\psi(t)$ be defined as in* (4.1)*, then*

$$t\psi''(t) + \psi'(t) > 0,\ for\ t < 1, \tag{4.5}$$

$$t\psi''(t) - \psi'(t) > 0,\ for\ t > 1, \tag{4.6}$$

$$\psi'''(t) < 0,\ for\ t > 0, \tag{4.7}$$

$$2\psi''(t)^2 - \psi'''(t)\psi'(t) > 0,\ for\ t < 1, \tag{4.8}$$

$$\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0,\ for\ t > 1, \beta > 1. \tag{4.9}$$

Proof. For (4.5) and (4.6), using (4.2) and (4.3), we have

$$t\psi''(t) + \psi'(t) = 4t + \frac{2}{t^3} + (\frac{1}{t^2} + \frac{1}{t^3})e^{\frac{1}{t}-1} > 0\, \textit{for}\ t < 1, \tag{4.10}$$

and

$$t\psi''(t) - \psi'(t) = \frac{4}{t^3} + (\frac{1}{t^3} + \frac{3}{t^2})e^{\frac{1}{t}-1} > 0,\ \textit{for}\ t > 1. \tag{4.11}$$

For (4.7), it is clear from (4.4), $\psi'''(t) < 0\ for\ t > 0$.

For the (4.8), we have

$$2\psi''(t)^2 - \psi'''(t)\psi'(t) = K(t) + H(t)e^{2(\frac{1}{t}-1)} + Q(t)e^{(\frac{1}{t}-1)} > 0\, \textit{for}\ t < 1, \tag{4.12}$$

where $K(t) = 8 + \frac{48}{t^4} + \frac{6}{t^8},\ H(t) = (\frac{2}{t^6} + \frac{2}{t^7} + \frac{1}{t^8})$ and $Q(t) = (\frac{28}{t^3} + \frac{20}{t^4} + \frac{2}{t^5} + \frac{6}{t^7} + \frac{6}{t^8} - \frac{1}{t^9})$.

Finally, to obtain (4.9), using (4.6) and (4.7). □

Now, we introduce the proximity measure $\delta(V)$, as follows

$$\delta(V) \ = \ \tfrac{1}{2}\| - \psi'(V)\| = \tfrac{1}{2}\sqrt{Tr(\psi'(V)^2)} = \tfrac{1}{2}\|D_X + D_Y\|, \tag{4.13}$$

note that

$$\delta(V) = 0 \Leftrightarrow V = I \Leftrightarrow \Psi(V) = 0.$$

Lemma 4.2. *For $\psi(t)$, the following assertions holds*

1. $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2))$ *for* $t_1, t_2 > 0$,
2. $\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}(\psi'(t))^2$, *for* $t > 0$,
3. $\psi(t) \leq 4(t-1)^2$, *for* $t \geq 1$.

Proof. For the first item, using ([10], lemma 2.1.2) and (4.5), we get the exponential property of $\psi(t)$. For the first inequality in item (2), using $(\psi''(t) > 1)$, we have

$$\psi(t) = \int_1^t \int_1^\zeta \psi''(\xi) d\xi d\zeta \geq \int_1^t \int_1^\zeta d\xi d\zeta = \frac{1}{2}(\psi'(t))^2.$$

The second inequality is obtained, as follows

$$\psi(t) = \int_1^t \int_1^\zeta \psi''(\xi) d\xi d\zeta \leq \int_1^t \int_1^\zeta \psi''(\xi)\psi''(\zeta) d\xi d\zeta = \frac{1}{2}(t-1)^2.$$

For item (3), using Taylor's theorem with $\psi(1) = \psi'(1) = 0$, $\psi''(t) = 8$ and $\psi'''(t) < 0$, we obtain

$$\psi(t) \leq \frac{1}{2}(\psi''(1))(t-1)^2 = 4(t-1)^2.$$

□

Now, let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$, for $t \geq 1$ then we have the following lemma.

