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ON A BIVARIATE KATZ'S DISTRIBUTION

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ABSTRACT. In this paper, we propose the bivariate distribution of the univariate Katz distribution [7] using the technique of the product of marginal distributions by a multiplicative factor. This method has been examined in [11] and used in [9] to construct a bivariate Poisson distribution. The obtained model is a good way to unify bivariate Poisson, bivariate binomial and bivariate negative binomial distributions and has interesting properties. Among others, the correlation coefficient of the obtained model can be either positive, negative, or null, and the necessary condition of zero correlation is a necessary and sufficient condition for independence. We used two methods to estimate the parameters: the method of moments and the maximum likelihood method. An application to concrete insurance data has been made. This data concerns natural events insurance in the USA and third-party liability automobile insurance in France [13].

1. INTRODUCTION

One of the most important questions in the modeling and analysis of count data is how to formulate a probability distribution, and one method of constructing multivariate distributions is the technique of the product of marginal distributions

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by a multiplicative factor. Developed in [11] and used in [9] to construct a bivariate Poisson distribution, the technique of the product of marginal distributions by a multiplicative factor consists of constructing a multivariate distribution whose random variables can be dependent or independent with a correlation coefficient which can be positive as well as negative or null. This method is thus one of the alternatives to the method of trivariate reduction (see [3]) as well as parallel works such as [5, 6, 12]), whose multivariate distributions obtained admit only a positive correlation [5]. In the bivariate case, the technique of the product of marginal distributions by a multiplicative factor stipulates that the random pair (X, Y), whose random variables X and Y have for probability mass functions $p(x) = P(X = x)$ and $p(y) = P(Y = y)$ respectively, accepts for probability mass function pmf $p(x, y) = P(X = x, Y = y)$ given by

$$p(x,y) = p(x)p(y)[1 + \alpha\left(g_1(x) - E\left[g_1(X)\right]\right)\left(g_2(y) - E\left[g_2(Y)\right]\right)], \tag{1.1}$$

where g_1 and g_2 must be bounded functions of $(x, y) \in \mathbb{R}^2$ and α any real number in a suitable range choosen such that $1+\alpha\left(g_1(x) - E\left[g_1(X)\right]\right)\left(g_2(y) - E\left[g_2(Y)\right]\right) \geq 0$ for all $x, y \geq 0$, with $p(x)$ and $p(y)$ as marginals. In [9], the authors constructed the bivariate Poisson distribution by taking $g_1(t) = g_2(t) = e^{-t},\ t \geq 0$.

With this in mind, we construct a bivariate Katz distribution using the technique of the product of marginal distributions by a multiplicative factor by taking the same g_1 and g_2 functions as [9]. Indeed, in [7] author formulated one of the most important families of probability distributions in the analysis and modeling of count data. Defined from the successive probability ratios

$$\frac{p(z+1)}{p(z)} = \frac{\lambda + \beta z}{z+1},\ z = 0, 1, \ldots, \tag{1.2}$$

with $p(0) \neq 0$ and $p(z) = P(Z = z)$, where $\lambda > 0$ and $\beta < 1$, it is understood that if $\lambda + \beta z < 0$ then $p(z) = 0$ for $z = 1, 2, \ldots$ [1,2], its pmf is given by [14]

$$p(z) = \begin{cases} \dfrac{\lambda^z}{z!}e^{-\lambda} & \text{if } \beta = 0, \\ \dfrac{(\lambda/\beta)_z \beta^z}{z!}(1-\beta)^{\lambda/\beta} & \text{otherwise,} \end{cases} \qquad z = 0, 1, \ldots,$$

where $(\alpha)_z$ is the Pochhammer symbol and defined to be $(\alpha)_z = \alpha(\alpha+1)\ldots(\alpha+z-1)$ for $z = 0, 1, \ldots,$ and α any real number with $(\alpha)_0 = 1$. This distribution is a good way to unif Poisson, binomial, and negative binomial distributions when $\beta = 0,\ \beta < 0$ and $\beta > 0$, respectively [2]. Thus, the bivariate Katz distribution is unified with the bivariate Poisson, bivariate binomial, and bivariate negative binomial distributions.

