Advances in Mathematics: Scientific Journal **11** (2022), no.11, 1071–1084
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.11.8

A NOTE ON NONDEGENERATE POISSON STRUCTURE

Ange Maloko Mavambou[1], Norbert Mahoungou Moukala, and Vann Borhen Nkou

ABSTRACT. The Schouten-Nijenhuis bracket on the module of Kähler differentials is introduced. We recover Lichnerowicz's notion of Poisson manifold by using the universal property of derivations. We prove using the Schouten-Nijenhuis bracket that a nondegenerate Poisson structure corresponds exactly to a symplectic structure. Finally, we explore the notion of Hamiltonian vector fields on a nondegenerate Poisson manifold in terms of derivations.

1. INTRODUCTION

Poisson algebras were introduced by Lichnerowicz [6] as the algebraic structure on the ring of C^∞ functions on a certain kind of smooth manifolds, called Poisson manifolds. Poisson algebras are a generalization of symplectic algebras. The Poisson bracket is a derivation on the commutative algebra endowed with a Lie bracket.

All the objects that we consider are assumed to be C^∞-smooth and we follow the usual notation of differential geometric literature [4]. Let M be a smooth manifold, $C^\infty(M)$ the commutative algebra of smooth functions on M and E a

[1]*corresponding author*
2020 *Mathematics Subject Classification.* 17B10, 17B15, 16W25.
Key words and phrases. Schouten-Nijenhuis bracket, nondegenerate Poisson structure, symplectic manifold, Hamiltonian vector fields.
Submitted: 19.10.2022; *Accepted:* 04.11.2022; *Published:* 22.11.2022.

certain $C^\infty(M)$-module. A derivation on $C^\infty(M)$ with coeffcients in $\mathbb{R}$ is a $\mathbb{R}$-linear map $\varphi : C^\infty(M) \longrightarrow E$ such that

$$\varphi(f \cdot g) = \varphi(f) \cdot g + f \cdot \varphi(g)$$

for any $f, g \in C^\infty(M)$. We denote by $Der_{\mathbb{R}}[C^\infty(M), E]$ the $C^\infty(M)$-module of derivations on $C^\infty(M)$ with coeffcients in $\mathbb{R}$ and $Der_{\mathbb{R}}[C^\infty(M)]$ the $C^\infty(M)$-module of derivations on $C^\infty(M)$. Let $\mathfrak{X}(M)$ be the well-known $C^\infty(M)$-module and vector fields act as derivations on smooth functions, the map

$$D : \mathfrak{X}(M) \times C^\infty(M) \longrightarrow C^\infty(M), (X, f) \longmapsto D_X(f) := X(f)$$

satisfies the equation $D_X(f \cdot g) = D_X(f) \cdot g + f \cdot D_X(g)$, for any $f, g \in C^\infty(M)$.

The aim of this paper is to study the nondegenerate Poisson structures by using the universal property of derivations.

The paper is organized as follows. Section 2 contains some generalities about the universal property of derivations and the Schouten-Nijenhuis bracket on the module of Kähler differentials. In section 3, we recall the notion of Poisson structure and we recover Lichnerowicz's notion of Poisson manifold by using the universal property of derivations. In section 4, we introduce the Koszul bracket associated with a Kähler 2-form and we give the relation between Schouten-Nijenhuis bracket and Koszul bracket associated with a Kähler 2-form. In section 5, we prove using the Schouten-Nijenhuis bracket that a nondegenerate Poisson structure corresponds exactly to a symplectic structure. Finally, in section 6, we explore the notion of hamiltonian vector fields on a Poisson manifold defined by the symplectic manifold in terms of derivations.

2. Preliminaries

Let M be a smooth manifold and denote by $\Omega_{\mathbb{R}}[C^\infty(M)]$ the module of Kähler differentials of commutative algebra $C^\infty(M)$, that is, the quotient space $\Omega_{\mathbb{R}}[C^\infty(M)] = I/I^2$, where I is the $C^\infty(M)$-submodule of $C^\infty(M) \bigotimes_{\mathbb{R}} C^\infty(M)$ generated by the elements of the form $f \otimes 1_{C^\infty(M)} - 1_{C^\infty(M)} \otimes f$ with $f \in C^\infty(M)$ (see [1], [9]). The linear map $\delta_M : C^\infty(M) \longrightarrow \Omega_{\mathbb{R}}[C^\infty(M)]$ defined by

$$\delta_M(f) = \overline{f \otimes 1_{C^\infty(M)} - 1_{C^\infty(M)} \otimes f}$$

is the canonical derivation which the image of δ_M generates the $C^\infty(M)$-module $\Omega_{\mathbb{R}}[C^\infty(M)]$, that is, for $\alpha \in \Omega_{\mathbb{R}}[C^\infty(M)]$, we have $\alpha = \sum_{i\in I:finite} f_i \cdot \delta_M(g_i)$, with $f_i, g_i \in C^\infty(M)$.

