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MAXIMUM NORM CONVERGENCE OF NEWTON-MULTIGRID METHODS FOR ELLIPTIC QUASI-VARIATIONAL INEQUALITIES WITH NONLINEAR SOURCE TERMS

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ABSTRACT. In this paper, Newton-multigrid scheme on adaptive finite element discretisation is employed for solving elliptic quasi-variational inequalities with nonlinear source terms. We use Newton's method as the outer iteration for the standard linearization, and using standard multigrid as the inner iteration for the solution of the Jacobian system at each step. The uniform convergence of Newton-multigrid methods is shown in the sense that the multigrid methods have a contraction number with respect to the maximum norm.

1. INTRODUCTION

We apply Newton-multigrid methods for solving obstacle problems based on reformulating the nonlinear quasi-variational inequality (QVI) as a Hamilton-Jacobi-Bellman (HJB)-equation.

For the discretization, the finite element approximation is used to derive a discret system, and then we use an iterative procedure proposed by Hoppe [22] to solve the obtained system. Then we describe how to apply Newton-multigrid

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methods to solve the nonlinear algebraic systems obtained by the HJB reformulation 4.1, we do this by first linearizing the nonlinear system by Newton's method and then apply multigrid methods for the solution of the Jacobian system in each iteration.

For the inner iteration, we describe the maximum norm convergence analysis of the multigrid methods proposed by Arnold for elliptic PDEs [2]. According to Hackbusch [26], the proof of these results is based on the approximation and smoothing properties. For the outer iteration, we will see that Newton's method converges quadratically when the approximation solution is close to the actual solution of the nonlinear system.

The remainder of this paper is briefly summarized as follows. In §2, we state some assumptions and we introduce a continuous problem. In §3 the standard finite element discretizations is applied to derive a system of equations, and in §4 we describe a Newton-multigrid method for the solution of the algebraic systems obtained by the HJB reformulation. In §5 we present a uniforme convergence results of these multigrid methods.

2. Continuous problem

2.1. **Notations and assumptions.** Let Ω be an open in $\mathbb{R}^N$, with sufficiently smooth boundary $\partial\Omega$ for $u, v \in V$ $(V = H_0^1(\Omega))$, $a(u,v)$ be a variational form associated with the continuous non-linear operator A. Then, given a nonlinear right-hand side $f(u)$ such that:

$$f(u) \in L_\infty \cap C^1(\bar{\Omega}), \quad \frac{\partial f}{\partial u} \geq 0 \quad \text{in} \quad \bar{\Omega} \times \{u : \; u \geq 0\}.$$

Moreover, assume that A satisfies the coerciveness assumption

$$\exists v \in V \text{ such that } \|u - v\|^{-1} \langle Au - Av, u - v \rangle \longrightarrow \infty, \quad \|u\| \longrightarrow \infty, \qquad \forall u \in V,$$

and that A is continuous surjective M-function and strictly T-monotone, i.e., $\forall u, v \in V, (u - v)^+ \neq 0$

$$\langle Au - Av, (u - v)^+ \rangle \geq 0.$$

We defined the operator $M : V \cap L^\infty(\Omega) \longrightarrow V \cap L^\infty(\Omega)$ by:

$$
\begin{aligned}
(2.1) \qquad Mu &= k + \inf_{\varepsilon \geq 0, x+\varepsilon \in \overline{\Omega}} u\left(x+\varepsilon\right), \text{ k is a positive constant,} \\
M &\in W^{2,p}\left(\Omega\right), \ Mu \geq 0, \ on\ \partial\Omega : 0 \leq g \leq Mu,
\end{aligned}
$$

where g is a regular function defined on $\partial\Omega$. Let $K_g(u)$ an implicit convex and non empty set given by

$$K_g\left(u\right) = \left\{v \in V, \ v = g \ \ on \ \ \partial\Omega, \ v \leq Mu, \ in\ \Omega\right\}.$$

Consider the following problem: Find $u \in K_g\left(u\right)$ solution of

$$
(2.2) \qquad \begin{cases} a\left(u, v-u\right) \geq \langle f(u), v-u\rangle \qquad v \in K_g(u), \\ \qquad u \leq Mu \qquad Mu \geq 0, \\ \qquad u = g \qquad on \ \ \partial\Omega. \end{cases}
$$

It is well known that under the previous hypothesis the problem (2.2) has a unique solution.

