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A LYAPUNOV-TYPE INEQUALITY FOR A HIGHER ORDER FRACTIONAL BOUNDARY VALUE PROBLEM

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ABSTRACT. In this work, we obtain a Lyapunov-type inequality for a fractional differential equation involving Caputo fractional derivatives of higher order and subject to nonlocal boundary conditions. An application to eigenvalue problems is also given.

1. INTRODUCTION

The aim of this paper is to establish a new Lyapunov type inequality for the following higher order fractional boundary value problem:

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t) + q\,(t)\,u(t) = 0, \quad a \leq t \leq b, \\ u\,(a) = u'(a) = u'''\,(a) = u''\,(b) = 0, \end{cases} \tag{1.1}$$

where $3 < \alpha \leq 4$, $q : [a,b] \to \mathbb{R}$ is a continous function and ${}^{c}D_{0+}^{\alpha}$ denotes the Caputo's fractional derivative.

In [12], Lyapunov proved that if the boundary value problem

$$\begin{cases} u''\,(t) + q\,(t)\,u(t) = 0, \quad a \leq t \leq b, \\ u\,(a) = u\,(b) = 0, \end{cases} \tag{1.2}$$

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has a nontrivial solution, then

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a}, \tag{1.3}$$

where $q:[a,b]\to\mathbb{R}$ is a continous function.

Inequality (1.3) has been proved to be very useful in various problems related with differential and difference equations such that oscillation theory, disconjugacy, asymptotic theory, eigenvalue problems, see $[1,2,3,7,9,14,17]$ and the references therein.

Lyapunov-type inequalities for differencial equation involving fractional derivatives have attracted attention recently, the first work in this direction is due to Ferreira $[5]$, where he considered the fractional boundary value problem

$$\begin{cases} D_{a^+}^{\alpha}u(t)+q(t)\,u(t)=0, \quad a\le t\le b, \\ u(a)=u(b)=0, \end{cases} \tag{1.4}$$

and proved that if the above problem has a nontrivial solution, then

$$\int_a^b |q(s)|\, ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}, \tag{1.5}$$

where $(a,b)\in\mathbb{R}^2$, $a<b$, $\alpha\in(1,2)$, $q:[a,b]\to\mathbb{R}$ is a continuous function, and $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α.

Obviously, if we set $\alpha=2$ in (1.5), we obtain the classical Lyapunov inequality (1.3).

Recently, in $[10]$ Jleli et al., considered the fractional boundary value problem

$${}^{c}D_{a^+}^{\alpha}u(t)+q(t)\,u(t)=0, \quad a\le t\le b, \tag{1.6}$$

under the mixed boundary conditions

$$\begin{gathered} u(a)=u'(b)=0, \\ \text{or} \\ u'(a)=u(b)=0, \end{gathered} \tag{1.7}$$

where ${}^{c}D_{0^{+}}^{q}$ denotes the Caputo's fractional derivative of ordre $1 < \alpha \leq 2$. Then the following two types Lyapunov-type inequalities were derived respectively

$$(1.8)\qquad \begin{cases} \int_a^b (b-s)^{\alpha-2} |q(s)|\, ds \geq \frac{\Gamma(\alpha)}{\max\{\alpha-1, 2-\alpha\}(b-a)}, \\ \text{and} \\ \int_a^b (b-s)^{\alpha-1} |q(s)|\, ds \geq \Gamma(\alpha), \end{cases}$$

More recently, Guezane-Lakoud et al. in [8] considered the fractional boundary value problem

$$(1.9)\qquad \begin{cases} {}^{c}D_{b^{-}}^{\alpha} D_{a^{+}}^{\beta} u(t) + q(t) u(t) = 0, \quad a \leq t \leq b, \\ u(a) = D_{a^{+}}^{\beta} u(b) = 0, \end{cases}$$

where $\alpha > 0$, $\beta \leq 1$, $1 < \alpha + \beta \leq 2$, ${}^{c}D_{b^{-}}^{\alpha}$ denotes the right Caputo derivative, $D_{a^{+}}^{\beta} u(t)$ denotes the left Riemann–Liouville derivative. The authors proved that If the fractional boundary value problem (1.9) has a nontrivial continuous solution, then

$$(1.10)\qquad \int_a^b |q(s)|\, ds \geq \frac{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha+\beta-1}}.$$

For other related results, we refer to $[4, 6, 11, 13]$ and the references therein.