On this, the rest of the paper is presented as follows. First, we present successively the following notions: probability mass function, moment generating function, moments, correlation and independence. Secondly, we estimate the parameters of the model using two methods: the method of moments and the maximum likelihood method. Finally, we make an application to concrete insurance data followed by a conclusion. This data concerns natural events insurance in the USA and third-party liability automobile insurance in France [13].

2. Bivariate Katz's distribution

In this section, we present and study the bivariate Katz distribution from a probabilistic and statistical point of view. In the rest of this section, as in the others, we do not consider the particular case where one of the univariate Katz distributions reduces to the Poisson distribution, because otherwise it is sufficient to consider the Poisson distribution as a limit of the Katz distribution when the dispersion parameter tends to zero. And in particular, the case where the two univariate Katz distributions reduce to the univariate Poisson distributions, comes down to the bivariate Poisson distribution (cf. [9]).

2.1. **Probability mass function.** Let be consider two univariate Katz random variables, X and Y, with parameters (λ_1, β_1) and (λ_2, β_2), respectively. From (1.1), the random pair (X, Y) follows a bivaraite Katz distribution with parameters $(\lambda_1, \lambda_2, \beta_1, \beta_2, \alpha)$ which the pmf $p(x, y)$ is

$$\begin{aligned} p(x,y) &= (1-\beta_1)^{\lambda_1/\beta_1}(1-\beta_2)^{\lambda_2/\beta_2}\frac{(\lambda_1/\beta_1)_x(\lambda_2/\beta_2)_y\beta_1^x\beta_2^y}{x!y!} \\ &\times\left[1+\alpha\left(e^{-x}-\left[\frac{1-\beta_1e^{-1}}{1-\beta_1}\right]^{-\lambda_1/\beta_1}\right)\left(e^{-y}-\left[\frac{1-\beta_2e^{-1}}{1-\beta_2}\right]^{-\lambda_2/\beta_2}\right)\right], \end{aligned} \tag{2.1}$$

where $p(x)$ and $p(y)$ $((x, y) \in \mathbb{N}^2)$ are pmf of univariate Katz distributions as marginals with parameters (λ_1, β_1) and (λ_1, β_2), respectively.

2.2. **Moment generating function.** Let consider two random variables X and Y with joint probability given by (1.1), the moment generating function $M_{X,Y}$ of (X,Y) is:

$$\begin{aligned}
M_{X,Y}(t_1,t_2) =& E\left[e^{t_1X+t_2Y}\right] \\
=& \sum_{x,y\geq 0} e^{t_1x+t_2y}p(x)p(y)[1+\alpha\left(e^{-x}-E\left[e^{-X}\right]\right)\left(e^{-y}-E\left[e^{-Y}\right]\right)] \\
=& \sum_{x\geq 0} e^{t_1x}p(x)\sum_{y\geq 0} e^{t_2Y}p(y)+\alpha\left(\sum_{x\geq 0} e^{(t_1-1)x}p(x)-E\left[e^{-X}\right]\sum_{x\geq 0} e^{t_1x}p(x)\right) \\
&\times\left(\sum_{y\geq 0} e^{(t_2-1)y}p(y)-E\left[e^{-Y}\right]\sum_{y\geq 0} e^{t_2y}p(y)\right) \\
=& M_X(t_1)M_Y(t_2)+\alpha\left[M_X(t_1-1)-M_X(-1)M_X(t_1)\right] \\
&\cdot\left[M_Y(t_2-1)-M_Y(-1)M_Y(t_2)\right], \qquad (2.2)
\end{aligned}$$