3. Discrete problem

Let a decreasing sequence (mesh size parameter) $\{h_k\}_{k=0}^{l}$ such that

$$h_{k+1} < h_k, 0 \leq k \leq m-1.$$

We present a nested quasi-uniform triangulations family $\{\mathcal{T}_k, k \in N\}$ of

$$\Omega_k = \bigcup_{\mathcal{T} \in \mathcal{T}_k} T.$$

For all $\mathcal{T}_k$ we have

$$
\begin{gathered}
\Omega_k \subset \Omega_{k+1} \subset \Omega. \\
dist\left(\partial\Omega_k, \partial\Omega\right) \leq c_0 h_k^2. \\
h_k h_{k+1} \leq c_1.
\end{gathered}
$$

On each level k, we select a piecewise linear finite element space

$$(3.1) \qquad V_k = \left\{v_k \in C\left(\Omega\right) \cap H^1\left(\Omega\right) \mid v_{k|\Omega_k} \in P_1\right\},$$

and we associate on each h_k an analogous discretization of the problem (2.2) by a finite element method (FEM). To facilitate the notation, put

$$\Omega_k = \Omega_{h_k}, \ V_k = V_{h_k}, \ A_k = A_{h_k}.$$

We define the standar basis functions $\varphi_k^i, i \in (1, \ldots, m(h_k))$ as

$$\varphi_k^i\left(x_k^j\right) = \delta_{ij}.$$

x_k^i denote a vertex of the triangulation $\mathcal{T}_k$. Let $U_k = \mathbb{R}^{m_k}$, the usual finite element restriction operator from U_k into V_k is bijection defined by

$$r_k v(x) = \sum_{i=1}^{m(h_k)} v\left(x_k^i\right)\varphi_k^i(x). \tag{3.2}$$

On U_k we use a scaled euclidean scalar product

$$\langle u, v\rangle_k = h_k^2 \sum_{i=1}^{m_k} u_i v_i, \quad \text{and the associated norm } \|u\|_k = \langle u, u\rangle_k^{1/2},$$

furthermore, the adjoint operator $r_k^* : V_k \longrightarrow U_k$ satisfies

$$\langle r_k u, v\rangle_{L^2} = \langle u, r_k^* v\rangle, \ \forall u \in U_k, \ v \in V_k.$$

The maximum norm $\|.\|_\infty$ (on U_k) and the norm$\|.\|_{L^\infty}$ (on V_k) are equivalent, which are denoted by $\|.\|_\infty$.

Lemma 3.1 ([2]). *Let r_k the restriction operator defined by (3.2), then there exist constants $\mathcal{C}_1$ and $\mathcal{C}_2$ independent of k such that*

$$\begin{aligned} \|r_k(u)\|_{L^\infty} &= \|u\|_\infty, \ \forall u \in U_k, \\ \mathcal{C}_1 \|v\|_{L^\infty} &\leq \|r_k^*(v)\|_\infty \leq \mathcal{C}_2 \|v\|_{L^\infty}, \ \forall v \in V_k. \end{aligned}$$

The numerical approximation of the QVI (2.2) by finite elements leads to the solution of the following discrete QVI in finite dimension. Find $u_k \in K_{g,k}$ such that

$$\begin{cases} \langle A_k u_k, v_k - u_k\rangle \geq \langle f(u_k), v_k - u_k\rangle, & \forall v_k \in K_{g,k}, \\ u_k \leq M_k u_k, \quad v_k \leq M_k u_k, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} f(u_k) &\in L^\infty(\Omega), \\ M_k u_k &= K + \inf_{\epsilon \geq 0, \ (x+\epsilon) \in \bar{\Omega}} u_k(x+\epsilon), \ k \text{ is a positive constant}, \\ K_{g,k} &= \{v \in V_k : v = \pi_k g \text{ on } \partial\Omega, v \leq M_k u_k \text{ in } \Omega\}. \end{aligned}$$

And π_k define the interpolation operator on $\partial\Omega$.

The regularity results described by Ph. Cortey-Dumont in [19] are valid for our problem. Assuming that the hypothesis in [19] for the case of the non-lineair operator are satisfied, then the existence and uniqueness of the solution of the discrete problem (3.3) is well-known. Moreover, we have the following regularity result:

Theorem 3.1 ([19]). *Let u and u_k are solutions of the problems (2.2) and (3.3) respectively, then there exists a constant $\mathcal{C}$ independent of h_k such that:*

$$\|u - u_k\|_{L^\infty(\Omega)} \leq \mathcal{C}h^2 \left|\log h_k\right|^2. \tag{3.4}$$

4. Description of Newton-Multigrid Methods for QVIs

4.1. **The well defined HJB-formulation of the discret problem.** Formally, the QVI (3.3) can be written as the following HJB equation.