Lemma 4.3. *For $\psi(t)$, we have*

$$\sqrt{1+s} \leq \varrho(s) \leq 1 + \sqrt{2s}, \textit{ for } s \geq 0.$$

Proof. Let $s = \psi(t)$ for $t \geq 0$, since $\psi_b(t)$ is monotonically decreasing and $\psi_b(1) = 0$ then $s = t^2 - 1 + \psi_b(t) \leq t^2 - 1$, where $\psi_b(t) = \frac{1}{2}(t^{-2} - 3) + e^{\frac{1}{t}-1}$ denotes the barrier term. This implies that $t = \varrho(s) \geq \sqrt{1+s}$. According to the second inequality in Lemma 4.2, we have $t = \varrho(s) \leq 1 + \sqrt{2s}$, $s \geq 0$. □

Theorem 4.1. *Let $0 \leq \theta \leq 1$ and $V_+ = \frac{V}{\sqrt{1-\theta}}$. If $\Psi(V) \leq \tau$, then we have*

$$\Psi(V_+) \leq \frac{4}{1-\theta}(\sqrt{2\tau} + \sqrt{n}\theta)^2.$$

Proof. From ([2], Theorem 3.3.2) with $\beta = \frac{1}{\sqrt{1-\theta}}$, Lemma 4.3, we have

$$\Psi(V_{+)} \leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\varrho(\frac{\Psi(V)}{n}\right) \leq 4n\left(\frac{1+\sqrt{2(\frac{\tau}{n})}-\sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 \leq \frac{4}{1-\theta}\left(\sqrt{2\tau}+\sqrt{n}\theta\right)^2 = \Psi_0,$$

since $1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, for $0 \leq \theta < 1$. the last inequality is hold Ψ_0 is an upper bound of $\Psi(V)$. □

Theorem 4.2. *[9] Suppose that V_1 and V_2 are symmetric positive definite and Ψ is the real valued matrix function induced by the matrix function ψ. Then,*

$$\Psi\left(\left[(V_1^{\frac{1}{2}}V_2V_1^{\frac{1}{2}})^{\frac{1}{2}}\right]\right) \leq \frac{1}{2}(\Psi(V_1)+\Psi(V_2)).$$

Lemma 4.4. *For any $V \succ 0$,*

$$\delta \geq \sqrt{\frac{\Psi(V)}{2}}. \tag{4.14}$$

Proof. Using the second inequality in Lemma 4.2 and (4.14), we have

$$\delta^2 = \frac{1}{4}Tr(\psi'(V)^2) \geq \frac{2}{4}\sum_{i=1}^{n}\psi(\lambda_i(V)) \geq \frac{1}{2}\Psi(V),$$

hence $\delta \geq \sqrt{\frac{\Psi(V)}{2}}$. □

During the algorithm, we assume that $\tau \geq 1$. Using $\Psi(V) \geq \tau$ and (4.14), we have $\delta \geq \sqrt{\frac{1}{2}}$.

5. Complexity Analysis

5.1. **An estimation of the step size.** The important idea of this section is to obtain a new complexity results for an (SDLCP) problem by using the proximity function defined by new kernel function. During an inner iteration, we compute a default step size α, the decrease of the proximity function and give the complexity results of the algorithm. Taking a step size α, we have new iterates

$$X_+ = \sqrt{\mu}D(V+\alpha D_X)D \quad \text{and} \quad Y_+ = \sqrt{\mu}D^{-1}(V+\alpha D_Y)D^{-1},$$

where D_X, D_Y and D are defined by (3.6), so we have

$$V_+^2 = (V+\alpha D_X)^{\frac{1}{2}}(V+\alpha D_Y)(V+\alpha D_X)^{\frac{1}{2}}.$$

Since the proximity after one step is defined by

$$\Psi(V_+) = \Psi([(V+\alpha D_X)^{\frac{1}{2}}(V+\alpha D_Y)(V+\alpha D_X)^{\frac{1}{2}}]^{\frac{1}{2}}).$$

By Theorem (4.2), we have $\Psi(V_+) \leq \frac{1}{2}[\Psi((V+\alpha D_X) + \Psi(V+\alpha D_Y)]$.