where M_X and M_Y are the moment generating functions of variables X and Y, respectively. Since, the moment generating function of univariate Katz random variable Z with parameters (λ,β) is equal to $M_Z(t)=\left(\frac{1-\beta e^t}{1-\beta}\right)^{-\lambda/\beta}$ (see [1,2]), from (2.2) we deduce the moment generating function $M_{X,Y}$ of the Katz random pair (X,Y) with parameters $(\lambda_1,\lambda_2,\beta_1,\beta_2,\alpha)$,

$$\begin{aligned}
M_{X,Y}(t_1,t_2) =& \left(\frac{1-\beta_1e^{t_1}}{1-\beta_1}\right)^{-\lambda_1/\beta_1}\left(\frac{1-\beta_2e^{t_2}}{1-\beta_2}\right)^{-\lambda_2/\beta_2} \\
&+\alpha\left[\left(\frac{1-\beta_1e^{t_1-1}}{1-\beta_1}\right)^{-\lambda_1/\beta_1}-\left(\frac{1-\beta_1e^{-1}}{1-\beta_1}\right)^{-\lambda_1/\beta_1}\left(\frac{1-\beta_1e^{t_1}}{1-\beta_1}\right)^{-\lambda_1/\beta_1}\right] \\
&\times\left[\left(\frac{1-\beta_2e^{t_2-1}}{1-\beta_2}\right)^{-\lambda_2/\beta_2}-\left(\frac{1-\beta_2e^{-1}}{1-\beta_2}\right)^{-\lambda_2/\beta_2}\left(\frac{1-\beta_2e^{t_2}}{1-\beta_2}\right)^{-\lambda_2/\beta_2}\right].
\end{aligned}$$

2.3. **Moments.** Since the marginal pmf of X and Y are univariate Katz distributions with parameters (λ_1,β_1) and (λ_2,β_2), then the mean vector is

$$E\begin{bmatrix} X \\ Y \end{bmatrix}=\begin{bmatrix} \dfrac{\lambda_1}{1-\beta_1} \\ \dfrac{\lambda_2}{1-\beta_2} \end{bmatrix},$$

and the dispersion matrix

$$D = \begin{bmatrix} \dfrac{\lambda_1}{(1-\beta_1)^2} & cov(X,Y) \\ cov(X,Y) & \dfrac{\lambda_2}{(1-\beta_2)^2} \end{bmatrix}, \tag{2.3}$$

where

$$\begin{aligned}
cov(X,Y) &= E[XY] - E[X]E[Y] \\
=&\sum_{x,y\geq 0} xyp(x)p(y)\left[+\alpha\left(e^{-x} - E\left[e^{-X}\right]\right)\left(e^{-y} - E\left[e^{-Y}\right]\right)\right] - E[X]E[Y] \\
=&\alpha\left(E\left[Xe^{-X}\right] - E[X]E\left[e^{-X}\right]\right)\left(E\left[Ye^{-Y}\right] - E[Y]E\left[e^{-Y}\right]\right) \\
=&\alpha \cdot cov\left(X, e^{-X}\right) cov\left(Y, e^{-Y}\right) \\
=&\alpha\left(\frac{\lambda_1 e^{-1}}{1-\lambda_1 e^{-1}}\left[\frac{1-\beta_1 e^{-1}}{1-\beta_1}\right]^{-\lambda_1/\beta_1} - \frac{\lambda_1}{1-\beta_1}\left[\frac{1-\beta_1 e^{-1}}{1-\beta_1}\right]^{-\lambda_1/\beta_1}\right) \\
&\times\left(\frac{\lambda_2 e^{-1}}{1-\lambda_2 e^{-1}}\left[\frac{1-\beta_2 e^{-1}}{1-\beta_2}\right]^{-\lambda_2/\beta_2} - \frac{\lambda_2}{1-\beta_2}\left[\frac{1-\beta_2 e^{-1}}{1-\beta_2}\right]^{-\lambda_2/\beta_2}\right), \\
=&\frac{\alpha\lambda_1\lambda_2\left(1-e^{-1}\right)^2}{(1-\beta_1)(1-\beta_2)\left(1-\beta_1 e^{-1}\right)\left(1-\beta_1 e^{-1}\right)} \\
&\cdot\left[\frac{1-\beta_1 e^{-1}}{1-\beta_1}\right]^{-\lambda_1/\beta_1}\left[\frac{1-\beta_2 e^{-1}}{1-\beta_2}\right]^{-\lambda_2/\beta_2}.
\end{aligned}$$