Theorem 2.1. *[1] The pair $(\Omega_{\mathbb{R}}[C^\infty(M)], \delta_M)$ satisfies the following universal property: for every $C^\infty(M)$-module E and for every derivation $D : C^\infty(M) \longrightarrow E$, there exists a unique $C^\infty(M)$-linear map $\widetilde{D} : \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow E$ such that $\widetilde{D} \circ \delta_M = D$. Moreover, the linear mapping*

$$Hom_{C^\infty(M)}(\Omega_{\mathbb{R}}[C^\infty(M)], E) \longrightarrow Der_{\mathbb{R}}(C^\infty(M), E), \psi \longmapsto \psi \circ \delta_M$$

is an isomorphism of $C^\infty(M)$-modules. In particular,

$$(\Omega_{\mathbb{R}}[C^\infty(M)])^* \simeq Der_{\mathbb{R}}[C^\infty(M)].$$

For any $p \in \mathbb{N}$, $\Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)]) = \mathfrak{L}^p_{sks}(\Omega_{\mathbb{R}}[C^\infty(M)], C^\infty(M))$ denotes the $C^\infty(M)$-module of skew-symmetric multilinear forms of degree p from $\Omega_{\mathbb{R}}[C^\infty(M)]$ into $C^\infty(M)$ and

$$\Lambda(\Omega_{\mathbb{R}}[C^\infty(M)]) = \bigoplus_{p\in\mathbb{N}} \Lambda^p(\Omega_{\mathbb{R}}[C^\infty(M)]),$$

the exterior $C^\infty(M)$-algebra of $\Omega_{\mathbb{R}}[C^\infty(M)]$. Denote by $\mathcal{D}er^p_{sks}[C^\infty(M)]$ the $C^\infty(M)$-module of the skew-symmetric p-derivations of $C^\infty(M)$, that is, $D \in \mathcal{D}er^p_{sks}[C^\infty(M)]$ if the map

$$D^i = D\left(f_1, \ldots, \widehat{f_i}, \ldots, f_p\right) : C^\infty(M) \longrightarrow C^\infty(M)$$
$$f_i \longmapsto D\left(f_1, f_2, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_p\right)$$

is a derivation [9].

Theorem 2.2. *[9] For any $D \in \mathcal{D}er^p_{sks}[C^\infty(M)]$, there exists a unique skew-symmetric $C^\infty(M)$-multilinear map of degree p, $\tilde{D} : [\Omega_{\mathbb{R}}[C^\infty(M)]]^p \longrightarrow C^\infty(M)$ such that*

$$\tilde{D}\left(\delta_M\left(f_1\right), \ldots, \delta_M\left(f_p\right)\right) = D\left(f_1, \ldots, f_p\right) \tag{2.1}$$

and there exists a unique $C^\infty(M)$-linear map $\overline{D} : \Lambda^p\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right) \longrightarrow E$ such that

$$\overline{D}\left(\delta_M\left(f_1\right) \wedge \delta_M\left(f_2\right) \wedge \ldots \wedge \delta_M\left(f_p\right)\right) = D\left(f_1, f_2, \ldots, f_p\right), \tag{2.2}$$

for any $f_1, \ldots, f_p \in C^\infty(M)$.

When we denote $\tilde{D} = P \in \Lambda^p\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$ and $i_P = \bar{D}$, then for any $f_1, \ldots, f_p \in C^\infty(M)$,

$$\begin{aligned} i_P\left(\delta_M(f_1) \wedge \delta_M(f_2) \wedge \ldots \wedge \delta_M(f_p)\right) &= P\left(\delta_M(f_1), \delta_M(f_2), \ldots, \delta_M(f_p)\right) \\ &= D\left(f_1, \ldots, f_p\right). \end{aligned}$$

For $\pi \in \Lambda^2\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$ and $f, g, h \in C^\infty(M)$, we have

$$\begin{aligned} & i_\pi\left(\delta_M(f) \wedge \delta_M(g) \wedge \delta_M(h)\right) \\ &= i_\pi\left(\delta_M(f) \wedge \delta_M(g)\right) \cdot \delta_M(h) - i_\pi\left(\delta_M(f) \wedge \delta_M(h)\right) \cdot \delta_M(g) \\ &\quad + i_\pi\left(\delta_M(g) \wedge \delta_M(h)\right) \cdot \delta_M(f) \\ &= \pi\left(\delta_M(f), \delta_M(g)\right) \cdot \delta_M(h) - \pi\left(\delta_M(f), \delta_M(h)\right) \cdot \delta_M(g) \\ &+ \pi\left(\delta_M(g), \delta_M(h)\right) \cdot \delta_M(f) \end{aligned}$$

