ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **11** (2022), no.11, 1019–1031
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.11.3

A COMMON FIXED POINT THEOREM IN MENGER SPACE WITH WEAKLY COMPATIBLE MAPPINGS OF TYPE (*P*)

Ajay Kumar Chaudhary[1], Kanhaiya Jha, K.B. Manandhar, and H.K. Pathak

ABSTRACT. We define weakly compatible mappings of type (P) in Menger space and establish the common fixed point theorem for four self-mappings in this space with an appropriate example. Our result generalizes and extends a number of similar results in the literature.

1. INTRODUCTION

One of the most significant generalizations of metric space was introduced by Karl Menger in 1942 called statistical metric space [10], often known as probabilistic metric space after 1964. The concept of a probabilistic metric space applies to circumstances in which we do not precisely know the distance between two points but only the probabilities of different values for this distance. In note [10], Menger outlined how to replace the numerical distance between two points x and y by a distribution function $F_(x,y)$ whose value $F_(x,y)(t)$ at a real number t is interpreted as the probability that the distance between $x$and y is less than t. Due to B. Schweizer and A. Skalar [16], [13] in 1960, the study of this domain was significantly broadened. This space becomes very active when V.M. Sehgal and A.T.

[1]*corresponding author*
2020 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Menger Space, Common fixed point, Compatible mappings, Compatible mappings of type (P).
Submitted: 17.09.2022; *Accepted:* 03.10.2022; *Published:* 03.11.2022.

Barucha Reid [14] 1972, obtained a contraction mapping in Menger probabilistic metric space as a generalization of S. Banach's [1] well-known Banach contraction principle in metric space and developed fixed point theorems. In the study of Menger space, S. N. Mishra [11] 1991 developed compatible mapping in the probabilistic metric space, and then many researchers worked on a large number of compatible mappings. Recently, in 2021, A.K. Chaudhary, K. Jha, K.B. Manandhar, and P.P. Murthy [6] introduced a new notion of compatible mapping of type (P) in Menger space and established a common fixed point theorem by using compatible mapping of type (P) in Menger space which is earlier introduced in metric space by H.K. Pathak, Y.J. Cho, S.S. Chang and S.M. Kang [12] in 1996. And continuing this space study on weakly compatible by [8], [15], [17], and [18]. The purpose of this paper is to define a new notion of weakly compatible mapping of type (P) in Menger space and establish a common fixed point theorem with suitable examples to justify the main result which also generalizes a number of well-established findings in the literature.

2. Preliminaries

Definition 2.1. *[16] If a function $F : \mathbb{R} \to \mathbb{R}^+$ is*

(i) *is non-decreasing,*
(ii) *is left continuous, and*
(iii) $inf_{x\in\mathbb{R}}F(x) = 0$ *and* $sup_{x\in\mathbb{R}}F(x) = 1$.

*Then, it is said to be **distribution function**.*

Example 1. *Let $H(x)$ stands for the heavy side function, which is defined as:*

$$H(x) = \begin{cases} 0, & \textit{if } x\leq 0, \\ 1, & \textit{if } x > 0. \end{cases}$$

Definition 2.2. *[16] Let $F : K \times K \to L$ (the set of all distribution functions) be a distribution function and K be a non-empty set. Then, a pair (K, F) is said to be **Probabilistic Metric Space** (briefly, PM-space) if the distribution function$F(p, q)$, where $(p, q) \in K \times K$, also denoted by$F(p, q)$ or by $F_{p,q}$ satisfies following conditions:*

(i) $F_{p,q}(x) = 1$, *for every $x > 0$ if and only if $p = q$,*
(ii) $F_{p,q}(0) = 0$; *for every $p, q \in K$,*

(iii) $F_{p,q}(x) = F_{q,p}(x)$ *, for every* $p, q \in K$*, and*

(iv) *For every* $p, q, r \in K$ *and for every*

$$x,\ y > 0,\ F_{p,r}(x) = 1,\ F_{r,q}(y) = 1 \ \Rightarrow F_{p,q}(x + y) = 1.$$

Here, $F_(p, q)(x)$ *represents the value of* $F_(p, q)$ *at* $x \in \mathbb{R}$*.*

Definition 2.3. *[5] A function* $T : [0, 1] \times [0, 1] \to [0, 1]$ *is referred to as* ***Triangular norm*** *(shortly T-norm) if it satisfies the following conditions:*

(i) $T(0, 0) = 0$ *and* $T(a, 1) = a$ *for every* $a \in [0, 1]$*,*

(ii) $T(a, b) = T(b, a)$ *for every* $a, b \in [0, 1]$*,*

(iii) $T(a, b) \leq T(c, d)$ *whenever* $a \leq$ c *and* $b \leq d$*, and*

(iv) $T(a, T(b, c)) = T(T(a, b), c))$ *, for every* $a, b, c \in [0, 1])$.