2.4. **Correlation and independence.** From (2.3), the correlation coefficien of X and Y is

$$\rho_{XY} = \frac{\alpha\sqrt{\lambda_1\lambda_2}\left(1-e^{-1}\right)^2}{\left(1-\beta_1 e^{-1}\right)\left(1-\beta_1 e^{-1}\right)}\left[\frac{1-\beta_1 e^{-1}}{1-\beta_1}\right]^{-\lambda_1/\beta_1}\left[\frac{1-\beta_2 e^{-1}}{1-\beta_2}\right]^{-\lambda_2/\beta_2}. \tag{2.4}$$

Following [9], α can be bounded as follows

$$|\alpha| \leq \frac{1}{\left(1-\left[\dfrac{1-\beta_1 e^{-1}}{1-\beta_1}\right]^{-\lambda_1/\beta_1}\right)\left(1-\left[\dfrac{1-\beta_2 e^{-1}}{1-\beta_2}\right]^{-\lambda_2/\beta_2}\right)},$$

and ρ_{XY},

$$|\rho_{XY}| \leq \frac{\dfrac{\sqrt{\lambda_1\lambda_2}\,(1-e^{-1})^2}{(1-\beta_1 e^{-1})\,(1-\beta_1 e^{-1})}\left[\dfrac{1-\beta_1 e^{-1}}{1-\beta_1}\right]^{-\lambda_1/\beta_1}\left[\dfrac{1-\beta_2 e^{-1}}{1-\beta_2}\right]^{-\lambda_2/\beta_2}}{\left(1-\left[\dfrac{1-\beta_1 e^{-1}}{1-\beta_1}\right]^{-\lambda_1/\beta_1}\right)\left(1-\left[\dfrac{1-\beta_2 e^{-1}}{1-\beta_2}\right]^{-\lambda_2/\beta_2}\right)}.$$

From (2.4), it follows that for this model, $\rho_{XY} = 0$ if and only if $\alpha = 0$. This shows that the condition of zero correlation is a necessary and sufficient condition for the independence of the random variables X and Y. Moreover, the correlation coefficient of this model is positive, negative, or null according to α is positive, negative, or null.

3. Parameters estimating

In this section, we are interested in the estimation of the parameters, and we examine two methods of estimation: the method of moments and the maximum likelihood method. The asymptotic behavior of the obtained estimators is not studied because the asymptotic properties are directly derived from them (cf. [10]). In practice, the moment estimators (MME) can be used as initial values in the determination of the maximum likelihood estimators (MLE).

3.1. **Method of moments.** Let be considered a $n-$sample $(x_i, y_i),\ i = 1, 2, \ldots, n$ and note

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,\ \overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i,\ \widehat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2,\ \widehat{\sigma}_Y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i-\overline{y})^2$$

and

$$s_{XY} = \frac{1}{n}\sum_{x,y}^{n} n_{xy}(x-\overline{x})(y-\overline{y})$$

where $n_{x,y}$ is the frequency of the random pair (x, y) for $x = 0, 1, \ldots,\ y = 0, 1, \ldots$ and $\sum_{x,y} n_{xy} = n$.