2. Preliminaries

In this section, we present some definitions and lemmas from fractional calculus theory, which will be needed later. For more details, we refer to $[16]$.

Definition 2.1. *If $g \in C([a,b])$ and $\alpha > 0$, then the Riemann-Liouville fractional integral is defined by*

$$I_{a^{+}}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2.2. *Let $\alpha \geq 0, n = [\alpha] + 1$. If $g \in C^{n}[a,b]$ then the Caputo fractional derivative of order α of g defined by ${}^{c}D_{a^{+}}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds$ exists almost everywhere on $[a,b]$ ($[\alpha]$ is the entire part of α).*

Lemma 2.1. *Let $n-1 < \alpha < n$, $g \in C^{n}([a,b])$, then $I_{a^{+}}^{\alpha}\, {}^{c}D_{a^{+}}^{\alpha} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!} (t-a)^{k}$.*

Lemma 2.2. *Let* $p, q \geq 0$, $g \in L_1[a,b]$. *Then* $I_{0+}^{p} I_{0+}^{q} g(t) = I_{0+}^{p+q} g(t) = I_{0+}^{q} I_{0+}^{p} g(t)$ *and* ${}^{c}D_{a^+}^{q} I_{0+}^{q} g(t) = g(t)$, *for all* $t \in [a,b]$.

3. Lyapunov inequality

We transform the problem (1.1) to an equivalent integral equation.

Lemma 3.1. *Assume that* $3 < \alpha \leq 4$, *the function* u *is a solution to the boundary value problem* (1.1) *if and only if* u *satisfies the integral equation*

$$u(t) = \int_a^b G(t,s)\, q(s)\, u(s)\, ds \tag{3.1}$$

where the Green function $G(t,s)$ *is defined by*

(3.2)
$$G(t,s) = \frac{1}{2\Gamma(\alpha)} \begin{cases} (\alpha-1)(\alpha-2)(t-a)^2 (b-s)^{\alpha-3} - 2(t-s)^{\alpha-1}, & a \leq s \leq t \leq b \\ (\alpha-1)(\alpha-2)(t-a)^2 (b-s)^{\alpha-3}, & a \leq t \leq s \leq b. \end{cases}$$

Proof. Using Lemmas 2.3, we obtain that u is solution of (1.10) if and only if it satisfies the following equation

$$u(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + c_3(t-a)^3 - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)\, u(s) ds. \tag{3.3}$$

Using the initial conditions $u(a) = u'(a) = u'''(a) = 0$, we get $c_0 = c_1 = c_3 = 0$. The condition $u''(b) = 0$, yields $c_2 = \frac{(\alpha-1)(\alpha-2)}{2\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-3} q(s)\, u(s) ds$. Substituting c_0, c_1, c_2 and c_3 by their values in (3.3), we obtain

$$\begin{aligned} u(t) = {} & \tfrac{(\alpha-1)(\alpha-2)(t-a)^2}{2\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-3} q(s)\, u(s) ds \\ & - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)\, u(s) ds, \end{aligned} \tag{3.4}$$

that is

$$u(t) = \int_a^b G(t,s)\, q(s)\, u(s)\, ds, \quad t \in [a,b].$$

where $G(t,s)$ is as in (3.2). □

Set

$$G(t,s) = \begin{cases} g_1(t,s), & a \leq s \leq t \leq b \\ g_2(t,s), & a \leq t \leq s \leq b \end{cases} \tag{3.5}$$

Lemma 3.2. *The Green function G satisfies* (1) $G(t,s) \geq 0$ *for all* $a \leq t, s \leq b$.