Definition 2.1. *For $P \in \Lambda^p\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$ and $Q \in \Lambda^q\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$, the Schouten-Nijenhuis bracket of P and Q is a mapping*

$$[.,.]_S : \Lambda^p\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right) \times \Lambda^q\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right) \longrightarrow \Lambda^{p+q-1}\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$$

such that

$$[P, Q]_S = P \circ Q - (-1)^{(p-1)(q-1)}\, Q \circ P, \tag{2.3}$$

where

$$\begin{aligned} & (Q \circ P)\left(\delta_M(f_1), \delta_M(f_2), \ldots, \delta_M(f_{p+q-1})\right) \\ = & \sum_{\sigma \in S_{p,q-1}} (-1)^\sigma\, \tilde{Q}\left[\tilde{P}\left(f_{\sigma(1)}, f_{\sigma(2)}, \cdots f_{\sigma(p)}\right), f_{\sigma(p+1)}, \ldots, f_{\sigma(p+q-1)}\right], \end{aligned}$$

and $\tilde{P} = D \in \mathcal{D}er^p_{\mathbb{R}}[C^\infty(M)]$ is a unique skew-symmetric p-derivation such that

$$D\left(f_1, \ldots, f_p\right) = P\left(\delta_M\left(f_1\right), \ldots, \delta_M\left(f_p\right)\right).$$

Throughout this section, we denote $[.,.]_S$ by an unadorned bracket $[,]$. The description of interior product $P \in \Lambda\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$ with the Schouten bracket is similar to the interior product defined in [5] and [8]. Then, if P and Q are two elements of the $\Lambda\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$, then

$$i_{[P,Q]} = \left[\left[i_P, \delta_M\right], i_Q\right]. \tag{2.4}$$

If $P \in \Lambda^p\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$, then i_P is of degree $-p$. So,

$$[i_P, \delta_M] = i_P \circ \delta_M - (-1)^{-p}\, \delta_M \circ i_P. \tag{2.5}$$

Now, assuming $Q \in \Lambda^q\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$, we have

$$\begin{aligned} \left[[i_P, \delta_M], i_Q\right] &= [i_P, \delta_M] \circ i_Q - (-1)^{-q(1-p)}\, i_Q \circ [i_P, \delta_M] \\ &= i_P \circ \delta_M \circ i_Q - (-1)^{-p}\, \delta_M \circ i_{P\wedge Q} \\ &\quad - (-1)^{-q(1-p)}\, i_{P\wedge Q} \circ \delta_M + (-1)^{-q(1-p)-p}\, i_Q \circ \delta_M \circ i_P. \end{aligned} \tag{2.6}$$

Proposition 2.1. *For any* $\pi \in \Lambda^2\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$ *and* $\eta \in [\Omega_{\mathbb{R}}[C^\infty(M)]]^p$*, we have*

$$i_{[\pi,\pi]}\eta = 2 i_\pi \delta_M i_\pi \eta. \tag{2.7}$$

Proof. When we use equations (2.4) and (2.6) with $P = Q = \pi$, we get

$$\begin{aligned} i_{[\pi,\pi]}\eta &= i_\pi \delta_M i_\pi \eta + (-1)^{-2}\, \delta_M i_{\pi\wedge\pi}\eta \\ &\quad - (-1)^{-2(1-2)}\, i_{\pi\wedge\pi}\delta_M \eta + (-1)^{-2(1-2)-2}\, i_\pi \delta_M i_\pi \eta \\ &= i_\pi \delta_M i_\pi \eta + (-1)^{2-2}\, i_\pi \delta_M i_\pi \eta \\ &= 2 i_\pi \delta_M i_\pi \eta. \end{aligned}$$

□

Proposition 2.2. *If* $\pi \in \Lambda^2\left(\Omega_{\mathbb{R}}[C^\infty(M)]\right)$ *and* $f, g, h \in C^\infty(M)$*, then*

$$\frac{1}{2}[\pi,\pi]\left(\delta_M(f), \delta_M(g), \delta_M(h)\right) = \oint \pi\left(\delta_M\left(\pi\left(\delta_M(f), \delta_M(g)\right)\right), \delta_M(h)\right) \tag{2.8}$$

where the symbol $\oint$ *means the cyclic sum in* f, g, h.

Proof. For $\eta = \delta_M(f) \wedge \delta_M(g) \wedge \delta_M(h)$, we have

$$\begin{aligned} i_\pi \eta &= \pi\left(\delta_M(f), \delta_M(g)\right) \cdot \delta_M(h) - \pi\left(\delta_M(f), \delta_M(h)\right) \cdot \delta_M(g) \\ &\quad + \pi\left(\delta_M(g), \delta_M(h)\right) \cdot \delta_M(f), \\ \delta_M i_\pi \eta &= \delta_M\left(\pi\left(\delta_M(f), \delta_M(g)\right)\right) \wedge \delta_M(h) - \delta_M\left(\pi\left(\delta_M(f), \delta_M(h)\right)\right) \wedge \delta_M(g) \\ &\quad + \delta_M\left(\pi\left(\delta_M(g), \delta_M(h)\right)\right) \wedge \delta_M(f), \end{aligned}$$

and

$$\begin{aligned} i_\pi \delta_M i_\pi \eta &= \pi\left(\delta_M\left(\pi\left(\delta_M(f), \delta_M(g)\right)\right), \delta_M(h)\right) + \pi\left(\delta_M\left(\pi\left(\delta_M(h), \delta_M(f)\right)\right), \delta_M(g)\right) \\ &\quad + \pi\left(\delta_M\left(\pi\left(\delta_M(g), \delta_M(h)\right)\right), \delta_M(f)\right). \end{aligned}$$