Let the unique solution u_k^ν of the discrete HJB equation

$$\max_{1\leq i\leq N}\left(A_{k,i}[u_k^\nu]u_{k,i}^\nu - f[u_k^\nu]_{k,i}\,,\, u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}\right) = 0. \tag{4.1}$$

Choose an initial vector $u_k^0 \in U_k$. Given the iterate $u_k^\nu \in U_k,\ \nu \geq 0,$ we may split the set

$$\mathcal{J}_k = \{1, 2, \ldots, m_k\} \text{ by } \mathcal{J}_k = \bigcup_{p=1}^{3} J_k^p\left(u_k^\nu\right),$$

as

$$\begin{aligned}
\mathcal{J}_k^1\left(u_k^\nu\right) &= \left\{i \in \mathcal{J}_k \mid (A_k[u_k^\nu]u_k^\nu - f_k[u_k^\nu])_i > u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}\right\},\\
\mathcal{J}_k^2\left(u_k^\nu\right) &= \left\{i \in \mathcal{J}_k \mid (A_k[u_k^\nu]u_k^\nu - f_k[u_k^\nu])_i < u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}\right\},\\
\mathcal{J}_k^3\left(u_k^\nu\right) &= \left\{i \in \mathcal{J}_k \mid (A_k[u_k^\nu]u_k^\nu - f_k[u_k^\nu])_i = u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}\right\}.
\end{aligned} \tag{4.2}$$

And compute $u_k^{\nu+1} \in U_k$ as the solution of the nonlinear equation

$$A_k^\nu[u_k^\nu]u_k^{\nu+1} = f_k^\nu[u_k^\nu], \tag{4.3}$$

where

$$A_k^\nu[u_k^\nu] = \begin{cases} A_{k,i}, & \text{if } i \in \mathcal{J}_k^1\left(u_k^\nu\right),\\ I_{k,i}, & \text{if } i \in \mathcal{J}_k^2\left(u_k^\nu\right) \cup \mathcal{J}_k^3\left(u_k^\nu\right). \end{cases} \tag{4.4}$$

$$f_k^{\nu}[u_k^{\nu}] = \begin{cases} f_{k,i}, & \text{if } i \in \mathcal{J}_k^1(u_k^{\nu}), \\ M_k u_{k,i}^{\nu-1}, & \text{if } i \in \mathcal{J}_k^2(u_k^{\nu}) \cup \mathcal{J}_k^3(u_k^{\nu}), \end{cases} \tag{4.5}$$

with $A_{k,i}$ (resp. $f_{k,i}$)is the i^{th} row of $A_k[u_k^{\nu}]$ (resp. the ith component of the right-hand side $f_k[u_k^{\nu}]$) of our discret problem, and $I_{k,i}$ is the i^{th} row of the identity matrix I_k.

We can prove the monotone convergence of the iterates if we assume A_k to be continuously differentiable.

Theorem 4.1 ([22]). *Let (u_k^{ν}) be the iterate obtained by the previous iterative scheme so it satisfies the H.J.B equation above, moreover we suppose that A_k to be continuously differentiable, then the sequence $(u_k^{\nu})_{v\geq 0}$ converges monotonely decreasingly towards the unique solution u_k^* of (3.3).*

4.2. **Newton-Multigrid Algorithm.** Multigrid methods can be used to efficiently solve the nonlinear partial differential equations (PDE). To do this for solving the nonlinear system (4.3), we use Newton-multigrid method, in which a standard linearization is applied, such as in Newton's method. After the linearization of the problem (4.3), the standard multigrid method can be used for solving the Jacobian system in each linearization step.

Let v_k^{ν} an approximation to the exact solution u_k^{ν} of the nonlinear system (4.3), denote by e the error $e_k = u_k^{\nu} - v_k^{\nu}$.

Defining the residual to be $\mathcal{R}_k = f_k^{\nu}[u_k^{\nu}] - A_k^{\nu}[u_k^{\nu}]v_k^{\nu}$. Subtracting the original equation (4.3) from the residual, we obtain

$$A_k^{\nu}[u_k^{\nu}]u_k^{\nu} - A_k^{\nu}[u_k^{\nu}]v_k^{\nu} = \mathcal{R}_k. \tag{4.6}$$

Since $A_k^{\nu}[u_k^{\nu}]$ is nonlinear, $A_k^{\nu}[u_k^{\nu}](e_k) \neq \mathcal{R}_k$, this means that for the nonlinear problem, we can not determine the error by solving a linear equation on the coarse grid, as in standard multigrid. However, we must use (4.6) as the residual equation.

By applying Newton's method to the system (4.3), We can use (4.6) as a basis for the multigrid solver: for simplicity, we choose to use $\mathcal{F}_h$ as a nonlinear operator, and we define the system of equations (4.3) on fine grid as:

$$\mathcal{F}_k(u_k^{\nu}) = A_k^{\nu}[u_k^{\nu}]u_k^{\nu} - f_k^{\nu}[u_k^{\nu}] = 0. \tag{4.7}$$

The residual on the fine grid is rewritten as:

$$\mathcal{R}_k(u_k^\nu) = -\mathcal{F}_k(u_k^\nu).$$

Then Newton iteration for solving (4.7), is described as

$$u_k^{\nu+1} = u_k^\nu + (\mathbb{J}_k(u_k^\nu))^{-1}\mathcal{R}_k(u_k^\nu), \quad k = 0, 1, 2, 3, \ldots.$$

Here

$$\mathbb{J}_k(u_k^\nu)) = \mathcal{F}_k'(u_k^\nu),$$

is the Jacobian matrix of the nonlinear system.

