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A RECURRENCE RELATION FOR THE NUMBER OF REPRESENTATIONS OF A POSITIVE INTEGER AS A SUM OF SQUARES

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ABSTRACT. We exhibit a recurrence relation for $r_k(n)$, that is, for the number of representations of a positive integer as a sum of squares, so it is possible to determine $r_k(n)$ if we know $r_k(m), m = 0, 1, 2, \ldots, n-1$.

In [1] it was showed the following recurrence relation:

$$r_k(n) = -\frac{2\,k}{n} \sum_{j=1}^{n} (-1)^j\, j\, D(j)\, r_k(n-j), \qquad k, n \geq 1, \tag{1}$$

where $D(n) = \sum_{odd\ d/n} \frac{1}{d}$ and $r_k(n)$ is the number of representations of a positive integer n as a sum of k squares, such that representations with different orders and signs are counted as distinct [2-5].

Here we want exhibit certain relations that suggest the existence of (1). In fact, Jha [6, 7] obtained the expression:

$$\frac{1}{n}\,\sigma(n) = \sum_{k=1}^{n} \frac{(-1)^k}{k} \binom{n}{k} p_k(n), \qquad n \geq 1, \tag{2}$$

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for the sum of divisors function in terms of colour partitions [8-10], verifying the Gandhi's identity [8, 11, 12]:

$$p_k(n) = -\frac{k}{n} \sum_{j=1}^{n} \sigma(j)\, p_k(n-j), \qquad k, n \geq 1, \tag{3}$$

On the other hand, Jha [2, 13, 14] deduced the property:

$$2\,(-1)^n\, D(n) = \sum_{k=1}^{n} \frac{(-1)^k}{k} \binom{n}{k} r_k(n), \tag{4}$$

We see that (2) and (4) have a similar structure, then it is natural to ask if (4) is associated with a recurrence relation as (3); the answer is yes [1, 2] because into (2) we can realize the connections:

$$p_k(n) \;\to\; r_k(n), \qquad\qquad \sigma(j) \;\to\; 2\,(-1)^j\, j\, D(j), \tag{5}$$

to obtain the mentioned recurrence relation (1), that is, it is possible to determine $r_k(n)$ if we know $r_k(m)\,, m = 0,\, 1,\, 2, \ldots,\, n-1$; we must remember that $r_k(0) = 1, \;\; k \geq 1$. Furthermore, the expression (1) with $n = 1, 2, 3 \ldots,$ implies the following formulae:

$$\begin{aligned}
& r_k(1) = 2\,k\,, \qquad r_k(2) = 2\,k\,(k-1)\,, \qquad r_k(3) = \frac{4}{3}\,k\,(k-1)\,(k-2), \\
& r_k(4) = \frac{2}{3}\,k\,[3\,(2\,k-1) + k\,(k-1)\,(k-5)]\,, \\
& r_k(5) = \frac{4}{15}\,k\,(k-1)\,[3\,(2\,k-3) + k\,(k-4)\,(k-5)]\,, \\
& r_k(6) = \frac{4}{45}\,k\,(k-1)\,(k-2)\,[45 + (k-3)\,(k-4)\,(k-5)]\,, \\
& r_k(7) = \frac{8}{315}\,k\,(k-1)\,(k-2)\,(k-3)\,(k^3 - 15\,k^2 + 74\,k - 15)\,, \ldots,
\end{aligned} \tag{6}$$

which can be deduced via another approach. Similarly, from (3) with $p_k(0) = 1,\, k \geq 1$ [15]:

$$\begin{aligned}
& p_k(1) = -k\,, \qquad p_k(2) = \frac{1}{2!}\,k\,(k-3)\,, \qquad p_k(3) = -\frac{1}{3!}\,k\,(k-1)\,(k-8), \\
& p_k(4) = \frac{1}{4!}\,k\,(k-1)\,(k-3)(k-14)\,, \\
& p_k(5) = -\frac{1}{5!}\,k\,(k-3)\,(k-6)(k^2 - 21\,k + 8)\; \ldots
\end{aligned} \tag{7}$$

If $t_k(n)$ is the number of representations of n as the sum of k triangular numbers, such that representations with different orders are counted as unique, then it is valid the following recurrence relation [1]:

$$(8)\quad t_k(n) = -\frac{k}{n}\sum_{j=1}^{n} j\,T(j)\,t_k(n-j), \qquad T(j) = \sum_{d/j}\frac{1+2(-1)^d}{d} = \frac{1}{j}\sum_{d/j}(-1)^d\,d,$$

with the corresponding inversion:

$$(9)\qquad T(n) = \sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k}\,t_k(n)\,.$$

Let us indicate three interesting recurrences:

$$(10)\qquad \begin{aligned} p_{k+1}(n) &= \sum_{j=0}^{n} a(j)\,p_k(n-j), \quad r_{k+1}(n) = \sum_{j=0}^{n} b(j)\,r_k(n-j),\\ t_{k+1}(n) &= \sum_{j=0}^{n} c(j)\,t_k(n-j), \end{aligned}$$

where:

$$(11)\qquad \begin{aligned} a(j) &= \begin{cases} 0, & j \neq \frac{m}{2}\,(3\,m+1)\,, \\ (-1)^m, & j = \frac{m}{2}\,(3\,m+1)\,, \end{cases} \qquad m = 0\,,\pm1\,,\pm2\,\ldots\ ,\\ b(j) &= \begin{cases} 2, & n = m^2\,, \qquad m \geq 1, \\ 1, & \qquad\qquad n = 0, \\ 0, & \qquad \textit{otherwise}, \end{cases}\ ,\\ c(j) &= \begin{cases} 1, & n = \frac{m}{2}\,(m+1)\,, \qquad m \geq 0, \\ 0, & \qquad\qquad \textit{otherwise}, \end{cases} \end{aligned}$$

which are immediate from the following result for any analytic function $F(q)$:

$$(12)\qquad \text{" If }\ (F(q))^k = \sum_{n=0}^{\infty} f_k(n)\,q^n \qquad \text{then} \qquad f_{k+1}(n) = \sum_{j=0}^{n} f_1(j)\,f_k(n-j)\text{ ".}$$