Define for $\alpha > 0$, $f(\alpha) = \Psi(V_+) - \Psi(V)$ and $f_1(\alpha) = \frac{1}{2}[\Psi((V+\alpha D_X) + \Psi(V+\alpha D_Y)] - \Psi(V)$. It is easily seen that, $f_1(0) = f(0) = 0$ and $f(\alpha) \leq f_1(\alpha)$. Furthermore, $f_1(\alpha)$ is a convex function.

Now, to estimate the decrease of the proximity during one step, we need the two successive derivatives of $f_1(\alpha)$ with respect to α. By using the rule of differentiability in [5, 9], we obtain

$$f_1'(\alpha) = \frac{1}{2}Tr(\psi'((V+\alpha D_X)D_X + \psi'(V+\alpha D_Y)D_Y),$$

and

$$f_1''(\alpha) = \frac{1}{2}Tr(\psi''((V+\alpha D_X)D_X^2 + \psi''(V+\alpha D_Y)D_Y^2).$$

Hence, using (4.13) and (3.7), we obtain

$$f_1'(0) = \frac{1}{2}Tr(\psi'((V)(D_X+D_Y)) = \frac{1}{2}Tr(-\psi'(V)^2) = -2\delta^2(V). \tag{5.1}$$

In what follows, we use the short notation $\delta(V) := \delta$.

Lemma 5.1. *[2, Lemma 4.4] Let $\rho : [0,\infty) \to (0,1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$, and $\overline{\alpha}$, is given by*

$$\overline{\alpha} = \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)),$$

then

$$\overline{\alpha} \geq \tilde{\alpha} = \frac{1}{\psi''(\rho(2\delta))}.$$

Lemma 5.2. *One has*

$$\overline{\alpha} \geq \frac{1}{6 + 2(6\delta+1)(1+\log(4\delta+1))^2}. \tag{5.2}$$

Proof. We need to compute $\rho(2\delta) = s$, where $\rho : [0,\infty) \to (0,1]$ be the inverse of $-\frac{1}{2}\psi'(t)$ for $t \in [0,1)$. This implies

$$\begin{aligned} -\psi'(t) = 4\delta \quad &\Leftrightarrow \quad -2t + \tfrac{1}{t^3} + \tfrac{1}{t^2}e^{\frac{1}{t}-1} = 4\delta \\ &\Leftrightarrow \quad e^{\frac{1}{t}-1} = t^2(4\delta + 2t - \tfrac{1}{t^3}) \\ &\Rightarrow \quad t \geq \tfrac{1}{1+\log(4\delta+1)}. \end{aligned} \tag{5.3}$$

Using the definition of $\psi''(t)$ and equation (5.3), if $s \leq 1$, we have $\frac{s-1}{s^5} \leq 0$ and $\frac{1}{s^2} \leq (1+\log(4\delta+1))^2$. Then,

$$\psi''(t) = 2 + \frac{3}{t^4} + (\frac{2}{t^3} + \frac{1}{t^4})e^{\frac{1}{t}-1} \leq 6 + 2(6\delta+1)(1+\log(4\delta+1))^2$$

$$\overline{\alpha} \geq \frac{1}{6+2(6\delta+1)(1+\log(4\delta+1))^2.}$$

As a default step size, we take

$$\tilde{\alpha} = \frac{1}{6+2(6\delta+1)(1+\log(4\delta+1))^2.} \tag{5.4}$$

□

Following the same procedure as in [2], we have the following lemma

Lemma 5.3. *Let $\tilde{\alpha}$ be a step size as defined in* (5.4) *and $\Psi(V) \geq 1$, then*

$$f(\tilde{\alpha}) \leq -\frac{\sqrt{\Psi_0}}{33(1+\log(2\sqrt{2\Psi_0}+1))^2}. \tag{5.5}$$