Since

$$i_{[\pi,\pi]}\eta = 2i_\pi \delta_M i_\pi \eta,$$

then

$$\frac{1}{2}\left[\pi,\pi\right]\left(\delta_M(f),\delta_M(g),\delta_M(h)\right) = \oint \pi\left(\delta_M\left(\pi\left(\delta_M(f),\delta_M(g)\right)\right),\delta_M(h)\right).$$

□

3. POISSON STRUCTURES

We recall that a Poisson bracket on a manifold M is a Lie bracket $\{,\}$ on $C^\infty(M)$ satisfying the Leibniz identity

$$\{f, g\cdot h\} = \{f,g\}\cdot h + g\cdot\{f,h\} \tag{3.1}$$

for any $f,g,h \in C^\infty(M)$. A Poisson manifold is a manifold equipped with a Jacobi bracket (see [3], [6]). The Leibniz identity means that, for a given function $f \in C^\infty(M)$ on a Poisson manifold M, the inner derivation $ad(f) : C^\infty(M) \longrightarrow C^\infty(M), g \longmapsto \{f,g\}$ is a derivation of a commutative algebra $C^\infty(M)$.

If M is a Poisson manifold, the bracket $\{,\}$ is a skew-symmetric 2-derivation. By the Theorem 2.2, there exists $\pi \in \Lambda^2\left(\Omega_{\mathbb{R}}\left[C^\infty(M)\right]\right)$ such that

$$\{f,g\} = \pi(\delta_M(f),\delta_M(g)), \tag{3.2}$$

for any f and g in $C^\infty(M)$.

Consider the Jacobiator $J\left(.,.,.\right)$ defined as

$$J\left(f,g,h\right) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}$$

for $f,g,h \in C^\infty(M)$.

Lemma 3.1. *For all $f,g,h \in C^\infty(M)$, we have*

$$J\left(f,g,h\right) = \left(\frac{1}{2}[\pi,\pi]\right)\left(\delta_M\left(f\right),\delta_M\left(g\right),\delta_M\left(h\right)\right).$$

Proof. Using the equation (3.2), for any $f,g,h \in C^\infty(M)$, we have

$$\{\{f,g\},h\} = \pi(\delta_M(\{f,g\}),\delta_M\left(h\right)).$$

Using the skew-symmetry of π and grouping relevant terms together, we get

$$J(f,g,h) = \oint \pi\left(\delta_M\left(\pi\left(\delta_M(f), \delta_M(g)\right)\right), \delta_M(h)\right)$$
$$= \left(\frac{1}{2}[\pi,\pi]\right)\left(\delta_M(f), \delta_M(g), \delta_M(h)\right)$$

□

Theorem 3.1. *For all $f, g \in C^{\infty}(M)$, the bracket*

$$\{f,g\} = \pi(\delta_M(f), \delta_M(g))$$

satisfies the Jacobi identity if and only if

$$[\pi,\pi] = 0. \tag{3.3}$$

Proof. Assume that the bracket $\{,\}$ satisfies the Jacobi identity, then from the Lemma 3.1, $[\pi,\pi] = 0$.

Conversely, assume the equations (3.3), then from the Lemma 3.1, we have,

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

□

The skew-symmetric 2-form π on $\Omega_{\mathbb{R}}[C^{\infty}(M)]$ is called Poisson 2-form of the Poisson manifold M and the pair (M,π) is called Poisson manifold.

4. Koszul bracket

If (M,π) is a Poisson manifold, then the map

$$ad : C^{\infty}(M) \longrightarrow Der_{\mathbb{R}}[C^{\infty}(M)], f \longmapsto ad(f)$$

is a derivation. Thus, by the Theorem 2.1, there exists a unique $C^{\infty}(M)$-linear map

$$\overline{ad} : \Omega_{\mathbb{R}}[C^{\infty}(M)] \longrightarrow Der_{\mathbb{R}}[C^{\infty}(M)]$$

such that,

$$\overline{ad} \circ \delta_M = ad. \tag{4.1}$$

Then, from (3.2), we have, for any $f \in C^{\infty}(M)$ and $\alpha, \beta \in \Omega[C^{\infty}(M)]$

$$\left[\overline{ad}(\alpha)\right](f) = \pi\left(\alpha, \delta_M(f)\right), \tag{4.2}$$

$$\overline{\left[\overline{ad}\,(\alpha)\right]}\,(\beta) = \pi\,(\alpha, \beta)\,. \tag{4.3}$$

For any derivation $D : C^{\infty}(M) \longrightarrow C^{\infty}(M)$, the Lie derivative with respect to D is the map

$$\mathfrak{L}_D : \Lambda^p(\Omega_{\mathbb{R}}[C^{\infty}(M)]) \longrightarrow \Lambda^p(\Omega_{\mathbb{R}}[C^{\infty}(M)])$$

such that for $\eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^{\infty}(M)])$ and $x_1, \ldots, x_p \in \Omega_{\mathbb{R}}[C^{\infty}(M)]$,

$$(\mathfrak{L}_{D)}\eta)(x_1, \ldots, x_p) = D\,[\eta(x_1, \ldots, x_p)] - \sum_{i=1}^{p} \eta(x_1, \ldots, [x, x_i]\,, x_{i+1}, \ldots, x_p).$$

