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MODELLING OF DYNAMIC SYSTEMS USING A NUMERICAL ALGORITHM BASED ON AN ALGEBRAIC SUBSPACE APPROACH

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ABSTRACT. For the purpose of providing a linear model in state space form of a dynamic system which is appropriate for control and optimization research, this paper presents a systematic method based on algebraic notions for dynamical systems modeling from operational data using numerical methods.

1. INTRODUCTION

The following form describes the mathematical formulation of a linear discrete-time dynamic multivariate state space model [1]

$$\begin{cases} x(h+1) = Ax(h) + Bu(h) + w(h) \\ \quad y(h) = Cx(h) + Du(h) + v(h) \end{cases}, \tag{1.1}$$

with $x(h) \in \mathbb{R}^n$ denotes the states vector, $u(h) \in \mathbb{R}^{m\times 1}$ the input vector, $y(h) \in \mathbb{R}^{l\times 1}$, the output vector, $v(h) \in \mathbb{R}^{l\times 1}$, $w(h) \in \mathbb{R}^{n\times 1}$ are zero-mean unmeasurable vector sequences, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$ and $D \in \mathbb{R}^{l\times m}$ are regular matrices of standard dimensions.

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2. Description of the Numerical algorithm for system identification

The Hankel block matrices [15] constructed by the system operational data are used to generate the subspaces in the numerical technique [2, 3], which has recently been introduced as a potent alternative to the traditional system identification method based on iterative techniques. Several algebraic techniques, including SVD (Singular Value Decomposition), are used to determine the system's order and the Observability matrix, which contains the parameters of the predicted model. In this study, we employ a numerical methodology based on operational data to present the key mathematical methods used in the subspace identification method [16]. In fact, it is typically not able to measure the system's state completely; therefore, in order to stabilize the system, it is frequently necessary to take into consideration the curves provided by the standard equation given by (1.1). With the Extended Observability Matrix Γ_i associated with the state representation given by equation (1.1) is given by [4, 5]:

$$\Gamma_i = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{i-1} \end{pmatrix}, \tag{2.1}$$

where i provide the block's line number, the system matrices are denoted by A and C. the inverted extended controllability matrix of Δ_i^d of equation (1.1) is given by [6] [7]:

$$\Delta_i^d = \begin{pmatrix} A^{i-1}B & A^{i-2}B & \cdots & AB & B \end{pmatrix},$$

with A et B are the system matrices, and i denotes the number of columns. The block is set lower than the triangular Toeplitz matrix H_i , determined by the following formula, which according to it, the block is positioned beneath the triangular Toeplitz matrix:

$$H_i = \begin{pmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ CA^{i-2}B & CA^{i-3}B & CA^{i-4}B & \cdots & D \end{pmatrix},$$

with A, B, C and D denote the system matrices. The input and output Henckel matrices are defined by [8, 10]:

$$U_{0|i-1} = \begin{pmatrix} u_0 & u_1 & u_2 & \dots & u_{j-1} \\ u_1 & u_2 & u_3 & \dots & u_j \\ \vdots & \vdots & \vdots & & \vdots \\ u_{i-1} & u_i & u_{i+1} & \dots & u_{i+j-2} \end{pmatrix},$$

$$U_{i|2i-1} = \begin{pmatrix} u_i & u_{i+1} & u_{i+2} & \dots & u_{i+j-1} \\ u_{i+1} & u_{i+2} & u_{i+3} & \dots & u_{i+j} \\ \vdots & \vdots & \vdots & & \vdots \\ u_{2i-1} & u_{3i} & u_{i+1} & \dots & u_{2i+j-2} \end{pmatrix},$$

$$Y_{0|i-1} = \begin{pmatrix} y_0 & y_1 & y_2 & \dots & y_{j-1} \\ y_1 & y_2 & y_3 & \dots & y_j \\ \vdots & \vdots & \vdots & & \vdots \\ y_{i-1} & y_i & y_{i+1} & \dots & y_{i+j-2} \end{pmatrix},$$

$$Y_{i|2i-1} = \begin{pmatrix} y_i & y_{i+1} & y_{i+2} & \dots & y_{i+j-1} \\ y_{i+1} & y_{i+2} & y_{i+3} & \dots & y_{i+j} \\ \vdots & \vdots & \vdots & & \vdots \\ y_{2i-1} & y_{2i} & y_{2i+1} & \dots & y_{2i+j-2} \end{pmatrix},$$

where $U_{0|i-1}$ denotes preceding input, $U_{i|2i-1}$ the following input, and $Y_{0|i-1}$ the preceding output, $Y_{i|2i-1}$ the following output, u denotes the elements of the input vector, y denotes the output vector, to simplify more, $U_{0|i-1} = U_P$, $U_{i|2i-1} = U_S$, $Y_{0|i-1} = Y_P$ and $Y_{i|2i-1} = Y_S$ The preceding and following state matrices X_P and X_S are explained as:

$$X_p = \begin{pmatrix} x_0 & x_1 & x_2 & \dots & x_{j-1} \end{pmatrix} \in \mathbb{R}^{n\times j},$$

$$X_S = \begin{pmatrix} x_i & x_{i+1} & x_{i+2} & \dots & x_{i+j-1} \end{pmatrix} \in \mathbb{R}^{n\times j},$$