Definition 2.4. *[13]* ***Menger Space****, also known as Menger Probabilistic Metric Space, is a triplet* (K, F, T)*, where* (K, F) *is a PM space,* T *is a* $T-$ *norm and also satisfying following conditions:*

(v) $F_{p,q}(x + y) \geq T(F_{p,r}(x),\ F_{r,q}(y))$*, for all* $p, q, r \in K$ *and* $x, y \in \mathbb{R} > 0$.

Definition 2.5. *[2] A mapping* $Q : K \to K$ *in Menger space* (K, F, t) *is said to be* ***Continuous*** *at a point* $p \in K$ *if for every* $\epsilon > 0$ *and* $\lambda > 0$*,there exists* $\epsilon_1 > 0$ *and* $\lambda_1 > 0$ *such that if* $F_{p,q}(\epsilon_1) > 1 - \lambda_1$*, then* $F_{Qp,Qq}(\epsilon) > 1 - \lambda$*.*

Definition 2.6. *[2] Let* (K, F, T) *be a* ***Menger Space*** *and* t *be a continuous T-norm. Then,*

(a) *A sequence* $\{k_n\}$ *in* K *is said to be* ***converge*** *to a point* k *in* K *(written* $k_n \to k$*) iff for every* $\epsilon > 0$ *and* $\lambda > 0$*, there exists an integer* $N = (N, \epsilon) > 0$ *such that* $F_{k_n,k}(\epsilon) > 1 - \lambda$ *for all* $n \geq N$*. In this case , we write* $\lim_{n\to\infty} k_n = k$*.*

(b) *A sequence* $\{k_n\}$ *in* K *is said to be a* ***Cauchy sequence*** *if for every* $\epsilon > 0$ *and* $\lambda > 0$*, there exists an integer* $N = (N, \epsilon) > 0$ *such that* $F_{k_n,k_m}(\epsilon) > 1 - \lambda$ *for all* $n, m \geq N$*.*

(c) *A Menger space*(K, F, T) *is said to be* ***Complete*** *if every Cauchy sequence in* K *converges to a point in* K*.*

Definition 2.7. *[7] Let* K *be a non-empty set and* $Q, R : K \to K$ *be arbitrary mappings, then* $k \in K$ *is said to be a common fixed point of* Q *and* R *if* $Q(k) = R(k) = k$*.*

Example 2. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions such that $f(x) = x^3$ and $g(x) = Sinx$, then $x = 0$ is a common fixed point.*

The notion of compatible mapping in Menger Space was first introduced by S.N. Mishra [11] in 1991 as an extension of the compatible mapping in metric space introduced by G. Jungck [7] in 1986.

Definition 2.8. *Two mappings $Q, R : K \to K$ are said to be* ***Compatible Mappings*** *in Menger space (K, F, t) iff*

$$\lim_{n\to\infty} F_{QRk_n,RQk_n}(x) = 1 \quad \textit{for all} \quad x > 0$$

whenever $\{k_n\}$ is a sequence in K such that $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = k$ for some k in K.

The weakly commuting mappings were introduced by G. Jungck in 1996 as:

Definition 2.9. *[8] Two mappings $Q, R : K \to K$ are said to be* ***Weakly Commuting*** *in Menger space (K, F, t) iff $F_{QRk,RQk}(x) \geq F_{Qk,Rk}(x)$ for all k in K and $x > 0$*

Definition 2.10. *[15]: Two mappings $Q, R : K \to K$ are said to be* ***Weakly Compatible or coincidentally commuting*** *in Menger space (K, F, t) if they commute at their coincidence points i.e.if $Qk = Rk$ for some $k \in K$, then $QRk = RQk$.*

In 2021, A.K. Chaudhary, K. Jha, K. B. Manandhar, and P.P. Murthy [6] have introduced the following compatible mapping of type (P) in Menger space as an extension of H.K. Pathak et.al [12] as follows:

Definition 2.11. *[6] Two mappings $Q, R : K \to K$ are said to be* ***Compatible Mappings of type*** *(P) in Menger space (K, F, t) iff*

$$\lim_{n\to\infty} F_{QQk_n,RRk_n}(x) = 1 \quad \textit{for all} \quad x > 0$$

whenever $\{k_n\}$ is a sequence in K such that $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = k$ for some k in K.