Let M be a manifold. The 2-form π induces on $\Omega_{\mathbb{R}}[C^{\infty}(M)]$ a bracket $[.,.]_{\pi}$ called the Koszul bracket:

$$[\alpha, \beta]_{\pi} = \mathfrak{L}_{\overline{ad}(\alpha)}\beta - \mathfrak{L}_{\overline{ad}(\beta)}\alpha - \delta_M(\pi(\alpha, \beta)), \tag{4.4}$$

for all $\alpha, \beta \in \Omega_{\mathbb{R}}\,[C^{\infty}(M)]$. Let

$$d_{\overline{ad}} : \mathfrak{L}^k_{sks}(\Omega_{\mathbb{R}}[C^{\infty}(M)], C^{\infty}(M)) \longrightarrow \mathfrak{L}^{k+1}_{sks}(\Omega_{\mathbb{R}}[C^{\infty}(M)], C^{\infty}(M)), \quad Q \longmapsto d_{\overline{ad}}Q$$

be the operator associated to $\overline{ad}$ such that, for any $\alpha_1, \ldots, \alpha_{k+1} \in \Omega_{\mathbb{R}}[C^{\infty}(M)$,

$$\begin{aligned} d_{\overline{ad}}Q(\alpha_1, \ldots, \alpha_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i-1}\overline{ad}(\alpha_i) \cdot Q(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1}) \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} Q([\alpha_i, \alpha_j]_{\pi}, \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1}). \end{aligned}$$

For any $\pi \in \mathfrak{L}^2_{alt}(\Omega_{\mathbb{R}}[C^{\infty}(M)], C^{\infty}(M))$ and $Q \in \mathfrak{L}^k_{alt}(\Omega_{\mathbb{R}}[C^{\infty}(M)], C^{\infty}(M))$, we have

$$d_{\overline{ad}}Q = -[\pi, Q]. \tag{4.5}$$

In particular, if $Q = \pi$, then for any $\alpha, \beta, \gamma \in \Omega_{\mathbb{R}}[C^{\infty}(M)]$,

$$[\pi, \pi](\alpha, \beta, \gamma) = -\oint \overline{ad}(\alpha) \cdot \pi(\beta, \gamma) + \oint \pi([\alpha, \beta]_{\pi}, \gamma) \tag{4.6}$$

where the symbol $\oint$ means the cyclic sum in α, β, γ.

For any $\alpha, \beta, \gamma \in \Omega_{\mathbb{R}}[C^{\infty}(M)]$,

$$\pi\,(\delta_M(\pi(\alpha, \beta)), \gamma) = -\overline{ad}(\gamma) \cdot \pi(\alpha, \beta), \tag{4.7}$$

$$\pi\left(\mathfrak{L}_{\overline{ad}(\alpha)}\beta, \gamma\right) = \overline{ad}(\alpha) \cdot \pi(\beta, \gamma) - \left(\overline{\left[\overline{ad}(\gamma), \overline{ad}(\alpha)\right]}\right)(\beta)\,. \tag{4.8}$$

Proposition 4.1. *For any* $\alpha,\beta,\gamma\in\Omega_{\mathbb{R}}[C^\infty(M)]$,

$$\pi([\alpha,\beta]_\pi,\gamma)-\left(\overline{[\overline{ad}(\alpha),\overline{ad}(\beta)]}\right)(\gamma)=\frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma). \tag{4.9}$$

Proof. For $\alpha,\beta,\gamma\in\Omega[C^\infty(M)]$, by (4.4), we have

$$\pi([\alpha,\beta]_\pi,\gamma)=\pi\left(\mathfrak{L}_{\overline{ad}(\alpha)}\beta,\gamma\right)-\pi\left(\mathfrak{L}_{\overline{ad}(\beta)}\alpha,\gamma\right)-\pi\left(\delta_M(\pi(\alpha,\beta)),\gamma)\right).$$

Put

$$\phi(\alpha,\beta,\gamma)=\left(\left[\overline{\overline{ad}[\alpha,\beta]_\pi}\right]-\left(\overline{[\overline{ad}(\alpha),\overline{ad}(\beta)]}\right)\right)(\gamma) \tag{4.10}$$

for $\alpha,\beta,\gamma\in\Omega[C^\infty(M)]$. From (4.7) and (4.8), the formula (4.10) becomes

$$\begin{aligned}\phi(\alpha,\beta,\gamma)&=\pi([\alpha,\beta]_\pi,\gamma)-\left(\overline{[\overline{ad}(\alpha),\overline{ad}(\beta)]}\right)(\gamma)\\&=\overline{ad}(\alpha)\cdot\pi(\beta,\gamma)+\overline{ad}(\beta)\cdot\pi(\gamma,\alpha)+\overline{ad}(\gamma)\cdot\pi(\alpha,\beta)\\&\quad-\left(\overline{[\overline{ad}(\beta),\overline{ad}(\gamma)]}\right)(\alpha)-\left(\overline{[\overline{ad}(\gamma),\overline{ad}(\alpha)]}\right)(\beta)\\&\quad-\left(\overline{[\overline{ad}(\alpha),\overline{ad}(\beta)]}\right)(\gamma),\end{aligned}$$

i.e.,

$$\phi(\alpha,\beta,\gamma)=\oint\overline{ad}(\alpha)\cdot\pi(\beta,\gamma)-\oint\left(\overline{[\overline{ad}(\beta),\overline{ad}(\gamma)]}\right)(\alpha).$$