The method of moments consists of equaling the theoretical moments with the empirical moments in order to determine the estimators. The system of equaling the theoretical moments to the practical moments is

$$
\left\{
\begin{array}{l}
\dfrac{\lambda_1}{1-\beta_1} = \overline{x}, \\
\dfrac{\lambda_2}{1-\beta_2} = \overline{y}, \\
\dfrac{\lambda_1}{(1-\beta_1)^2} = \widehat{\sigma}^2_X, \\
\dfrac{\lambda_2}{(1-\beta_2)^2} = \widehat{\sigma}^2_Y, \\
\dfrac{\alpha\lambda_1\lambda_2\left(1-e^{-1}\right)^2}{(1-\beta_1)(1-\beta_2)\left(1-\beta_1 e^{-1}\right)\left(1-\beta_1 e^{-1}\right)} \\
\quad \cdot \left[\frac{1-\beta_1 e^{-1}}{1-\beta_1}\right]^{-\lambda_1/\beta_1} \left[\frac{1-\beta_2 e^{-1}}{1-\beta_2}\right]^{-\lambda_2/\beta_2} = s_{XY}.
\end{array}
\right.
\tag{3.1}
$$

From (3.1), we derive the moments estimators:

$$
\left\{
\begin{array}{l}
\widehat{\lambda}_1 = \dfrac{\overline{x}^2}{\widehat{\sigma}^2_X}, \\
\widehat{\lambda}_2 = \dfrac{\overline{y}^2}{\widehat{\sigma}^2_Y}, \\
\widehat{\beta}_1 = 1 - \dfrac{\overline{x}}{\widehat{\sigma}^2_X}, \\
\widehat{\beta}_2 = 1 - \dfrac{\overline{y}}{\widehat{\sigma}^2_Y}, \\
\widehat{\alpha} = \dfrac{(1-\widehat{\beta}_1)(1-\widehat{\beta}_2)\left(1-\widehat{\beta}_1 e^{-1}\right)\left(1-\widehat{\beta}_1 e^{-1}\right) s_{XY}}{\widehat{\lambda}_1\widehat{\lambda}_2\left(1-e^{-1}\right)^2} \\
\quad \cdot \left[\frac{1-\widehat{\beta}_1 e^{-1}}{1-\widehat{\beta}_1}\right]^{\widehat{\lambda}_1/\widehat{\beta}_1} \left[\frac{1-\widehat{\beta}_2 e^{-1}}{1-\widehat{\beta}_2}\right]^{\widehat{\lambda}_2/\widehat{\beta}_2}.
\end{array}
\right.
$$

3.2. **Maximum likelihood method.** In this section, we start with the maximum likelihood of the univariate Katz distribution in order to deduce the maximum likelihood of the bivariate Katz distribution.

3.2.1. *Maximum likelihood method for the univariate Katz distribution.* Let Z be a Katz random variable with parameters (λ, β). We propose an expression for the log-likelihood function of the univariate Katz distribution for any $\beta < 1$ while keeping in mind that for $\beta < 0$ if $\lambda + \beta z < 0$ for all $z = 1, 2, \ldots, p(z) = 0$.

Firstly, since $p(0) = (1-\beta)^{\lambda/\beta}$ then from (1.2), we have:

$$\begin{aligned} p(z) &= \frac{\lambda+\beta(z-1)}{z} p(z-1), & z = 1, 2, \dots, \\ &= \frac{\lambda+\beta(z-1)}{z} \times \dots \times \frac{\lambda+\beta}{2} \times \frac{\lambda}{1} p(0), & z = 1, 2, \dots, \\ &\vdots \\ &= \prod_{k=1}^{z} \frac{\lambda+\beta(k-1)}{k} (1-\beta)^{\lambda/\beta}, & z = 1, 2, \dots, \end{aligned}$$

i.e.,

$$p(z) = \frac{(1-\beta)^{\lambda/\beta}}{z!} \prod_{k=1}^{z} [\lambda+\beta(k-1)], \ z = 0, \dots, \tag{3.2}$$

with convention $\prod_{k=1}^{z} = 1$ for $z = 0$.