The multigrid approache for the nonlinear system of equations (4.7) can be obtained by solving the Jacobian linear system for e_k^ν as:

$$\mathbb{J}_k(u_k^\nu)e_k^\nu = \mathcal{R}_k(u_k^\nu). \tag{4.8}$$

In the linear multigrid, we choose an iterate $e_k^\nu, \nu > 0$, we get $\bar{e}_k^\nu$ by α applications of an iterative method for the solution of the system (4.8), denoted by

$$\bar{e}_k^\nu = \mathcal{S}_k^\alpha\left(e_k^\nu\right). \tag{4.9}$$

$\mathcal{S}$ is the iteration matrix of smoothing method, and α is the number of iterations performed.

Denote by e_k^* the solution of (4.8). Setting the error $\mathcal{E}_k^\nu = \bar{e}_k^\nu - e_k^*$, and the residual $d_k^{(\nu)} = \mathcal{R}_k(u_k^\nu) - \mathbb{J}_k(u_k^\nu)\bar{e}_k^\nu$, we can write the equation (4.8) as

$$\mathbb{J}_k(u_k^\nu)\left(\bar{e}_k^\nu + \mathcal{E}_k^\nu\right) = \mathcal{R}_k(u_k^\nu).$$

Which results in the residual equation

$$\mathbb{J}_k(u_k^\nu)\mathcal{E}_k^\nu = \mathcal{R}_k(u_k^\nu) - \mathbb{J}_k(u_k^\nu)\bar{e}_k^\nu = d_k^{(\nu)}.$$

So to determine $\mathcal{E}_k^\nu$ completely, we need to calculate $\mathcal{E}_{k-1}^\nu$ at level $(k-1)$ as the solution of the coarse grid system

$$\mathbb{J}_{k-1}(e_{k-1}^\nu)\mathcal{E}_{k-1}^\nu = d_{k-1}^{(\nu)}. \tag{4.10}$$

Here $\mathcal{E}_{k-1}^\nu$ $\left(\text{resp } \mathbb{J}_{k-1}^\nu(e_{k-1}^\nu), d_{k-1}^{(\nu)}\right)$ define an approximation at the level $k-1$ of $\mathcal{E}_k^\nu$ $\left(\text{resp } \mathbb{J}_k^\nu(e_k^\nu), d_k^{(\nu)}\right)$:

$$\mathcal{E}_{k-1}^\nu = R_k\mathcal{E}_k^\nu, \qquad \mathbb{J}_{k-1}^\nu(e_{k-1}^\nu) = R_k\mathbb{J}_k^\nu(e_k^\nu)P_k, \qquad d_{k-1}^{(\nu)} = R_k d_k^{(\nu)}.$$

Consequently, we determine an improved iterate for (4.8) at the level k by

$$e_k^{\nu+1} = \bar{e}_k^{\nu} + P_k\left(\mathcal{E}_{k-1}^{\nu}\right). \tag{4.11}$$

We use the identity operator

$$\begin{aligned} \Pi &: V_{k-1} \longrightarrow V_k \\ \Pi v &= v, \end{aligned}$$

to define the restriction and the prolongation operators, i.e.,

$$P_k = r_k^{-1} r_{k-1}, \qquad R_k = P_k^t. \tag{4.12}$$

Remark 4.1. *The previous algorithm describes one cycle of a multigrid method (as the inner iteration) to solve (4.8) for two hierarchy of grids Ω_k and Ω_{k-1}, and one iteration of Newton's method (as the outer iteration) to solve (4.7). Thus we can solve the system (4.10) approximately by applying the two-grid iteration recursively to all hierarchy of grids $\{\Omega_k,\ k = 0, \ldots, m_k\}$ and we terminate the iteration process of the Newton-multigrid when the iteration error is small.*

The Newton-multigrid iteration may be described as the following algorithms 1and 2.

Algorithm 1 Newton-Muligrid methods

Choose an initial guess u_k^0 and a desired tolerance η.
while $(\mathcal{R}_k < \eta)$ **do**
  compute the Jacobian matrix $\mathbb{J}_k$ and the residual vector $\mathcal{R}_k$
  Solve the linear system by a Muligrid method $e_k^{\nu} \leftarrow MGM(\mathbb{J}_k, \mathcal{R}_k, e_k^{\nu})$
  Set $u_k^{\nu} \leftarrow u_k^{\nu} + e_k^{\nu}$;
  Set $\mathcal{R}_k \leftarrow f_k^{\nu}[u_k^{\nu}] - A_k^{\nu}[u_k^{\nu}]u_k^{\nu}$;
end while

5. Multigrid Convergence on L_∞-Norm

In this paragraph, the analysis of the uniform convergence for the multigrid algorithm (the inner iteration) is described as well as the convergence of Newton's method (the outer iteration) under assumptions which are similar to those that have been imposed on the multigrid methods for the solution of nonlinear equations.