From (4.6), we have

$$\begin{aligned}\phi(\alpha,\beta,\gamma)&=-[\pi,\pi](\alpha,\beta,\gamma)+\pi([\alpha,\beta]_\pi,\gamma)+\pi([\beta,\gamma]_\pi,\alpha)+\pi([\gamma,\alpha]_\pi,\beta)\\&\quad-\left(\overline{[\overline{ad}(\beta),\overline{ad}(\gamma)]}\right)(\alpha)-\left(\overline{[\overline{ad}(\gamma),\overline{ad}(\alpha)]}\right)(\beta)\\&\quad-\left(\overline{[\overline{ad}(\alpha),\overline{ad}(\beta)]}\right)(\gamma)\\&=-[\pi,\pi](\alpha,\beta,\gamma)+\phi(\beta,\gamma,\alpha)+\phi(\gamma,\alpha,\beta)+\phi(\alpha,\beta,\gamma).\end{aligned}$$

Hence, since ϕ is an alternating map,we get

$$[\pi,\pi](\alpha,\beta,\gamma)=2\phi(\alpha,\beta,\gamma).$$

Thus,

$$\pi([\alpha,\beta]_\pi,\gamma)-\left(\overline{[\overline{ad}(\alpha),\overline{ad}(\beta)]}\right)(\gamma)=\frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma).$$

□

5. NONDEGENERATE POISSON STRUCTURE

A smooth manifold M is called a symplectic manifold, if there is defined on M a closed nondegenerate 2-form η, that is, an $\eta \in \Lambda^2(M)$ such that

(i) $d\eta = 0$,

(i) the map $\eta^\flat : \mathfrak{X}(M) \longrightarrow (\Lambda^1(M))^*, X \longmapsto i_X\eta$ is an isomorphism of $C^\infty(M)$-modules, where $(i_X\eta)(Y) = \eta(X,Y)$ pour any $Y \in \mathfrak{X}(M)$ (see [2], [7]) for more details.

We denote (M,η) a symplectic manifold.

A Poisson structure is nondegenerate if the Poisson 2-form π is nondegenerate i.e., the map

$$\pi^\flat : \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow \Omega_{\mathbb{R}}[C^\infty(M)]^*, \alpha \longmapsto i_\alpha\pi$$

is an isomorphism of $C^\infty(M)$-modules, where $(i_\alpha\pi)(\beta) = \pi(\alpha,\beta)$.

If the Poisson 2-form π is nondegenerate, then the map $\overline{ad}$ is an isomorphism of $C^\infty(M)$-modules. Indeed, the map

$$\overline{ad} = K \circ \omega_M^\flat : \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow Der_{\mathbb{R}}[C^\infty(M)]$$

is an isomorphism of $C^\infty(M)$-modules, where

$$K : \Omega_{\mathbb{R}}[C^\infty(M)]^* \longrightarrow Der_{\mathbb{R}}[C^\infty(M)], \varphi \longmapsto \varphi \circ \delta_M$$

is an isomorphism of $C^\infty(M)$-modules defined by the Theorem 2.1, and for any $f, g \in C^\infty(M)$,

$$\left(K \circ \pi^\flat\right)(\delta_M(f))(g) = \left[i_{\delta_M(f)}\pi \circ \delta_M\right](g) = \overline{ad}(\delta_M(f))(g).$$

Proposition 5.1. *If (M,π) is a Poisson manifold and if π is nondegenerate, then there exists $\eta \in \Lambda^2(M)$ such that for any $X, Y \in \mathfrak{X}(M)$ and $\alpha, \beta \in \Omega_{\mathbb{R}}[C^\infty(M)]$,*

$$\eta(X,Y) = \pi(\alpha,\beta). \tag{5.1}$$

Proof. If $\pi : \Omega_{\mathbb{R}}[C^\infty(M)] \times \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow C^\infty(M)$ is nondegenerate, then the map

$$\overline{ad} : \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow \Omega_{\mathbb{R}}[C^\infty(M)]^* \cong \mathcal{D}er_{\mathbb{R}}[C^\infty(M)]$$

is an isomorphism. Let $\overline{ad}^{-1}$ be the inverse isomorphism of $\overline{ad}$. Thus, for any $X, Y \in \mathfrak{X}(M)$, there exists $\alpha, \beta \in \Omega_{\mathbb{R}}[C^\infty(M)]$ such that $\alpha = \overline{ad}^{-1}(X)$, $\beta = \overline{ad}^{-1}(Y)$.