In fact:

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} f_{k+1}(n)\,q^n &= F^{k+1} = F\,F^k = (\textstyle\sum_{m=0}^{\infty} f_1(m)\,q^m)(\textstyle\sum_{l=0}^{\infty} f_k(l)\,q^l),\\ &= \textstyle\sum_{n=0}^{\infty}(\textstyle\sum_{j=0}^{n} f_1(j)\,f_k(n-j)\,)\,q^n, \end{aligned}$$

in accordance with (12). If $F(q) = (q;\, q)_\infty$ then $f_k(n) = p_k(n)$ and $f_1(n) = p_1(n) = a(n)$ [15], thus (12) implies the first relation in (10); for $F(q) = \prod_{j=1}^{\infty} \frac{1-q^j}{1+q^j}$ we obtain that $f_k(n) = (-1)^n\, r_k(n)$ and $f_1(n) = (-1)^n\, b(n)$, so (12) gives the second recurrence in (10); and if $F(q) = \prod_{j=1}^{\infty} (1+q^j)^2\,(1-q^j)$ then $f_k(n) = t_k(n)$ and $f_1(n) = c(n)$ [16], therefore (12) generates the third identity in (10).

Remark 1. *The second relation in (10) can be written in the form:*

$$r_{k+1}(n) = r_k(n) + 2 \sum_{m=1}^{[\sqrt{n}]} r_k(n-m^2)\,; \tag{13}$$

if p is a prime number, then [5]:

$$\text{"}\ p \equiv 1\ (mod\ 4) \quad \Longleftrightarrow \quad p = \textit{sum of two squares}\ \text{"}, \tag{14}$$

that is, $p = 4N+1,\ N \geq 0$*, then with (13) for* $k=1$ *and (14) it is easy to prove the result:*

$$r_2(p) = \begin{cases} 4, & p = 1 \\ 8, & p \neq 1 \end{cases}, \qquad p \equiv 1\ (mod\ 4)\,. \tag{15}$$

Furthermore, we have [4, 5] that $r_2(p) = 0$ if $p \equiv 3\ (mod\ 4)$.

Remark 2. *It is known the property [4, 5]:*

$$r_3(n) = 0 \quad \Longleftrightarrow \quad n = 4^M\,(8\,N+7), \quad M,\, N \geq 0, \tag{16}$$

which can be applied to the second expression in (10) with $k=2$*:*

$$r_3(n) = \sum_{j=0}^{n} b(j)\, r_2(n-j), \tag{17}$$

to obtain the interesting result:

$$r_2(n-m^2) = r_2(4^M\,(8\,N+7) - m^2) = 0, \quad m,\, M,\, N \geq 0, \quad n - m^2 > 0, \tag{18}$$

that is, $r_2(q) = 0$ *for* $q = 3,\ 6,\ 7,\ 11,\ 14,\ 15,\ 19,\ 22,\ 23,\ 27,\ 28,\ 30,\ 31, \ldots$*, hence this numbers have the same quantity of divisors* $\equiv 1$ *and* $\equiv 3,\ (mod\ 4)$.

Remark 3. *The inversion of* $r_{k+1}(n) = \sum_{j=0}^{n} b(j)\, r_2(n-j)$ *is given by:*

$$\begin{aligned} r_k(n) &= \sum_{j=0}^{n} (-1)^j\, h(j)\, r_{k+1}(n-j), \\ h(n) &= -\sum_{j=1}^{n} (-1)^j\, b(j)\, h(n-j),\ n \geq 1,\ h(0) = 1, \end{aligned} \tag{19}$$

or in terms of the incomplete exponentials Bell polynomials [17], for $m \geq 0$*:*

$$\begin{aligned} h(m) = \frac{1}{m!} \sum_{k=0}^{m} (-1)^k\, k!\, B_{m,k}\, &(-b(1),\, 2!\, b(2),\, -3!\, b(3), \ldots, \\ &(m-k+1)!\, (-1)^{m-k+1}\, b(m-k+1)\big)\,, \end{aligned} \tag{20}$$

thus $h(n) = 1,\, 2,\, 4,\, 8,\, 14,\, 24,\, 40,\, 64, \ldots$*, for* $n = 0,\, 1,\, 2,\, 3,\, 4,\, 5,\, 6,\, 7, \ldots$*, respectively.*

Remark 4. *From (10) with* $k = 1$*:*

$$r_2(n) = b(n) + \sum_{j=1}^{n} b(j)\, b(n-j), \tag{21}$$

which is an alternative for the known expression [4, 5]:

$$r_2(n) = 4\, \left(d_1(n) - d_3(n) \right), \tag{22}$$

where $d_j(n)$ *is the number of divisors of n with the structure* $4k + j,\ j = 1, 3$*; the relation (21) is very easy to apply and does not need the divisors of* n *. In the literature we do not find, explicitly, the results (10), (12), and (18)-(21).*

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