Now, we introduce weakly compatible mappings of type (P) in Menger space with an example as follows:

Definition 2.12. *Two mappings $Q, R : K \to K$ are said to be* ***Weakly Compatible Mappings of type*** (P) *in Menger space* (K, F, t) *if and only if* $\lim_{n\to\infty} F_{QQk_n,RRk_n}(x) \geq F_{Qk_n,Rk_n}(x)$ *for all* $x > 0$ *whenever* $\{k_n\}$ *is a sequence in* K *such that* $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = k$ *for some* k *in* K.

Example 3. *Let* (K, d) *be a metric space where* $K = [0, 2]$ *with usual metric* $d(x, y) = |x - y|$ *and let* (K, F) *be PM space with*

$$F_{x,y}(t) = \begin{cases} e^{\frac{d(x,y)}{t}} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

for all $x, y \in K$. *Let* $Q, R : K \to K$ *be defined by*

$$Q(x) = \begin{cases} 1 - x & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 2] \end{cases}$$

and

$$R(x) = \begin{cases} x & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 2] \end{cases}$$

Taking sequence $\{k_n\}$ *where* $k_n = \frac{1}{2} - \frac{1}{n}$. *Then,* $Qk_n = \lim_{n\to\infty} Q(1 - (\frac{1}{2} - \frac{1}{n})) = \frac{1}{2} + \frac{1}{n} = \frac{1}{2} = k$, $Rk_n = \lim_{n\to\infty}(\frac{1}{2} - \frac{1}{n}) = \frac{1}{2} = k$. *Also,* $QQk_n = Q((\frac{1}{2} + \frac{1}{n})) = 1$ *and* $RRk_n = R((\frac{1}{2} - \frac{1}{n})) = \frac{1}{2} - \frac{1}{n} = \frac{1}{2}$. *So that* $\lim_{n\to\infty} F_{QQk_n,RRk_n}(t) = \lim_{n\to\infty} F_{1,\frac{1}{2}}(t) = e^{\frac{d(1,1/2)}{t}} = e^{\frac{1}{2t}} > 1 \neq 1$ *for all* $t > 0$ *and* $\lim_{n\to\infty} F_{Qk_n,Rk_n}(t) = \lim_{n\to\infty} F_{1,1}(t) = e^{\frac{d(1,1)}{t}} = 1$ *for all* $t > 0$.

Therefore, we have $\lim_{n\to\infty} F_{QQk_n,RRk_n}(x) \geq F_{Qk_n,Rk_n}(x)$ *for all* $x > 0$. *Hence,* (Q, R) *are weakly compatible mappings of type* (P) *but it is neither compatible mappings of type* (P) *nor compatible mappings.*

Theorem 2.1. *[2] Let* (K, F, t) *be Menger space with the continuous* $T-norm$ t *and* $Q : K \to K$ *be self mapping. Then, Q is continuous at a point* $k \in K$ *if and only if for every sequence* $\{k_n\}$ *in*K *converging to a point* k*, then sequence* $\{Qk_n\}$ *converges to the point* Qk*, i.e. if* $\{k_n\} \to k$ *then it implies* $Qk_n \to k$.

Proposition 2.1. *[12] In Menger Space*(K, F, t)*, if* $t(k, k) \geq k$ *for all* $k \in [0, 1]$ *then* $t(a, b)$ *=min (a,b) for all* $a, b \in [0, 1]$.

Lemma 2.1. *[15] Let* (K, F, t) *be a Menger space. If there exists* $k \in (0, 1)$ *such that for all* $p, q \in K$, $F_{p,q}(kx) \geq F_{p,q}(x)$ *then* $p = q$.

We need the following propositions for the establishment of our main result in the Menger space.

Proposition 2.2. *Let (K, F, t) be a Menger Space such that the $T-$norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $Q, R : K \to K$ be continuous mappings. Then, Q and R also written as (Q, R), are weakly compatible mappings of type (P) if they are compatible mappings of type(P).*

Proof. Suppose Q and R be compatible mappings of type (P). Then, we have, $1 = \lim_{n\to\infty} F_{QQk_n,RRk_n}(x) \geq F_{Qk_n,Rk_n}(x)$. So, (Q, R) be weakly compatible mappings of type (P). □

Proposition 2.3. *Let (K, F, t) be a Menger space such that the $T-$ norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $Q, R : K \to K$ be continuous mappings. Then, Q and R are compatible mappings of type (P) if they are weakly compatible mappings of type (P).*