The map

$$\eta = \pi \circ (\overline{ad}^{-1} \times \overline{ad}^{-1}) : Der_{\mathbb{R}}[C^\infty(M)] \times Der_{\mathbb{R}}[C^\infty(M)] \longrightarrow C^\infty(M)$$

satisfies

$$\eta(X, Y) = \pi \circ (\overline{ad}^{-1} \times \overline{ad}^{-1})(X, Y) = \pi(\overline{ad}^{-1}(X), \overline{ad}^{-1}(Y)) = \pi(\alpha, \beta).$$

□

Proposition 5.2. *For any $X, Y, Z \in \mathfrak{X}(M)$ and for any $\alpha, \beta, \gamma \in \Omega_{\mathbb{R}}[C^\infty(M)]$, we have*

$$\eta([X, Y], Z) = -\frac{1}{2}[\pi, \pi](\alpha, \beta, \gamma) + \pi([\alpha, \beta]_\pi, \gamma). \tag{5.2}$$

Proof. For $X, Y, Z \in \mathfrak{X}(M)$ and for $\alpha, \beta, \gamma \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$\eta([X, Y], Z) = \pi \circ (\overline{ad}^{-1} \times \overline{ad}^{-1})([X, Y], Z) = \pi(\overline{ad}^{-1}([\overline{ad}(\alpha), \overline{ad}(\beta)]), \gamma)$$

From (4.3), we have

$$\eta([X, Y], Z) = \overline{\left(\overline{ad}\left[\overline{ad}^{-1}([\overline{ad}(\alpha), \overline{ad}(\beta)])\right]\right)}(\gamma) = \left(\overline{[\overline{ad}(\alpha), \overline{ad}(\beta)]}\right)(\gamma).$$

By (4.9), it follows that

$$\eta([X, Y], Z) = -\frac{1}{2}[\pi, \pi](\alpha, \beta, \gamma) + \pi([\alpha, \beta]_\pi, \gamma).$$

□

Lemma 5.1. *For any $X, Y, Z \in \mathfrak{X}(M)$ and for any $\alpha, \beta, \gamma \in \Omega_{\mathbb{R}}[C^\infty(M)]$, we have*

$$(d\eta)(X, Y, Z) = \left(\frac{1}{2}[\pi, \pi]\right)(\alpha, \beta, \gamma). \tag{5.3}$$

Proof. For $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned} d\eta(X, Y, Z) &= X \cdot [\eta(Y, Z)] - Y \cdot [\eta(X, Z)] + Z \cdot [\eta(X, Y)] \\ &\quad - \eta([X, Y], Z) + \eta([X, Z], Y) - \eta([Y, Z], X). \end{aligned}$$

From expressions (5.1) and (5.2), we have

$$\begin{aligned} d\eta(X,Y,Z) &= \overline{ad}(\alpha)\cdot\pi(\beta,\gamma) - \overline{ad}(\beta)\cdot\pi(\alpha,\gamma) + \overline{ad}(\gamma)\cdot\pi(\alpha,\beta) \\ &+ \frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma) - \pi([\alpha,\beta]_\pi,\gamma) \\ &- \frac{1}{2}[\pi,\pi]\,(\alpha,\gamma,\beta) + \pi([\alpha,\gamma]_\pi,\beta) \\ &+ \frac{1}{2}[\pi,\pi]\,(\beta,\gamma,\alpha) - \pi([\beta,\gamma]_\pi,\alpha), \end{aligned}$$

that is,

$$\begin{aligned} d\eta(X,Y,Z) = & \oint \overline{ad}(\alpha)\cdot\pi(\beta,\gamma) - \oint \pi([\alpha,\beta]_\pi,\gamma) \\ & + \frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma) - \frac{1}{2}[\pi,\pi]\,(\alpha,\gamma,\beta) + \frac{1}{2}[\pi,\pi](\beta,\gamma,\alpha). \end{aligned}$$

From (4.6), we get

$$d\eta(X,Y,Z) = \frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma). \tag{5.4}$$

□

Theorem 5.1. *The pair (M,π) is a nondegenerate Poisson manifold if and only if the pair (M,η) is a symplectic structure.*

Proof. From the Lemma 5.1, we deduce that the identity $[\pi,\pi]=0$ is satisfied if and only if the identity $d\eta = 0$ is. □

6. Hamiltonian vector fields

When (M,π) is a Poisson manifold, then the map $\delta_M : C^\infty(M) \longrightarrow \Omega_{\mathbb{R}}[C^\infty(M)]$ is a Lie algebras homomorphism, that is

$$\delta_M\left(\{f,g\}\right) = \left[\delta_M(f),\delta_M(g)\right]_\pi, \tag{6.1}$$

for any $f,g\in C^\infty(M)$. From (5.2), since $[\pi,\pi]=0$, then

$$\eta([X,Y],Z) = \pi([\alpha,\beta]_\pi,\gamma), \tag{6.2}$$

for any $X=\overline{ad}(\alpha), Y=\overline{ad}(\beta)$ and $Z=\overline{ad}(\gamma)$ in $Der_{\mathbb{R}}[C^\infty(M)]$.

When (M,η) is a symplectic manifold, we recall that a vector field X on M is locally hamiltonian if the form $i_X\eta$ is closed for the de Rham cohomology. A vector field $X \in \mathfrak{X}(M)$ is globally hamiltonian if the 1-form $i_X\eta$ is d-exact that is there exists $f \in C^\infty(M)$ such that $i_X\omega = -df$ [2] and [7].