Algorithm 2 Muligrid methods

$MGM(\mathbb{J}_k, \mathcal{R}_k, e_k, \alpha_1, \alpha_2, \mu)$
$e_k \leftarrow smoother(\mathbb{J}_k, \mathcal{R}_k, e_k, \alpha_1);$ % (presmoothing)
$d_k \leftarrow \mathcal{R}_k - \mathbb{J}_k e_k;$ % Computat the residual
$R_k; P_k;$ % Define the prolongation and the restriction
$\mathbb{J}_{k-1} \leftarrow R_k \mathbb{J}_k P_k;$ % Restrict $\mathbb{J}_k$
$d_{k-1} \leftarrow R_k d_k;$ % Restrict d_k
$\mathcal{E}_{k-1} \leftarrow d_{k-1} \cdot 0;$ % Define a start value
if $size(\mathbb{J}_{k-1} \leq \mu)$ % Coarsest grid Ω_μ **then**
  $\mathcal{E}_{k-1} \leftarrow \mathbb{J}_{k-1}^{-1} d_{k-1};$ % The direct solve on the coarse grid
else
  $\mathcal{E}_{k-1} \leftarrow MGM(\mathbb{J}_{k-1}, d_{k-1}, \mathcal{E}_{k-1});$ % Solve the coarse problem
end if
$\mathcal{E}_k \leftarrow P\mathcal{E}_{k-1};$ % Prolongat $\mathcal{E}_{k-1}$
$e_k \leftarrow e_k + \mathcal{E}_k;$ % Add correction
$e_k \leftarrow smoother(\mathbb{J}_k, \mathcal{R}_k, e_k, \alpha_2);$ % (Postsmoothing)
return e_k

We now present the main hypotheses:

(1) There exist $u_k^* \in K_{g,k}$ such that $\mathcal{F}_k(u_k^*) = 0$.
(2) For any u_k in the neighborhood of u_k^* there exist a linear mapping $\mathcal{F}_k'(u_k)$ such that: for any small $\varepsilon > 0$ there exist an $\eta > 0$ such that

$$\|\mathcal{F}_k(u_k) - \mathcal{F}_k(u_k^*) - \mathcal{F}_k'(u_k^*)(u_k - u_k^*)\| \leq \varepsilon \|u_k - u_k^*\|,$$

whenever $\|u_k - u_k^*\| < \eta$.
(3) The derivative $\mathcal{F}_k'(u_k)$ is invertible and $(\mathcal{F}_k'(u_k))^{-1}$ is a bounded linear operator, for any u_k in the neighborhood of u_k^*, that is,

$$\|(\mathcal{F}_k'(u_k))^{-1}\| \leq \kappa,$$

with a constant κ. In addition, we assume that the mapping $(\mathcal{F}_k'(u_k))^{-1}$ is continuous in u_k. That is, for any $\varepsilon > 0$ there exist an $\eta > 0$ for which

$$\|I - \mathcal{F}_k'(u_k^*)(\mathcal{F}_k'(u_k))^{-1}\| \leq \varepsilon,$$

and

$$\|I - (\mathcal{F}_k'(u_k^*))^{-1}\mathcal{F}_k'(u_k)\| \leq \varepsilon,$$

hold, whenever $\|u_k - u_k^*\| < \eta$.

5.1. **The iteration matrix of the multigrid algorithm.** The iteration matrix of the two-grid method with α_1 presmoothing and α_2 postsmoothing iterations on level k is given by

$$TG_k(\alpha_1, \alpha_2) = \mathcal{S}_k^{\alpha_2}\left(\mathbb{J}_k^{-1} - P_k \mathbb{J}_{k-1}^{-1} R_k\right)\mathbb{J}_k \mathcal{S}_k^{\alpha_1}. \tag{5.1}$$

One can easily prove that the multigrid method is linear.

Theorem 5.1 ([3]). *The multigrid method is a linear iterative method with the iteration matrix MG_k given by*

$$\begin{aligned}
MG_0 &= 0, && \text{(5.2a)}\\
MG_k &= \mathcal{S}_k^{\alpha_2}\left(I_k - P_k\left(I_k - MG_{k-1}\right)\mathbb{J}_{k-1}^{-1}R_k\right)\mathbb{J}_k\mathcal{S}_k^{\alpha_1}, && \text{(5.2b)}\\
&= TG_k + \mathcal{S}_k^{\alpha_2}P_k(MG_{k-1})\mathbb{J}_{k-1}^{-1}R_k\mathbb{J}_k\mathcal{S}_k^{\alpha_1},\ k = 1, 2, \ldots. && \text{(5.2c)}
\end{aligned}$$

5.2. **Approximation property.** The proof of the approximation property is based on Theorem (3.1) and Lemma (3.1).