(2) $\max_{t\in[a,b]} G(t,s) = G(b,s)\,, \quad s \in [a,b]\,.$

(3) $G(b,s)$ *has a unique maximum given by*

$$\max_{t\in[a,b]} G(b,s) = \frac{1}{\Gamma(\alpha)} \left(\frac{(\alpha-3)^{\frac{1}{2}}(\alpha-2)^{\frac{3}{2}}}{2} \right)^{\alpha-3} (b-a)^{\alpha-1}\,. \tag{3.6}$$

Proof. The function $g_1(t,s)$ is positive and non-decreasing. Indeed, for $a \leq s \leq t \leq b$,

$$\begin{aligned} g_1(t,s) &= \frac{1}{2\Gamma(\alpha)} \left[(\alpha-1)(\alpha-2)(t-a)^2 (b-s)^{\alpha-3} - 2(t-s)^{\alpha-1} \right] \\ &\geq \frac{1}{2\Gamma(\alpha)} \left[(\alpha-1)(\alpha-2)(t-s)^2 (t-s)^{\alpha-3} - 2(t-s)^{\alpha-1} \right] \\ &= \frac{1}{2\Gamma(\alpha)} \left[\alpha(\alpha-3)(t-s)^{\alpha-1} \right] \geq 0. \end{aligned}$$

On the other hand

$$\begin{aligned} \frac{\partial g_1(t,s)}{\partial t} &= \frac{1}{\Gamma(\alpha)} \left[(\alpha-1)(\alpha-2)(t-a)(b-s)^{\alpha-3} - (\alpha-1)(t-s)^{\alpha-2} \right] \\ &\geq \frac{1}{\Gamma(\alpha)} \left[(\alpha-1)(\alpha-2)(t-s)(t-s)^{\alpha-3} - (\alpha-1)(t-s)^{\alpha-2} \right] \\ &= \frac{1}{\Gamma(\alpha)} \left[(\alpha-1)(\alpha-3)(t-s)^{\alpha-2} \right] \geq 0. \end{aligned}$$

Consequently

$$\max_{t,s\in[a,b]} g_1(t,s) = \max_{t,s\in[a,b]} g_1(b,s)\,.$$

In view of (3.2) and (3.3), $g_1(b,s)$ is defined by

$$g_1(b,s) = \frac{1}{2\Gamma(\alpha)} \left[(\alpha-1)(\alpha-2)(b-a)^2 (b-s)^{\alpha-3} - 2(b-s)^{\alpha-1} \right].$$

Its derivative with respect to s takes the form

$$\frac{\partial g_1(b,s)}{\partial s} = \frac{1}{2\Gamma(\alpha)} (\alpha-1)(b-s)^{\alpha-4} \left[-(\alpha-3)(\alpha-2)(b-a)^2 + 2(b-s)^2 \right],$$

then

$$\frac{\partial g_1(b,s)}{\partial s} = 0 \text{ for } s = s^* = b - \left(\frac{(\alpha-3)(\alpha-2)}{2} \right)^{\frac{1}{2}} (b-a)\,.$$

Hence

$$\max_{s\in[a,b]} g_1(b,s) = \frac{1}{2\Gamma(\alpha)}\left[(\alpha-1)(\alpha-2)(b-a)^2(b-s^*)^{\alpha-3} - 2(b-s^*)^{\alpha-1}\right],$$

by computation we get

$$\max_{s\in[a,b]} g_1(b,s) = g_1(b,s^*) = \frac{1}{\Gamma(\alpha)}\left(\frac{(\alpha-3)^{\frac{1}{2}}(\alpha-2)^{\frac{3}{2}}}{2}\right)^{\alpha-3}(b-a)^{\alpha-1}.$$