Proposition 6.1. *A locally hamiltonian vector field on M is a derivation of the Lie algebra $C^\infty(M)$ induced by the structure of Poisson defined by the symplectic manifold (M, η).*

Proof. Let X be a locally hamiltonian vector field, then $d\,(i_X\eta) = 0$. Thus, for any Y and Z in $\mathfrak{X}(M) \simeq Der_{\mathbb{R}}[C^\infty(M)]$, $(di_X\eta)\,(Y, Z) = 0$ i.e.,

$$\eta\,(X, [Y, Z]) = Y\,[\eta(X, Z)] - Z\,[\eta(X, Y)]\,.$$

From (5.1) and (6.2), we have

$$\pi\,(\alpha, [\beta, \gamma]_\pi) = \overline{ad}(\beta)\,[\pi(\alpha, \gamma)] - \overline{ad}(\gamma)\,[\pi(\alpha, \beta)]\,.$$

Since $\beta = \delta_M(f), \gamma = \delta_M(g)$ with $f, g \in C^\infty(M)$ and by (4.2) and (6.1), we get

$$\left[\overline{ad}(\alpha)\right](\{f, g\}) = ad(f)\left(\left[\overline{ad}(\alpha)\right](g)\right) - ad(g)\left(\left[\overline{ad}(\alpha)\right](f)\right).$$

Thus,

$$\begin{aligned} X(\{f, g\}) &= ad(f)\,(X(g)) - ad(g)\,(X(f)) = \{f, X(g)\} - \{g, X(f)\} \\ &= \{X(f), g\} + \{f, X(g)\}. \end{aligned}$$

for any $f, g \in C^\infty(M)$. □

Proposition 6.2. *A globally hamiltonian vector field on M is a derivation interior of the Lie algebra $C^\infty(M)$ induced by the structure of Poisson defined by the symplectic manifold (M, η).*

Proof. Let X be a globally hamiltonian vector field, there exists $f \in C^\infty(M)$ such that $i_X\omega = -df$. For any $Y \in \mathfrak{X}(M) \simeq Der_{\mathbb{R}}[C^\infty(M)]$, $\eta\,(X, Y) = -Y(f) = -\left[\overline{ad}(\beta)\right](f)$. Since $\beta = \delta_M(g)$ with $g \in C^\infty(M)$, we have

$$\eta\,(X, Y) = -\left[\overline{ad}(\delta_M(g))\right](f) = -ad(g)(f) = -\{g, f\} = ad\,(f)\,(g)\,.$$

On the other hand, from (5.1),

$$\eta\,(X, Y) = \pi\,(\alpha, \delta_M(g)) = \left[\overline{ad}(\alpha)\right](g).$$

Thus, $\left[\overline{ad}(\alpha)\right](g) = ad\,(f)\,(g)$, for any $g \in C^\infty(M)$, that is $X = ad\,(f)$. Thus, X is the derivation interior of the Poisson algebra $C^\infty(M)$. □

ACKNOWLEDGMENT

We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

REFERENCES

[1] N. BOURBAKI: *Algèbre, Chap 10, Algèbre homologique*, Masson, Paris, New York, Barcelone, Milan, 1980.

[2] L. CAMILLE, A.P. GENGOUX, P. VANHAECKE: *Poisson Structures*, Spinger-Verlag Berlin Heildelberg, 2013.

[3] A. KIRILLOV: *Local Lie algebras*, Russian Math. Surveys, **31** (1976), 55–75.

[4] S. KOBAYASHI, K. NOMIZU: *Foundations of Differential Geometry*, Interscience Publishers, vol II, New-York, London, Sydney, 1969.

[5] J.-L. KOSZUL: *Crochet de Schouten-Nijenhuis et Cohomologies*, Astérisques, **S131** (1985), 257-271.

[6] A. LICHNEROWICZ: *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Diff. Geom, **12** (1977), 253–300.

[7] P. LIBERMANN: *Sur le problème d'équivalence de certaines structures infinitésimales régulières*, Ann. Mat. Pura. Appl, **36** (1954), 27-120.

[8] C.-M. MARLE: *The Schouten-Nijenhuis bracket and interior products*, Journal of geometry and Physics, **23** (1997), 350-359.

[9] E. OKASSA: *Algèbres de Jacobi et Algèbres de Lie-Rinehart-Jacobi*, J.of Pure and Applied Algebra, **208** (2007), 1071-1089.

DEPARTMENT OF MATHEMATICS, MARIEN NGOUABI UNIVERSITY, BRAZZAVILLE, CONGO.
Email address: angemaloko@gmail.com

DEPARTMENT OF MATHEMATICS, MARIEN NGOUABI UNIVERSITY, BRAZZAVILLE, CONGO.
Email address: nmahomouk1@gmail.com

DEPARTMENT OF MATHEMATICS, MARIEN NGOUABI UNIVERSITY, BRAZZAVILLE, CONGO.
Email address: vannborhen@gmail.com