Proof. By (Lemma 4.5, [2]) and $\overline{\alpha} \geq \tilde{\alpha}$, we have

$$\begin{aligned} f(\tilde{\alpha}) &\leq -\frac{\delta^2}{6+2\delta(6+\sqrt{2})(1+\log(4\delta+1))^2} \\ &\leq -\frac{1}{2}\left(\frac{\Psi}{6+2\frac{\sqrt{\Psi}}{\sqrt{2}}(6+\sqrt{2})(1+\log(4\frac{\sqrt{\Psi}}{\sqrt{2}}+1))^2}\right) \\ &\leq -\frac{\sqrt{\Psi_0}}{33(1+\log(2\sqrt{2\Psi_0}+1))^2}. \end{aligned}$$

□

5.2. **Iteration bound.** To come back to the situation, where $\Psi(V) \leq \tau$ after $\mu-$update we need to count how many inner iterations. Let the value of $\Psi(V)$ after $\mu-$update be denoted by Ψ_0 and the subsequent values by Ψ_k, for $k = 0, 1, \ldots, K-1$, where K is the total number of inner iterations in the outer iteration. Then,

$$\Psi_{K-1} > \tau, 0 \leq \Psi_K \leq \tau. \tag{5.6}$$

Lemma 5.4. *Let K be the total number of inner iterations in the outer iteration. Then,*

$$K \leq 66\left(1+\log(2\sqrt{2\Psi_0}+1))^2\right)\Psi_0^{\frac{1}{2}}.$$

Proof. Using ([10], proposition 1.2.3), by taking $t_k = \Psi_k, \ \ \beta = \frac{1}{33(1+\log(2\sqrt{2\Psi_0}+1))^2}$ and $\gamma = \frac{1}{2}$ we get the result. □

Now, we estimate the total number of iterations of our algorithm.

Theorem 5.1. *If $\tau \geq 1$, the total number of iterations is not more than*

$$66(1+\log(2\sqrt{2\Psi_0}+1))^2)\Psi_0^{\frac{1}{2}}\frac{1}{\theta}\log\frac{n\mu^0}{\epsilon}.$$

Proof. In the algorithm, $n\mu \leq \epsilon$, $\mu^k = (1-\theta)^k\mu^0$ and $\mu^0 = \dfrac{x_0^t y_0}{n}$. By simple computation, we have

$$k \leq \frac{1}{\theta}\log\tfrac{n\mu^0}{\epsilon}.$$

By multiplying the number of outer iterations and the number of inner iterations, we get an upper bound for the total number of iterations, namely

$$\frac{K}{\theta}\log\tfrac{n\mu^0}{\epsilon} \leq \frac{66}{\theta}(1+\log(2\sqrt{2\Psi_0}+1))^2)\Psi_0^{\frac{1}{2}}\log\frac{n\mu^0}{\epsilon}.$$

This completes the proof. □

Remark 5.1. *We assume that $\tau = O(n)$, $\theta = \Theta(1)$ and $\Psi_0^{\frac{1}{2}} = O(\sqrt{n})$, we obtain the solution of the problem at most $O(\sqrt{n}(\log n)^2\log\dfrac{n}{\epsilon})$.*

6. Numerical results

The main purpose of this section, is to present three monotone SDLCPs for testing the effectiveness of algorithm. The implementation is manipulated in "Matlab". Here we use "inn" which means the iterations number produced by the algorithm. The choice of different values of the parameters shows their effect on reducing the number of iterations.

In all experiments, we use $\tau = 2$, $\epsilon = 10^{-6}$, $\alpha \in \{\ 0.9,\ 1\}$ and $\theta \in \{\ 0.95,\ 0.99\}$, the barrier parameter $\mu_0 \in \{Tr(XY)/n,\ 0.005,\ 0.0005\}$. We provide a feasible initial point (X_0, Y_0) such that IPC and $\Psi(X_0, Y_0, \mu_0) \leqslant \tau$ are satisfied.