Secondly, taking the logarithm of (3.2), we have:

$$\log p(z) = \frac{\lambda}{\beta} \log(1-\beta) + \sum_{k=1}^{z} \log[\lambda+\beta(k-1)] + \log(z!), \ z = 0, 1, \dots,$$

with convention $\sum_{k=1}^{z} = 0$ for $z = 0$.

Finally, under this convention, considering a $n-$sample $z = (z_1, \dots, z_n)$ and putting $\theta = (\lambda, \beta)$ the vector of the parameters, the log-likelihhod function $\log L(\theta; z)$ of the univariate Katz distribution is:

$$\log L(\theta; z) = \frac{n\lambda}{\beta} \log(1-\beta) + \sum_{i=1}^{n} \sum_{k=1}^{z_i} \log[\lambda+\beta(k-1)] + n\overline{\log(z!)}, \tag{3.3}$$

where $\overline{\log(z!)} = \frac{1}{n} \sum_{i=1}^{n} \log(z_i!)$.

3.2.2. *Maximum likelihood method for the bivaraite Katz distribution.* Now let us consider a $n-$sample $(x, y) = (x_1, y_1),\ ldots, (x_n, y_n)$and put $\theta_1 = (\lambda_1, \lambda_2, \beta_1, \beta_2, \alpha)$, $\theta_1 = (\lambda_1, \beta_1)$ and $\theta_2 = (\lambda_2, \beta_2)$ the vectors of the parameters. According to (2.1) and (3.3), the log-likelihood function $logL(\theta; x, y)$ of the bivariate Katz distribution is a function of the univariate Katz distribution's log-likelihood functions

$\log L(\theta_1; x)$ and $\log L(\theta_2; y)$ is equal to:

$$\log L(\theta; (x, y)) = \log L(\theta_1; x) + \log L(\theta_2; y)$$
$$+ \sum_{i=1}^{n} \log \left[1 + \alpha \left(e^{-x_i} - \left[\frac{1 - \beta_1 e^{-1}}{1 - \beta_1} \right]^{-\lambda_1/\beta_1} \right) \left(e^{-y_i} - \left[\frac{1 - \beta_2 e^{-1}}{1 - \beta_2} \right]^{-\lambda_2/\beta_2} \right) \right].$$

4. Applications

In this section, we make an application to concrete insurance data, such as that from [13] and concerning natural events insurance in the USA (application 1) and third-party liability automobile insurance in France (application 2). To estimate the parameters, we used the package maxLik for the R statistical environment [4].

4.1. **Application 1. Data description**: In order to study the joint distribution of the random pair (X, Y), the North Atlantic coastal states of the United States (from Texas to Maine) that can be affected by tropical cyclones have been divided into three geographical zones: Texas, Louisiana, and Mississippi (Zone 1), Alabama (Zone 2), and other states (Zone 3). To do so, we used the data from the Table 1, first row in each cell, which shows the realizations of (X, Y) observed during the 93 years from 1899 to 1991, where X and Y are the yearly frequencies of hurricanes affecting Zones 1 and 3 [13].

And the elementary statistics are

$$\begin{array}{ll} \overline{x} = 0.7419355 & \widehat{\sigma}_X^2 = 0.6283310 \\ \overline{y} = 0.4731183 & \widehat{\sigma}_Y^2 = 0.5345956 \end{array} \quad s_{XY} = 0.02532085$$

Table 1: Comparison of observed and theoretical yearly frequencies of Hurricanes (1899-1991) having affected Zone 1 and Zone 3 [13]

Zone 3 / Zone 1	0	1	2	3	Total
0	27 27.176007	9 10.8133049	3 2.8300534	2 0.60754107	41