Proof. Let $\{k_n\}$ be a sequence in K and since Q and R be continuous mappings. Then, by Theorem 2.1, we have $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = k$ for some k in K If Q and R are weakly compatible mappings of type (P). Then, we have $\lim_{n\to\infty} F_{QQk_n,RRk_n}(x) \geq F_{Qk_n,Rk_n}(x) = F_{k,k}(x) = 1$, for all $x > 0$. So, (Q, R) be compatible mappings of type (P). □

Proposition 2.4. *Let (K, F, t) be a Menger space such that the $T-$norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $Q, R : K \to K$ be mappings. If Q and R are weakly compatible mappings of type (P) and $Qk = Rk$ for some $k \in K$ then, $QQk = QRk = RQk = RRk$.*

Proof. Suppose $\{k_n\}$ is a sequence in K defined by $k_n = k$ where $n = 1, 2, 3, \ldots$ for some $k \in K$ and $Qk = Rk$. Then, we have $Qk_n, Rk_n \to$ Qk as $n \to \infty$. Since Q and R are weakly compatible mappings of type (P), then for every $\epsilon > O$, $F_{QQk,RRk}(\epsilon) = \lim_{n\to\infty} F_{QQk_n,RRk_n}(\epsilon) \geq F_{Qk_n,Rk_n}(\epsilon) = F_{Qk,Rk}(\epsilon) = F_{Qk,Qk}(\epsilon) = 1$.
So, $QQk = RRk$, since $Qk = Rk$ implies $QQk = QRk = RQk = RRk$. □

Proposition 2.5. *Let (K, F, t) be a Manger space such that the $T-$norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $Q, R : K \to K$ be mappings. Let Q and R*

be weakly compatible mappings of type (P) *and* $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = k$ *for some* k *in* K. *Then We have,*

(i) $\lim_{n\to\infty} RRk_n = Qk$, *if* Q *is continuous at* k,

(ii) $\lim_{n\to\infty} QQk_n = Rk$, *if* R *is continuous at* k,

(iii) $QRk = RQk$ *and* $Qk = Rk$ *if* Q *and* R *are continuous at* k.

Proof.

(i) Suppose that Q is continuous at k. Since, we have $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = k$ for some k in K. So, $\lim_{n\to\infty} QQk_n = Qk$, as n→∞. Again, since Q and R are weakly compatible mappings of type (P), So for every $\epsilon > 0$, $F_{QQk_n,RRk_n}(\epsilon)\geq F_{Qk_n,Rk_n}(\epsilon)$. Therefore, we have

$$F_{RRk_n,Qk}(\epsilon)\geq t(F_{RRk_n,QQk_n}(\frac{\epsilon}{2}), F_{QQk_n,Qk}(\frac{\epsilon}{2})),$$

by definition of Menger space or,

$$\begin{aligned} F_{RRk_n,Qk}(\epsilon) &\geq t(F_{Qk_n,Rk_n}(\frac{\epsilon}{2}), F_{Qk,Qk}(\frac{\epsilon}{2})) \\ &\geq t(F_{k,k}(\frac{\epsilon}{2}), 1)) \\ &\geq t(1,1)). \end{aligned}$$

This implies that $F_{RRk_n,Qk}(\epsilon) = 1$. So, $\lim_{n\to\infty} RRk_n = Qk$.

(ii) We may prove (ii) as we prove (i).

(iii) Suppose that $Q, R : K \to K$ are continuous at k. So, by (i) , $RRk_n \to Qk$ as $n \to \infty$.

On the other hand, since $\lim_{n\to\infty} Qk_n \to k$,as $n \to \infty$ and R is continuous at k. So, by proposition 2.5 (ii), we get $\lim_{n\to\infty} QQk_n = Rk$. Thus, we have $Qk = Rk$ by the uniqueness of the limit and so by preposition 2.4, we get $QRk = RQk$.

This completes the proof.

□

The following lemma needs to prove the main theorem:

Lemma 2.2. *15 Let* $\{k_n\}$ *be a sequence in Menger space* (K, F, t), *where* t *is continuous* $T-$*norm and* $t(x,x)\geq x$ *for all* $x \in [0,1]$. *If there exists a constant* $k \in [0,1]$ *such that* $\lim_{n\to\infty} F_{k_n,k_n+1}(kx)\geq F_{k_n-1,k_n}(x)$, *for all* $x > 0$ *and* $n \in N$, *then* $\{k_n\}$ *is a Cauchy sequence in* K.