The first example is the monotone SDLCP defined by two sided multiplicative linear transformation [1]. The second is monotone SDLCP which is equivalent to the symmetric semidefinite least squares (SDLS) problem and the third one is reformulated from nonsymmetric semidefinite least squares (NS-SDLS) problem [7], in the second and third example, L is Lyaponov linear transformation.

Example 1. *The data of the monotone SDLCP is given by $L(X) = AXA^T$, where*

$$A = \begin{pmatrix} 17.25 & -1.75 & -1.75 & -1.75 & -1.75 \\ -1.75 & 16.25 & -2 & 0 & 0 \\ -1.75 & -2 & 16.25 & -2 & 0 \\ -1.75 & 0 & -2 & 16.25 & -2 \\ -1.75 & 0 & 0 & -2 & 16.25 \end{pmatrix},$$

$$Q = \begin{pmatrix} -9.25 & 1.25 & 1.25 & 1.25 & 1.25 \\ 1.25 & -8.25 & 1.5 & 0 & 0 \\ 1.25 & 1.5 & -8.25 & 1.5 & 0 \\ 1.25 & 0 & 1.5 & -8.25 & 1.5 \\ 1.25 & 0 & 0 & 1.5 & -8.25 \end{pmatrix}.$$

The strictly feasible initial starting point $X^0 \succ 0$ is given by $X^0 = Diag(0.0620, \ldots, 0.0620)$. The unique solution $X^* \in S^5_+$, is given by

$$X^* = \begin{pmatrix} 0.0313 & 0.0020 & 0.0020 & 0.0020 & 0.0020 \\ 0.0020 & 0.0313 & 0.0019 & 0 & 0 \\ 0.0020 & 0.0019 & 0.0312 & 0.0019 & 0 \\ 0.0020 & 0 & 0.0019 & 0.0312 & 0.0019 \\ 0.0020 & 0 & 0 & 0.0019 & 0.0313 \end{pmatrix}.$$

The number of inner iterations for several choices of α, θ and μ obtained by algorithm are presented, in Table 1

TABLE 1. Number of inner iterations for several choices of α, θ and μ

$\alpha = 0.9$	μ		
θ	$Tr(XY)/n$	*0.005*	*0.0005*
0.95	*11*	*11*	*10*
0.99	*13*	*12*	*10*
$\alpha = 1$	μ		
θ	$Tr(XY)/n$	*0.005*	*0.0005*
0.95	*7*	*7*	*7*
0.99	*7*	*7*	*6*

Example 2. *The data of the monotone SDLCP which is equivalent to the symmetric semidefinite least squares (SDLS) problem, is given by*

$$L(X) = \frac{1}{2}(A^T AX + XA^T A) \quad \textit{and} \quad Q = -\frac{1}{2}(A^T B + B^T A),$$

where

$$A = \begin{pmatrix} 6 & -1 & 0 & 0 & 0 \\ -0.1 & 6 & -1 & 0 & 0 \\ 0 & -0.1 & 6 & -1 & 0 \\ 0 & 0 & -0.1 & 6 & -1 \\ 0 & 0 & 0 & -0.1 & 6 \\ 0 & 0 & 0 & 0 & -0.1 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.4 & 1 & 0 & 0 & 0 \\ -0.4 & -0.4 & 1 & 0 & 0 \\ -0.4 & 0 & -0.4 & 1 & 0 \\ -0.4 & 0 & 0 & -0.4 & 1 \\ -0.4 & 0 & 0 & 0 & -0.4 \end{pmatrix}.$$

The strictly feasible initial point $X^0 \succ 0$ *defined by* $X^0 = Diag(0.2369, \ldots, 0.2369)$. *The unique solution* $X^* \in S^5_+$ *of the proposed example is given by*

$$X^* = \begin{pmatrix} 0.1639 & -0.0215 & -0.0342 & -0.0328 & -0.0300 \\ -0.0215 & 0.1553 & -0.0227 & -0.0019 & -0.0027 \\ -0.0342 & -0.0227 & 0.1558 & -0.0194 & 0.0014 \\ -0.0328 & -0.0019 & -0.0194 & 0.1564 & -0.0189 \\ -0.0300 & -0.0027 & 0.0014 & -0.0189 & 0.1598 \end{pmatrix}$$

