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FARADAY AND ZILCH TENSORS IN MINKOWSKI GEOMETRY

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ABSTRACT. We show that, in Minkowski spacetime, any Faraday tensor verifying the Maxwell equations in vacuum allows construct a tensor of 4th order with the same algebraic symmetries than the Riemann tensor, which implies the presence of a third-order tensor with the algebraic and differential properties of the Lanczos generator, and finally this indicates the existence of the important zilch electromagnetic tensor.

Here we employ any Faraday tensor and its dual verifying the Maxwell equations in vacuum [1, 2]:

$$F^{ab}{}_{,b} = {}^*F^{ab}{}_{,b} = 0, \qquad \Box F^{ab} = \Box\, {}^*F^{ab} = 0, \tag{1}$$

in the Riemann-Lorenz gauge [3, 7], to construct a tensor with the algebraic symmetries of the Riemann curvature tensor [8, 9]. In fact, the following tensorial

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field:

$$
\begin{aligned}
A_{\mu\nu\alpha\beta} &\equiv F_{\mu\nu}F_{\alpha\beta} +^{*} F^{*}_{\mu\nu}F_{\alpha\beta}, \\
&= (g_{\mu\alpha}F_{\beta\lambda} - g_{\mu\beta}F_{\alpha\lambda})F^{\lambda}_{\nu} - (g_{\nu\alpha}F_{\beta\lambda} - g_{\nu\beta}F_{\mu\lambda})F^{\lambda}_{\nu} \\
&\quad + \frac{1}{2}(g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta})F_{\lambda\tau}F^{\lambda\tau},
\end{aligned} \tag{2}
$$

satisfies the properties:

$$
\begin{gathered}
A_{\mu\nu\alpha\beta} = -A_{\nu\mu\alpha\beta} = -A_{\mu\nu\beta\alpha} = A_{\alpha\beta\mu\nu}, \qquad A_{\mu\nu\alpha\beta} + A_{\mu\alpha\beta\nu} + A_{\mu\beta\nu\alpha} = 0, \\
A_{\mu}{}^{\nu}{}_{\alpha\nu} = 2F_{\mu}{}^{\nu}F_{\alpha\nu} - \frac{1}{2}g_{\mu\alpha}F_{\lambda\tau}F^{\lambda\tau}, \qquad A^{\mu\nu}_{\mu\nu} = 0.
\end{gathered} \tag{3}
$$

Then it is natural to obtain the corresponding simple dual:

$$
B_{\mu\nu\alpha\beta} \equiv -^{*}A_{\mu\nu\alpha\beta} = F_{\mu\nu}{}^{*}F_{\alpha\beta} - F_{\alpha\beta}{}^{*}F_{\mu\nu}, \tag{4}
$$

with the symmetries:

$$
B_{\mu\nu\alpha\beta} = -B_{\nu\mu\alpha\beta} = -B_{\alpha\beta\nu\mu} - B_{\alpha\beta\mu\nu}, \qquad B_{\mu}{}^{\nu}{}_{\alpha\nu} = 0, \tag{5}
$$

which allows introduce the following interesting third-order tensor:

$$
K_{\mu\nu\alpha} \equiv B_{\mu\nu\alpha}{}^{\beta}{}_{,\beta} = F_{\mu\nu,\beta}{}^{*}F_{\alpha}{}^{\beta} - F_{\alpha}{}^{\beta *}F_{\mu\nu,\beta}, \tag{6}
$$

verifying the same algebraic and differential symmetries as the Lanczos potential [10–14]:

$$
K_{\mu\nu\alpha} = -K_{\nu\mu\alpha}, \quad K^{\alpha}{}_{\nu\alpha} = 0, \quad K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0, \quad K^{\mu\nu\alpha}{}_{,\alpha} = 0. \tag{7}
$$

Thus we say that (6) is a Lanczos type spintensor in Minkowski spacetime [15]; in this case, we have the additional property $K^{\mu}{}_{\nu\alpha,\mu} = 0$. Therefore, (5) is a superpotential for the Lanczos type tensorial field (6), that is, $K_{\mu\nu\alpha}$ is an exact divergence. The concept of Minkowskian spintensor, that is, verifying (7), is not trivial because it was important to elucidate the physical meaning of Weert's potential [16, 17] for the bounded part of the Liénard-Wiechert electromagnetic field [1, 18], and to justify the López's splitting [19, 20] in angular momentum of the Maxwell field generated by a classical charged particle in arbitrary motion.

Now we apply the Maxwell equations in the expression (6) to obtain the splitting:

$$K_{\mu\nu\alpha} = F_{\beta\nu,\mu}{}^{*}F_{\alpha}{}^{\beta} - F_{\alpha}{}^{\beta *}F_{\beta\nu,\mu} - (F_{\beta\mu,\nu}{}^{*}F_{\alpha}{}^{\beta} - F_{\alpha}{}^{\beta *}F_{\beta\mu,\nu}) = Z_{\alpha\nu\mu} - Z_{\alpha\mu\nu}. \tag{8}$$

with the presence of the important zilch electromagnetic tensor [21–27]:

$$Z_{\mu\nu\alpha} = F_{\beta\nu,\alpha}{}^{*}F_{\mu}{}^{\beta} - F_{\mu}{}^{\beta *}F_{\beta\nu,\alpha}, \tag{9}$$

satisfying the properties:

$$Z_{\mu\nu\alpha} = Z_{\nu\mu\alpha}, \qquad Z^{\nu}_{\nu\alpha} = Z^{\nu}_{\alpha\nu} = 0, \qquad Z^{\mu\nu\alpha}{}_{,}\nu = Z^{\mu\alpha\nu}{}_{,}\nu = 0. \tag{10}$$

The relationship (8) of the zilch tensor with the Lanczos type potential gives support to a possible connection between $Z_{\mu\nu\alpha}$ and the helicity of the electromagnetic field in vacuum [23] because $K_{\mu\nu\alpha}$ represents some type of intrinsic rotation.

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