3. Main Theorem

Now, we prove our main theorem for weakly compatible mappings of type (P) in Complete Menger Space:

Theorem 3.1. *Let (K, F, t) be a complete Menger space with $t(x, y) = min(x, y)$ for all $x, y \in [0, 1]$ and $Q, S, R, T : K \to K$ be mappings such that*

(3.1) $Q(K) \subseteq T(K)$ *and* $S(K) \subseteq R(K)$,

(3.2) *the pairs (Q, R) and (S, T) are weakly compatible mappings of type (P),*

(3.3) *One of Q, S, R, T be continuous, and*

(3.4) *there exists a constant $k \in (0, 1)$ such that*

$$F_{Qx,Sy}(kt) \geq \min\{(F_{Rx,Qx}(t), F_{Ty,Sy}(t), F_{Ty,Qx}(\alpha t), F_{Rx,Sy}((2-\alpha)t), F_{Rx,Ty}(t)\}$$

for all $x, y \in K, \alpha \in (0, 2)$ and $t > 0$.

Then, Q, S, R, T have a unique common fixed point in K.

Proof. Consider $u_0 \in K$. Since $Q(K) \subseteq T(K)$, so there exists a point u_1 in K such that $Qu_0 = Tu_1 = v_0$. Again, since $S(K) \subseteq R(K)$, so for u_1, we may choose u_2 in K such that $Su_1 = Ru_2 = v_1$ and so on.

And inductively, we may construct sequence $\{u_n\}$ and $\{v_n\}$ in K such that $Qu_{2n} = Tu_{2n+1} = v_{2n}$, and $Su_{2n+1} = Ru_{2n+2} = v_{2n+1}$, for $n = 0, 1, 2, \ldots$. Putting $x = u_{2n}$ and $y = u_{2n+1}$ for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$ in (3.4), we get,

$$\begin{aligned} F_{Qu_{2n},Su_{2n+1}}(kt) \geq \min\{&(F_{Ru_{2n},Qu_{2n}}(t), F_{Tu_{2n+1},Su_{2n+1}}(t), F_{Tu_{2n+1},Qu_{2n}}((1-q)t), \\ &F_{Ru_{2n},Su_{2n+1}}(1+q)t), F_{Ru_{2n},Tu_{2n+1}}(t)\} \end{aligned}$$

or,

$$\begin{aligned} F_{v_{2n},v_{2n+1}}(kt) &\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n},v_{2n}}((1-q)t), \\ &\qquad F_{v_{2n-1},v_{2n+1}}(1+q)t), F_{v_{2n-1},v_{2n}}(t)\} \\ F_{v_{2n},v_{2n+1}}(kt) &\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), 1, F_{v_{2n-1},v_{2n+1}}(1+q)t), F_{v_{2n-1},v_{2n}}(t)\} \\ &\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(qt), F_{v_{2n-1},v_{2n}}(t)\} \\ &\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n},v_{2n+1}}(qt)\}. \end{aligned}$$

As $q \to 1$, we obtain

$$\begin{aligned} F_{v_{2n},v_{2n+1}}(kt) &\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n},v_{2n+1}}(t)\} \\ &\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t)\}. \end{aligned}$$

Hence, we get

$$F_{v_{2n},v_{2n+1}}(kt) \geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t)\},$$

i.e. $F_{v_{2n},v_{2n+1}}(kt) \geq F_{v_{2n-1},v_{2n}}(t)$.

Similarly, we obtain $F_{v_{2n+1},v_{2n+2}}(kt) \geq F_{v_{2n},v_{2n+1}}(t)$. Therefore, for every $n \in N$, $F_{v_n,v_{n+1}}(kt) \geq F_{v_{n-1},v_n}(t)$. So, using Lemma (2.2), $\{v_n\}$ is a Cauchy sequence in K. Since the Menger space (K, F, t) is complete, so $\{v_n\}$ converges to a point z in K and consequently the sub sequences $\{Qu_{2n}\}, \{Su_{2n+1}\}, \{Ru_{2n}\}, \{Tu_{2n+1}\}$, of $\{v_n\}$ also converges to z.