Theorem 5.2 ([15]). *Under the previous assumptions, the matrix*

$$\chi = \left[\mathbb{J}_k^{-1} - P_k\mathbb{J}_{k-1}^{-1}R_k\right],$$

satisfies the following approximation propertie:

$$\|\chi\|_\infty \le Ch_k^2\left|\log h_k\right|^2. \tag{5.3}$$

Proof. According to Theorem (3.1) we have

$$\|u - u_k\|_{L^\infty} \le Ch_k^2\left|\log h_k\right|^2\|\nabla(f)\|_{L^\infty}, \qquad u \in W^{2,p}.$$

Then

$$\|u - u_k^*\|_{L^\infty} \le Ch_k^2\left|\log h_k\right|^2\|\nabla(f)\|_{L^\infty}.$$

So we get

$$\begin{aligned}
\left\|u_k^* - u_{k-1}^*\right\|_{L^\infty} &\le \|u_k^* - u\|_{L^\infty} + \left\|u_{k-1}^* - u\right\|_{L^\infty}\\
&\le Ch_k^2\left|\log h_k\right|^2\|\nabla(f)\|_{L^\infty} + Ch_{k-1}^2\left|\log h_k\right|^2\|\nabla(f)\|_{L^\infty}\\
&\le Ch_k^2\left|\log h_k\right|^2\|\nabla(f)\|_{L^\infty}.
\end{aligned} \tag{5.4}$$

The Galerkin discretization results in

$$a\left(r_k u, r_k v\right)=\left\langle A_k u, v\right\rangle_{L^2}, \quad \forall u, v \in U_k.$$

Then

$$\left(a\left(r_k u, r_k v\right)\right)^{\prime}=\left\langle(\mathbb{J}_k) u, v\right\rangle_{L^2}, \quad \forall u, v \in U_k.$$

Also we have

$$\left(a\left(r_k^{-1}(A_k)^{-1}(f_k), v\right)\right)^{\prime}=\left\langle\left(r_k^*\right)^{-1} \nabla(f), v\right\rangle_{L^2}, \ \forall v \in K_{g,k}.$$

Let $u_k \in K_{g,k}$ and $u_{k-1} \in K_{g,k-1}$, be such that

$$a\left(u_k, v\right)=\left\langle\left(r_k^*\right)^{-1} f, v\right\rangle_{L^2},$$
$$a\left(u_{k-1}, v\right)=\left\langle\left(r_{k-1}^*\right)^{-1} f, v\right\rangle_{L^2},$$

it follow that $u_k^*=r_k^{-1}\mathbb{J}_k^{-1}\nabla(f)$ and $u_{k-1}^*=r_{k-1}^{-1}\mathbb{J}_{k-1}^{-1}R_k\nabla(f)$.

Using (5.4) and Lemma (3.1) we get

$$\left\|r_k^{-1}\mathbb{J}_k^{-1}\nabla(f)-r_{k-1}^{-1}\mathbb{J}_{k-1}^{-1}R_k\nabla(f)\right\|_{\infty} \leq C h_k^2\left|\log h_k\right|^2\|\nabla(f)\|_{L^\infty}.$$

Then

$$\left\|\mathbb{J}_k^{-1}-r_k^{-1}r_{k-1}\mathbb{J}_{k-1}^{-1}R_k\right\|_{\infty} \leq C h_k^2\left|\log h_k\right|^2.$$

This completes the proof

$$\left\|\mathbb{J}_k^{-1}-P_k\left(\mathbb{J}_{k-1}\right)^{-1}R_k\right\|_{\infty} \leq C h_k^2\left|\log h_k\right|^2.$$

□

5.3. **Smoothing prorperty.** To demonstrate a smoothing prorperty, we decompose $\mathbb{J}_k=E_k-N_k$, and use the following assumptions:

(5.5) $\qquad E_k$ is regular and $\left\|E_k^{-1}N_k\right\|_{\infty} \leq 1,$ for all k.

(5.6) $\qquad \|E_k\|_{\infty} \leq C h_k^{-2},$ for all k, with C independent of k,

For the pre&post-smoothing, we use a relaxation method with iteration matrix

$$\mathcal{S}_k=I_k-\omega E_k^{-1}N_k, \qquad \omega \in(0,1).$$

Theorem 5.3 ([2]). *Assume that the previous assumptions and notations are satisfied, then there exist a constant $\mathcal{C}$ independont of k and α such that*

$$\|(\mathbb{J}_k)\,\mathcal{S}_k^{\alpha}\|_\infty \le \mathcal{C}\frac{1}{\sqrt{\alpha}}h_k^{-2}, \tag{5.7}$$

holds.