Now if $a \le t \le s \le b$,then

$$g_2(t,s) = \frac{1}{2\Gamma(\alpha)}\left[(\alpha-1)(\alpha-2)(t-a)^2(b-s)^{\alpha-3}\right] \ge 0$$

and

$$\frac{\partial g_2(t,s)}{\partial t} = \frac{1}{\Gamma(\alpha)}\left[(\alpha-1)(\alpha-2)(t-a)(b-s)^{\alpha-3}\right] \ge 0.$$

Hence

$$\max_{t,s\in[a,b]} g_2(t,s) = \max_{s\in[a,b]} g_2(b,s) = g_2(s,s) =$$

$$\frac{1}{2\Gamma(\alpha)}\left[(\alpha-1)(\alpha-2)(s-a)^2(b-s)^{\alpha-3}\right],$$

and

$$\frac{\partial g_2(s,s)}{\partial s} = \frac{1}{2\Gamma(\alpha)}(\alpha-1)(\alpha-2)(s-a)(b-s)^{\alpha-4}\left[2b+\alpha a-3a-s(\alpha-1)\right].$$

We have

$$\frac{\partial g_2(s,s)}{\partial s} = 0 \text{ for } s = s^* = \frac{2b+\alpha a-3a}{(\alpha-1)}.$$

Hence

$$\max_{s\in[a,b]} g_2(s,s) = g_2(s^*,s^*) = \frac{2}{\Gamma(\alpha)}\frac{(\alpha-2)}{(\alpha-1)}\left(\frac{\alpha-3}{\alpha-1}\right)^{\alpha-3}(b-a)^{\alpha-1}$$

Now we need to compare $g_1(b,s^*)$ and $g_2(s^*,s^*)$. We have

$$g_1(b,s^*) \ge g_2(s^*,s^*)$$

then

$$\max_{s\in[a,b]} G(b,s) = g_1(b,s^*) = \frac{1}{\Gamma(\alpha)}\left(\frac{(\alpha-3)^{\frac{1}{2}}(\alpha-2)^{\frac{3}{2}}}{2}\right)^{\alpha-3}(b-a)^{\alpha-1}.$$

□

Now we are ready to give the Lyapunov type inequality for problem (1.1).

Theorem 3.1. *Let q is be a real continuous function. If the fractional boundary value problem*

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t)+q(t)\,u(t)=0, & a\le t\le b,\\ u(a)=u'(a)=u'''(a)=u''(b)=0, \end{cases} \tag{3.7}$$

has a nontrivial continuous solution, then

$$\int_a^b |q(t)|\,ds \ge \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\left(\frac{2}{(\alpha-3)^{\frac{1}{2}}(\alpha-2)^{\frac{3}{2}}}\right)^{\alpha-3}. \tag{3.8}$$

Proof. Let $X=C[a,b]$ be the Banach space with the norm $\|u(t)\|_\infty=\max_{a\le t\le b}|u(t)|$. Since

$$u(t)=\int_a^b G(t,s)\,q(s)\,u(s)\,ds, \qquad t\in[a,b],$$

then

$$\|u\|\le\int_a^b \max_{t,s\in[a,b]}|G(t,s)|\,|q(s)|\,ds\,\|u\|.$$

In view of Lemma 3.2, we get

$$1\le\frac{1}{\Gamma(\alpha)}\left(\frac{(\alpha-3)^{\frac{1}{2}}(\alpha-2)^{\frac{3}{2}}}{2}\right)^{\alpha-3}(b-a)^{\alpha-1}\int_a^b|q(s)|\,ds,$$

from which the inequality in (3.8) follows. □

4. Application to a fractional eigenvalue problem

We present an application of the obtained results to eigenvalue problems.

Corollary 4.1. *Assume that $3<\alpha\le 4$. If λ is an eigenvalue to the fractional boundary value problem (1.1), then*

$$|\lambda|\ge\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\left(\frac{2}{(\alpha-3)^{\frac{1}{2}}(\alpha-2)^{\frac{3}{2}}}\right)^{\alpha-3}.$$

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