Now, suppose that T is continuous. Then, since S and T are weakly compatible mappings of type (P) then by Proposition 2.5, SSu_{2n+1} , $TSu_{2n+1} \to Tz$ as $n \to \infty$. Putting $x = u_{2n}$ and $y = Su_{2n+1}$ in relation (3.4), we get

$$\begin{aligned} F_{Qu_{2n},SSu_{2n+1}}(kt) \geq \min\{(&F_{Ru_{2n},Qu_{2n}}(t), F_{TSu_{2n+1},SSu_{2n+1}}(t), F_{TSu_{2n+1},Qu_{2n}}((\alpha)t), \\ &F_{Ru_{2n},SSu_{2n+1}}(2-\alpha)t), F_{Ru_{2n},TSu_{2n+1}}(t)\}. \end{aligned}$$

Taking $n \to \infty$, we have

$$F_{z,Tz}(kt) \geq \min\{(F_{z,z}(t), F_{Tz,Tz}(t), F_{Tz,z}((\alpha)t), F_{z,Tz}(2-\alpha)t), F_{z,Tz}(t)\}.$$

Letting $\alpha = 1 - q$ with $q \in (0, 1)$ then

$$F_{z,Tz}(kt) \geq \min\{(F_{Tz,z}((1-q)t), F_{z,Tz}((1+q)t), F_{z,Tz}(t)\},$$

or,

$$\begin{aligned} F_{z,Tz}(kt) &\geq \min\{(F_{Tz,Tz}((1-q+1+q)t), F_{z,Tz}(t)\} \\ &\geq \min\{F_{z,Tz}(t)\}. \end{aligned}$$

Therefore, $F_{z,Tz}(kt) \geq F_{z,Tz}(t)$, which implies $z = Tz$ by Lemma 2.1.

Similarly, replacing x by u_{2n} and y by z in relation (3.4), we have

$$\begin{aligned} F_{Qu_{2n},Sz}(kt) \geq \min\{(&F_{Ru_{2n},Qu_{2n}}(t), F_{Tz,Sz}(t), F_{Tz,Qu_{2n}}((\alpha)t), F_{Ru_{2n},Sz}(2-\alpha)t), \\ &F_{Ru_{2n},Tz}(t)\}. \end{aligned}$$

Taking $n \to \infty$, we get

$$\begin{aligned} F_{z,Sz}(kt) &\geq \min\{(F_{z,z}(t), F_{z,Sz}(t), F_{z,z}((\alpha)t), F_{z,Sz}(2-\alpha)t), F_{z,z}(t)\} \\ F_{z,Sz}(kt) &\geq \min\{(F_{z,Sz}(t),, F_{z,Sz}(2-(1-q)t)\} \\ &\geq \min\{(F_{z,Sz}(t),, F_{z,Sz}(1+q)t)\} \end{aligned}$$

$$\geq \min\{(F_{z,Sz}(t), F_{z,z}(t), F_{z,Sz}(qt)\}$$
$$\geq \min\{(F_{z,Sz}(t), F_{z,Sz}(t)\} \text{ as } q \to 1$$

So that $F_{z,Sz}(kt) \geq \{(F_{z,Sz}(t)\}$, which implies $z = Sz$. Since, $S(K) \subseteq R(K)$, so there exists a point w in K such that $Sz = Rw = z$. By using relation (3.4) with $x = w, y = z$, we have

$$F_{Qw,z}(kt) \geq \min\{(F_{Rw,Qw}(t), F_{Tz,Sz}(t), F_{Tz,Qw}(\alpha t), F_{Rw,Sz}(2-\alpha)t), F_{Rw,Tz}(t)\}$$
$$\geq \min\{(F_{z,Qw}(t), F_{Tz,z}(t), F_{z,Qw}((1-q)t), F_{Rw,z}(1+q)t), F_{z,Tz}(t)\}$$
$$\geq \min\{(F_{z,Qw}(t), F_{Tz,z}(t), F_{Qw,z}((1-q)t), F_{Rw,z}(1+q)t), F_{z,Tz}(t)\}$$
$$\geq \min\{(F_{z,Qw}(t), F_{z,z}(t), F_{Qw,Rw}((1-q+1+q)t)\}$$
$$\geq \min\{(F_{z,Qw}(t), F_{Qw,z}(2t)\}.$$

Therefore, $F_{Qw,z}(kt) \geq F_{z,Qw}(t) = F_{Qw,z}(t)$, which implies $Qw = z$, by Lemma 2.1.