	28.799891	9.5517349	2.2731149	0.46203950	
1	24 23.049160 22.397167	13 9.9598363 11.0478636	1 2.6881501 3.0828705	0 0.58368569 0.66618815	38
2	8 7.915677 7.066103	2 3.5246400 4.0071084	1 0.9612066 1.1621309	0 0.20948894 0.25439741	11
3	1 1.351342 1.005382	0 0.6083693 0.5999297	2 0.1665229 0.1761737	0 0.03634042 0.03872171	3
Total	60	24	7	2	93

First row: observed frequency

Middle row : theoretical frequency for MME

Last row: theoretical frequency for MLE

Estimation and goodness-of-fit: Table 2 contains the parameter estimates for both methods, and the corresponding theoretical frequencies are presented in Table 1, second and third rows of each cell for MME and MLE, respectively. For the MLE, the log-likelihood corresponding is equal to -179.4414.

TABLE 2. Parameters estimation

Method	Parameters				
	λ_1	λ_2	β_1	β_2	α
MME	0.8760801	0.4187107	-0.1808036	0.1149978	0.2059890
MLE	0.9066641	0.4254498	-0.2346817	0.1057741	0.9799467

We have used the Pearson's χ^2 test to calculate the differences between the observed and expected values after grouping the values into 7 categories: (0,0), (0,1), (0,2 and 3), (1,0), (1,1 and above), (2,0) and (other cases). Table 3 contains the values of the corresponding χ^2 and $p-$value for the two estimation methods. This last table shows that we can't reject the idea that the data fits a bivariate Katz distribution with a type I error probability of 5% for both methods.

TABLE 3. Goodness-of-fit

Method	χ^2	$p-$value
MME	1.357	0.9684
MLE	2.7496	0.8396

4.2. **Application 2. Data description** In [13], author describes these data as follows. The claims experience of a large automobile portfolio in France including 181038 liability policies was observed during the year 1989. The corresponding yearly claim frequencies, collected in table 2 (first row in each cell), have been divided into material damage only (type 1) and bodily injury (type 2) claims.

And for this data, the elementary statistics are:

$$\overline{x} = 0.051005866 \quad \widehat{\sigma}_X^2 = 0.053884078 \qquad s_{XY} = 0.0001930154$$
$$\overline{y} = 0.005529226 \quad \widehat{\sigma}_Y^2 = 0.005520779$$

Table 4: Comparison of observed and theoretical yearly frequencies of Automobile third party liability insurance [13]

Type 2 / Type 1	0	1	2	Total
0	171345 171356.0 171339.3	918 916.51289192 918.77207313	2 1.812641 1.819196	172265
1	8273 8241.683 8256.275	73 75.78842897 76.14888396	0 0.1740909 0.1751368	8346
2	389 418.4827 418.1020	5 4.44364827 4.45319390	0 0.01047165 0.01050725	394

3	31	1	0	32
	21.62586	0.24097439	0.0005722265	
	21.51784	0.24051046	0.0005718383	
4	1	0	0	1
	1.127173	0.01277756	0.00003042178	
	1.116183	0.01269212	0.00003025611	
Total	180039	997	2	181038

First row: observed frequency
Middle row : theoretical frequency for MME
Last row: theoretical frequency for MLE

Estimation and goodness-of-fit. Table 5 contains the parameter estimates for both methods, and the corresponding theoretical frequencies are presented in Table 4, second and third rows of each cell for MME and MLE, respectively. For the MLE, the log-likelihood corresponding is equal to -42578.77.

TABLE 5. Parameters estimation

Method	Parameters				
	λ_1	λ_2	β_1	β_2	α
MME	0.04828139	0.00553768	0.05341488	-0.00153004	1.73265626
MLE	0.04837225	0.00555236	0.05305245	-0.00153998	1.73420781

The values of the corresponding χ^2 and $p-$values of the two estimation methods are presented in Table 6 after considering seven categories: (0,0), (0,1 and 2), (1,0), (1,1 and 2), (2,0), (2,1 and 2), and (other cases). With a probability of Type I error of 5%, it is apparent from Table 6 that the data fit the bivariate Katz distribution for both estimation methods.