5.4. **Convergence result of multigrid methods.** Besides the approximation and smoothing property, we also need to get the following stability bound:

$$\exists C_s : \|\mathcal{S}_k^{\alpha}\|_\infty \le C_s, \text{ for all } k \text{ and } \alpha. \tag{5.8}$$

The convergence analysis is based on the following splitting of the two-grid iteration matrix, with $\alpha_2 = 0$:

$$\begin{aligned}\|TG_k(\alpha_1, 0)\|_\infty &= \left\|\left((\mathbb{J}_k)^{-1} - P_k(\mathbb{J}_{k-1})^{-1} R_k\right)(\mathbb{J}_k)\,\mathcal{S}_k^{\alpha_1}\right\|_\infty, \\ &\le \left\|(\mathbb{J}_k)^{-1} - P_k(\mathbb{J}_{k-1})^{-1} R_k\right\|_\infty \|(\mathbb{J}_k)\,\mathcal{S}_k^{\alpha_1}\|_\infty.\end{aligned}$$

Theorem 5.4. *Under the previous assumptions, there exist a constant C independent of k and α, such that the iterate $u_k^{\nu}, \nu \ge 0$ for two grids k and $k-1$ satisfies:*

$$\left\|u_k^{\nu+1} - u_k^*\right\|_\infty \le \left(\frac{C}{\sqrt{\alpha}}\left|\log h_k\right|^2\right)\|u_k^{\nu} - u_k^*\|_\infty. \tag{5.9}$$

Proof. We have:

$$\begin{aligned}\left\|u_k^{\nu+1} - u_k^*\right\|_\infty &= \left\|\left(\left(I_k - P_k\left(I_k - MG_{k-1}\right)(\mathbb{J}_{k-1})^{-1} R_k\right)(\mathbb{J}_k)\,\mathcal{S}_k^{\alpha_1}\right)(u_k^{\nu} - u_k^*)\right\|_\infty \\ &\le \left\|I_k - P_k\left(I_k - MG_{k-1}\right)(\mathbb{J}_{k-1})^{-1} R_k\right\|_\infty \|(\mathbb{J}_k)\,\mathcal{S}_k^{\alpha_1}\|_\infty \|(u_k^{\nu} - u_k^*)\|_\infty \\ &\le \left(C_2\frac{1}{\sqrt{\alpha}}h_k^{-2}\right)\left(C_1 h_k^2\left|\log h_k\right|^2\right)\|u_k^{\nu} - u_k^*\|_\infty \\ &\le \frac{C_1 C_2}{\sqrt{\alpha}}\left|\log h_k\right|^2 \|u_k^{\nu} - u_k^*\|_\infty.\end{aligned}$$

□

Usually, we will choose a hierarchy of more than two grids. The iteration matrix (5.2) in this case can be defined by using the iteration matrix (5.1) for all levels k recursively.

Theorem 5.5 ([3]). *Consider the multigrid method with iteration matrix given in (5.2). Then under the previous assumptions and for parameter values $\alpha_2 = 0$, $\alpha_1 = \alpha > 0$, $\tau = 2$.*

For any $\zeta \in (0,1)$ there exists an α^* such that for all $\alpha \geq \alpha^*$

$$\|MG_k\|_\infty \leq \zeta, \qquad k = 0, 1, \dots \tag{5.10}$$

holds.

Proof. Combining by the smoothing property, the approximation property and the stability bound (5.8), then the same arguments as in [3, Theorem 7.20] can be applied. □

For the outer iteration, when we apply Newton's method one can see that it converges quadratically [5] once the approximation solution is close to the current solution of the nonlinear system.

$$\|u_k^{\nu+1} - u_k^*\| \leq C\|u_k^\nu - u_k^*\|^2. \tag{5.11}$$

Conclusion 5.6. *In this work, we have applied Newton-Multigrid methods for the nonlinear quasi-variational inequality related to HJB equation. we have proposed three numerical methods. For the discretization, the finite element method is used to construct the discrete system, and the two approaches of Newton-Multigrid methods: Newton's method as the outer iteration for the global linearization, and a standard multigrid methods for solving the Jacobian system. The uniform convergence of this Non-linear multigrid has been demonstrated successfully.*

It is well known that if the initial guess is accurate, the iteration of any nonlinear method is much faster. An interesting future case is to apply nested iteration, which produces a good initial guess by first solving the problem on a coarser grid, and doing this on all levels, it is the so-called FMG approach.

REFERENCES

[1] A .BRANDT: *Algebraic multigrid theory: The symmetric case*, Applied mathematics and computation, **19**(1-4) (1986), 23–56.

[2] A. REUSKEN: *On maximum norm convergence of multigrid methods for elliptic boundary value problems*, SIAM journal on numerical analysis, **31**(2) (1994), 378–392.

[3] A. REUSKEN: *Introduction to multigrid methods for elliptic boundary value problems*, Inst. f¨ur Geometrie und Praktische Mathematik, 2008.