Again, since Q and R are weakly compatible mappings of type (P) and $Qw = Rw = z$, by Proposition 2.4, we have for every $\epsilon > 0$, $1 = F_{QQw,RRw}(\epsilon) \geq F_{Qw,Rw}(\epsilon)$. Hence, $Qw = QQw = RRw = Rw$. Finally, by relation (3.4) with $x = z, y = Sz = z$, we have

$$F_{Qz,z}(kt) = F_{Qz,Sz}(kt) \geq \min\{(F_{Rz,Qz}(t), F_{Tz,z}(t), F_{Tz,Qz}(\alpha t), F_{Rz,z}(2-\alpha)t), F_{Rz,Tz}(t)\}$$
$$\geq \min\{(F_{Qz,Qz}(t), F_{z,z}(t), F_{z,Qz}(\alpha t), F_{Qz,z}(2-\alpha)t), F_{Qz,z}(t)\}$$
$$\geq \min\{(F_{Qz,z}(\alpha t), F_{z,Qz}(2-\alpha)t), F_{Qz,z}(t)\}$$
$$\geq \min\{(F_{Qz,Qz}(\alpha t + 2t - \alpha t), F_{Qz,z}(t)\}$$

or, $F_{Qz,z}(kt) \geq F_{Qz,z}(t)$. Therefore, $Qz = z$. Hence, $Qz = Sz = Rz = Tz = z$. That is, z is common fixed point of given mappings Q, S, R and T.

Uniqueness: Suppose z_1 be other point in K such that $z_1 = Qz_1 = Sz_1 = Rz_1 = Tz_1$. Then, putting $x = z$ and $y = z_1, \alpha = 1$ in (3.4), we get

$$F_{Qz,Sz_1}(kt) = F_{z,z_1}(kt) \geq \min\{(F_{Rz,Qz}(t), F_{Tz_1,Sz_1}(t), F_{Tz_1,Qz}(t), F_{Rz,Sz_1}(t), F_{Rz,Tz_1}(t)\}$$

or, $F_{z,z_1}(kt) \geq \min\{(F_{z,z_1}(t), F_{z,z}(t)\}$, or, $F_{z,z_1}(kt) \geq F_{z,z_1}(t)$. By Lemma 2.1, $z = z_1$.

Hence, $z = Qz = Sz = Rz = Tz$ and z is unique common fixed point for Q, S, R, T in K.

This completes the proof. □

4. Verification and Application

We verify our Main Theorem (3) with following example:

Example 4. *Let (K, d) be metric space where $K = [0, 2]$with usual metric $d(x, y) = |x - y|$ and distribution function F is defined by*

$$F_{x,y}(t) = \begin{cases} e^{\frac{d(x,y)}{t}} & \textit{for } t > 0 \\ 0 & \textit{for } t = 0 \end{cases},$$

for all $x, y \in K$. Let $Q, R, S, T : K \to K$ be defined by

$$Q(K) = \begin{cases} 1 - x & \textit{for } x \in [0, \frac{1}{2}) \\ 1 & \textit{for } x \in [\frac{1}{2}, 2] \end{cases},$$

$$S(K) = \begin{cases} \frac{1}{2} - x & \textit{for } x \in [0, \frac{1}{2}) \\ 1 & \textit{for } x \in [\frac{1}{2}, 2] \end{cases},$$

$$R(K) = \begin{cases} x & \textit{for } x \in [0, \frac{1}{2}) \\ 1 & \textit{for } x \in [\frac{1}{2}, 2] \end{cases}$$

and

$$T(K) = \begin{cases} 0 & \textit{for } x \in [0, \frac{1}{2}) \\ 1 & \textit{for } x \in [\frac{1}{2}, 2] \end{cases}.$$

We take sequence $\{k_n\}$ in K where $k_n = \frac{1}{2} - \frac{1}{n}$, $n \in N$. Then Q, S, R and T satisfy all the conditions of the above Theorem 3 and have a unique common fixed point $x = 1$ in K.

Our established Theorem 3 may apply in consequences results in metric space in four self mappings and also may use to prove following corollaries.

In the Theorem 3, if we take $Q = S, T = R$, then we have

Corollary 4.1. *Let (K, F, t) be a complete Menger space with continuous $t(x, y) = min(x, y)$ for all $x, y \in [0, 1]$ and $Q, R : K \to K$ be mappings such that*

(3.1) *$Q(K) \subseteq R(K)$,*
(3.2) *the pairs (Q, R) be weakly compatible mappings of type (P),*
(3.3) *R be continuous, and*
(3.4) *there exists a constant $k \in (0,1)$ such that*

$$F_{Qx,Qy}(kt) \geq \min\{(F_{Rx,Qx}(t), F_{Ry,Qy}(t), F_{Ry,Qx}(\alpha t), F_{Rx,Qy}(2-\alpha)t), F_{Rx,Ry}(t)\}$$

for all $x, y \in K, \alpha \in (0,2)$ and $t > 0$.