5. CONCLUSION

The product of marginal distributions by a multiplicative factor is an appealing method for constructing multivariate distributions, and the bivariate Katz distribution obtained by this method has some intriguing properties. Among others,

TABLE 6. Goodness-of-fit

Method	χ^2	$p-$value
MME	6.6488	0.3546
MLE	6.6624	0.3532

the correlation coefficient of the obtained model can be either positive, negative, or zero, and the necessary condition of zero correlation is a necessary and sufficient condition for the independence of the random variables X and Y. This property makes this model a suitable alternative to the bivariate Katz model using the trivariate reduction method, whose correlation coefficient can only be strictly positive (cf. [8]).

Also note that the marginal distributions of bivariate Katz distribution are Katz distributions for the technique of the product of marginal distributions by a multiplicative factor, which is still not the case for the trivariate reduction method. For this last method, the marginal distributions of bivariate Katz distribution are distributions of sums of random variables that follow Katz distributions (cf. [8]), and for these marginal distributions to be Katz distributions, it is necessary that the Katz dispersion parameters be equal (cf. theorem 3 in [14]).

REFERENCES

[1] M.A. BARADARAN GHAHFAROKHI, P. BARADARAN GHAHFAROKHI, R. BAHRAMIAN: *Application of Katz family of distributions for detecting and testing overdispersion in Poisson models*, Journal of Statistical Modeling and Analytics, **1**(1) (2010), 29–44.

[2] M.A. BARADARAN GHAHFAROKHI, H. IRAVANI, M. R. SEPEHRI: *Application of Katz family of distributions for detecting and testing overdispersion in Poisson regression models*, World Academy of Science, Engineering and Technology, **42** (2008), 514–519.

[3] J.T. CAMPBELL: *The Poisson correlation function*, Proceedings of the Edinburgh Mathematical Society, **4** (1934), 18–26.

[4] A. HENNINGSEN, O. TOOMET: *maxlik: A package for maximum likelihood estimation in R*, Comput Stat, **26**(3) (2011), 443–458.

[5] P. HOLGATE: *Estimation for the bivariate Poisson distribution*, Biometrika, **51**(1-2) (1964), 241–245.

[6] N.L. JOHNSON, S. KOTZ: *Distributions in statistics: discrete distributions*, John Wiley & Sons, 1970.

[7] L. KATZ: *Unfied of a broad class of discrete probability distributions*, Editor Patil, Michigan State University. Distribution Symposium, **1** (1965), 175–182.

[8] M. KOUKOUATIKISSA DIAFOUKA, R.O. MALOUATA, G.C. LOUZAYADIO, R. BIDOUNGA: *On a bivariate Katz's distribution using the trivaraite reduction method*, submitted.

[9] J. LAKSHMINARAYNA, N.N. PANDIT, K. SRINIVASA RAO: *On a bivariate Poisson distribution*, Comm. Statist. Theory methods, **28**(2) (1999), 267–276.

[10] W.K. NEWEY, D. MCFADDEN: *Chapter 36 Large sample estimation and hypothesis testing*, Handbook of econometrics, **4** (1994), 2111–2245.

[11] S.N.N. PANDIT, C.R. RAO, Y.V.S.S. SARMA: *On a bavariate binomial distribution*, Assam Statistical Review, **2**(2) (1988), 67–72.

[12] H. TEICHER: *On the multivariate Poisson distribution*, Kand. Aktttar. Tidskr., **35** (1954), 1–9.

[13] R. VERNIC: *On the bivariate generalized Poisson distribution*, Astin Bulletin, **27**(1) (1997), 23–31.

[14] A.R. WASHBURN: *Katz distributions, with applications to minefi eld clearance*, Naval Postgraduate School Monterey, CA 93943, (1996), 1–23.

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