The number of inner iterations for several choices of α*,* θ *and* μ *are presented,in table 2.*

TABLE 2. Number of inner iterations for several choices of α, θ and μ

$\alpha = 0.9$	μ		
θ	$Tr(XY)/n$	*0.005*	*0.0005*
0.95	*11*	*11*	*10*
0.99	*12*	*12*	*10*

$\alpha = 1$	μ		
θ	$Tr(XY)/n$	*0.005*	*0.0005*
0.95	*7*	*7*	*7*
0.99	*7*	*7*	*6*

Example 3. *We consider the monotone SDLCP which is reformulated from NS-SDLS problem, the matrices A and B of NS-SDLS are given by*

$$A = \begin{pmatrix} -0.3157 & 0.0330 & 0.0603 \\ -0.3274 & -0.0158 & 0.0625 \\ -0.3569 & 0.0787 & 0.0563 \\ -0.2994 & 0.0301 & 0.0496 \\ -0.3243 & -0.0048 & 0.0715 \\ -0.3447 & 0.0736 & 0.0545 \\ -0.2417 & 0.0709 & 0.0522 \\ -0.2063 & -0.0099 & 0.0233 \\ -0.3285 & 0.1585 & 0.0979 \\ -0.2484 & 0.0878 & 0.0622 \\ -0.2196 & 0.0023 & 0.0280 \\ -0.3148 & 0.1506 & 0.0922 \end{pmatrix},$$

and

$$B = \begin{pmatrix} -1.4257 & 0.1528 & 0.4398 \\ -1.4024 & -0.3092 & 0.4187 \\ -1.3766 & 0.4366 & 0.4197 \\ -1.4274 & 0.1424 & 0.4353 \\ -1.3994 & -0.3095 & 0.4206 \\ -1.3716 & 0.4285 & 0.4193 \\ -1.4269 & 0.1581 & 0.4335 \\ -1.4015 & 0.3229 & 0.4214 \\ -1.3767 & -0.4189 & 0.4333 \\ -1.4257 & 0.1515 & 0.4358 \\ -1.3989 & 0.3276 & 0.4217 \\ -1.3724 & 0.1454 & 0.4356 \end{pmatrix}.$$

Lyapunov linear transformation $L(X)$ is symmetric and strictly monotone given by

$$L(X) = \frac{1}{2}(G^{-1}X + XG^{-1}) \text{ and } Q = -\frac{1}{2}(G^{-1}A^TB + B^TAG^{-1}), \tag{6.1}$$

where $G = A^TA$. The unique solution X^* of monotone SDLCP is given by

$$X^* = \begin{pmatrix} 5.1571 & -0.2615 & 1.9198 \\ -0.2609 & 6.2655 & -4.0779 \\ 1.9199 & -4.0778 & 0.6117 \end{pmatrix}.$$

The number of inner iterations for several choices of $\alpha \in \{\ 0.3, 0.5, 0.9\}$, $\mu = Tr(XY)/n$ *with feasible starting point* $X^0 = I$ *are presented in the following table 3.*

TABLE 3. Number of inner iterations

α	$\mu = Tr(XY)/n$
0.3	*6*
0.5	*3*
0.9	*2*

The results in these tables show that the algorithm based on our kernel function $\psi(t)$ is effective. the number of iterations and the time produced depends on the values of the parameters α, θ and μ, for all possible combinations of this parameters in practical computation, we obtained the better results than [1], [7].

7. CONCLUSION

In this paper, we introduced our new kernel function, with a complete theoretical study related to their characteristics, which contributed well to creating a new design for primal-dual interior-point algorithms. We showed that the theoretical complexity of large-update interior point method is $O(\sqrt{n}(\log n)^2 \log \frac{n}{\epsilon})$, which improves the best iteration complexity. Finally, the numerical results obtained are excellent, which indicated that our kernel function used in algorithm is efficient.

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