Then, Q and R have a unique common fixed point in K.

5. Conclusions

In a conclusion our result extends and generalize the results of Jungck et.al [7], [9] and of Chaudhary et al. [6]. This result also generalizes and improve the result of Pathak et.al. [12], Stojavic, et.al. [2] and other similar results in literature.

Acknowledgment

First author is highly thankful to University Grant Commission, Nepal for providing PhD fellowship award PhD-76/77-S and T-3. And authors also would like to thanks referee for their kind suggestions for improvement of our paper.

References

[1] S. Banach: *Sur les Operations dans les ensembles abstraits et leur applications aux equations integral*, Fund Math, **3** (1922), 87-92.
[2] Y.J. Cho, P.P. Murthy, M. Stojakovic : *Compatible Mappings of type (A) and Common fixed point in Menger space*, Comm. Korean Math. Soc., **7**(2) (1992), 325-339.
[3] M. Frechet: *Sur quelques points du calcul fonctionnel*, Rendic. Circ. Mat. Palermo, (1906), 1-74.
[4] F. Hausdroff: *Grundzuge der Mengenlehre (German)*, Chelsea Publishing Company, New York, 1949.
[5] O. Hadzic, E. Pap : *Fixed-Point Theory in Probabilistic Metric Space*, Kluwer Academic Publisher, London, **536**, 2010.
[6] A.K. Chaudhary, K.B. Manandhar, K. Jha, P.P. Murthy: *A common fixed point theorem in Menger space with compatible mapping of type (P)*, International Journal of Math. Sci. Engg. Appls., **15**(2) (2021), 59-70.

[7] G. JUNGCK: *Compatible Mapping and common fixed points*, Internat. J. Math. Sci. **9**(4) (1986), 771-779.

[8] G. JUNGCK: *Common fixed points for non-continuous non self-maps on non-metric spaces*, Far East. J. Math. Sci. **4**(2) (1996), 199-212.

[9] G. JUNGCK , P.P. MURTHY, Y.J. CHO: *Compatible mappings of type (A) and common fixed points*, Math. Japonica **38** (1993), 381-390.

[10] K. MENGER: *Statistical Matrices*,Proceedings of National Academy of Sciences of USA, **28** (1942), 535-537.

[11] S.N. MISHRA: *Common fixed points of Compatible Mappings in probabilistic Metric Space*, Math.Japonica, **36** (1991), 283-289.

[12] H.K. PATHAK, Y.J. CHO , S.S. CHANG, S.M. KANG : *Compatible mappings of type (P) and fixed point theorem in metric spaces and Probabilistic metric spaces*, Novi Sad J. Math., **26**(2) (1996), 87-109.

[13] B. SCHWEIZER, A. SKLAR : *Statistical metric space*, Pacific J. of Math., **10** (1960), 314-334.

[14] V.M. SEHGAL, A.T. BHARUCHA-REID : *Fixed Point contraction mapping in Probabilistic Metric Space*, Math. System Theory, **6** (1972), 97-102.

[15] B. SINGH, S. JAIN : *Common Fixed Point theorem in Menger Space through Weak Compatibility*, J. Math. Anal. Appl., **301** (2005), 439-448.

[16] B. SCHWEIZER, A. SKLAR : *Probabilistic Metric space*, Dover Publications, INC, Mineola, New York, 2005.

[17] S. SESSA: *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. (Beograd) (N.S.), **32**(46) (1982), 149–153.

[18] S. SESSA, B.E. RHOADES, M.S. KHAN: *On common fixed points of compatible mappings in metric and Banach Spaces*, Internat. J. Math. Math. Sci., **11**(2) (1988), 375-392.

DEPARTMENT OF MATHEMATICS, TRICHANDRA MULTIPLE CAMPUS, TRIBHUVAN UNIVERSITY GHANTAGHAR, KATHMANDU, NEPAL AND DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE, KATHMANDU UNIVERSITY, KAVRE, NEPAL.
Email address: `akcsaurya81@gmail.com`

DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE, KATHMANDU UNIVERSITY, KAVRE, NEPAL.
Email address: `jhakn@ku.edu.np`

DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE,KATHMANDU UNIVERSITY, KAVRE, NEPAL.
Email address: `kb.manandhar@ku.edu.np`

DEPARTMENT OF MATHEMATICS, PT. RAVISHANKAR SHUKLA UNIVERSITY, INDIA
Email address: `akhilesh120078@gmail.com`