[4] B. HANOUZET, J.L. JOLY: *Convergence uniforme des iteres definissant la solution d'une inequation quasi variationnelle abstraite*, Journées équations aux dérivées partielles, 1978.

[5] C. REMANI: *Numerical methods for solving systems of nonlinear equations*, Lakehead University Thunder Bay, Ontario, Canada, 2013.

[6] L.C. EVANS: *Classical solutions of the hamilton-jacobi-bellman equation for uniformly elliptic operators*, Transactions of the American Mathematical Society, **275**(1) (1983), 245–255.

[7] M.A.B. LE HOCINE, M. HAIOUR: *L^∞-error analysis for parabolic quasi-variational inequalities related to impulse control problems*, Computational Mathematics and Modeling, **28**(1) (2017), 89–108.

[8] M.A.B. LE HOCINE, S. BOULAARAS, M. HAIOUR: *On finite element approximation of system of parabolic quasi-variational inequalities related to stochastic control problems*, Cogent Mathematics, **3**(1) (2016), art.id. 1251386.

[9] M. BEGGAS, M. HAIOUR: *The maximum norm analysis of schwarz method for elliptic quasi-variational inequalities*, Kragujevac Journal of Mathematics, **45**(4) (2021), 635–645.

[10] M. BOULBRACHENEM, B.CHENTOUF: *The finite element approximation of Hamilton–Jacobi–bellman equations: the noncoercive case*, Applied Mathematics and Computation, **158**(2) (2004), 585–592.

[11] M. BOULBRACHENE, M. HAIOUR: *The finite element approximation of hamiltonjacobibellman equations*, Computers & Mathematics with Applications, **41**(7- 8) (2001), 993–1007.

[12] M. BOULBRACHENE, M. HAIOUR, S. SAADI: *L^∞-error estimates for a system of quasivariational inequalities*, 2000.

[13] M. BOULBRACHENE, M.HAIOUR, S. SAADI: *L^∞-error estimate for a system of elliptic quasivariational inequalities*, International Journal of Mathematics and Mathematical Sciences, **2003**(24) (2003), :1547–1561.

[14] M.G. LARSON, F. BENGZON: *The finite element method: theory, implementation, and practice*, Texts in Computational Science and Engineering, **10** (2010), 23-44.

[15] M. HAIOUR: *Etude de la convergence uniforme de la méthode multigrilles appliquées aux problèmes frontières libres*, PhD thesis, Université de Annaba-Badji Mokhtar.

[16] M. HAIOUR, S. BOULAARAS: *Uniform convergence of schwarz method for elliptic quasi-variational inequalities related to impulse control problem*, An-Najah University Journal for Research-A (Natural Sciences), 2010.

[17] P-G. CIARLET, P-A. RAVIART: *Maximum principle and uniform convergence for the finite element method*, Computer methods in applied mechanics and engineering, **2**(1) (1973), 17–31.

[18] P. CORTEY-DUMONT: *Approximation numérique d'une inéquation quasi variationnelle liée à des problèmes de gestion de stock. RAIRO*, Analyse numérique, **14**(4) (1980), 335–346.

[19] P. CORTEY-DUMONT: *On finite element approximation in the L^∞-norm of variational inequalities*, Numerische Mathematik, **47**(1) (1985), 45–57.

[20] P.N. BROWN, P.S. VASSILEVSKI, C.S. WOODWARD: *On Mesh-Independent Convergence of an Inexact Newton–Multigrid Algorithm*, SIAM Journal on Scientific Computing, **25**(2) (2003), 570-590.

[21] R.H.W. HOPPE: *Multi-grid methods for hamilton-jacobi-bellman equations*, Numerische Mathematik, **49**(2) (1986), 239–254.

[22] R.H.W. HOPPE: *Multigrid algorithms for variational inequalities*, SIAM journal on numerical analysis, **24**(5) (1987), 1046–1065.

[23] R.H.W. HOPPE: *Une méthode multigrille pour la solution des problèmes d'obstacle*, ESAIM: Mathematical Modelling and Numerical Analysis, **24**(6) (1990), 711–735.

[24] S. BOULAARAS, M.A.B. LE HOCINE, M. HAIOUR: *The finite element approximation in a system of parabolic quasi-variational inequalities related to management of energy production with mixed boundary condition*, Computational Mathematics and Modeling, **25**(4) (2014), 530–543.

[25] S. BOULAARAS, M. HAIOUR: *The finite element approximation in parabolic quasivariational inequalities related to impulse control problem with mixed boundary conditions*, Journal of Taibah University for Science, **7**(3) (2013), 105–113.

[26] W. HACKBUSCH: *Multi-grid methods and applications*, Berlin and New York, Springer-Verlag, Springer Series in Computational Mathematics., **4**, 1985.

[27] W. HACKBUSCH, H-D. MITTELMANN: *On multi-grid methods for variational inequalities*, Numerische Mathematik, **42**(1) (1983), 65–76.

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