X_P and X_S can be determined by applying the (SVD) and (QR) decompositions to the data frame's Hankel vectors. The matrix of instrumental variables Z_p is presented by [10–12]:

$$Z_p = \begin{pmatrix} U_p & Y_p \end{pmatrix}^T,$$

with: U_p denotes the vectors of the preceding inputs, and Y_P the vectors of preceding outputs. The extended Observability matrix and the state matrix are obtained using the unified method proposed by Van Overschee and Bart De Moor in [3]. Following are definitions of the state matrix input-output equations and main projections:

$$\begin{aligned} X_S &= A^i X_P + \Delta_i U_P, \\ Y_P &= \Gamma_i X_P + H_i U_P, \\ Y_S &= \Gamma_i X_P + H_i U_S, \end{aligned}$$

where U_S is the following input, X_P and X_S are the preceding and following state matrices, Γ_i the extended Observability matrix and A^i is the Kalman filter's block matrix in a non-steady state.

The preceding and the following projections, Z_i and Z_{i+1} are expressed in the form:

$$\begin{cases} Z_i = \Gamma_i \hat{X}_i + H_i U_{i|2\mathrm{i}-1} \\ Z_{i+1} = \Gamma_{i-1} \hat{X}_{i+1} + H_{i-1} U_{i+1|2\mathrm{i}-1} \end{cases},$$

where $U_{i|2\mathrm{i}-1}$ is the following inputs matrix, $\hat{X}_i$ and $\hat{X}_{i+1}$ are the preceding and following states figured out in the following section using the Kalman filter's estimation. To estimate the extended Observability matrix, the subspace technique uses a matrix relation that represents the system's output as a linear function of state and inputs. By using the recursive Kalman filter's many versions, this state matrix can be constructed [3]:

$$\hat{X}_i = \left(A^i - Q_i \Gamma_i \left|\Delta_\mathrm{i} - Q_i H_i\right| Q_i\right) \left(\frac{SR^{-1} U_{0|2\mathrm{i}-1}}{\frac{U_{0|2\mathrm{i}-1}}{Y_{0|2\mathrm{i}-1}}} \right),$$

with: Γ_i the extended Observability matrix, $U_{i|2\mathrm{i}-1}$ is the following input, Q and R are the matrices of the QR decomposition, A^i , Q_i and S are the block matrices of the Kalman filter in the unsteady state.

Hence, the matrix of future state sequences $\hat{X}_{i+1}$ is obtained by the following relation:

$$\hat{X}_{i+1} = \left(A^{i+1} - Q_{i+1} \Gamma_{i+1} \left|\Delta_{i+1} - Q_{i+1} H_{i+1}\right| Q_{i+1}\right) \left(\frac{SR^{-1} U_{0|2\mathrm{i}-1}}{\frac{U_{0|\mathrm{i}}}{Y_{0|\mathrm{i}}}} \right),$$

with:

$$Q_i = \alpha_i \beta_i^{-1}.$$

Hence, $\alpha_i = A^i \left(P - SR^{-1}S^t\right) \Gamma_i^t + \Delta_i$ and $\beta_i = \Gamma_i \left(P - SR^{-1}S^t\right) \Gamma_i^t + L_i$ P is the Kalman filter covariance matrix, the element d represents the fully deterministic section., A_i , Q_i and S are the block matrices of the Kalman filter in the unsteady state, L denotes the SVD decomposition matrix [13, 14].

3. State representation

Using only the information from a few measurements of the inputs u_i and the outputs y_i that have been performed, this modeling technique aims to identify the system's order n and generate a realization (A, B, C, or D) of the dynamic system under study. In this work, we also provide a numerical method for determining these state matrices. We present the solution using the structural features of the extended Observability matrix provided by the equation (2.1), performed by reconstructing a sequence of the system state under examination. The four phases below make up the procedure.

Phase 1: Determine the projections: This step consists in determining the projections Z_i and Z_{i+1} in order to apply singular value composition (SVD), and are defined as following:

$$Z_i = Y_{i|2i-1} / \left(\frac{U_{0|i-1}}{U_{i|2i-1}^{U_{i-1}}} \right) = (\underbrace{L_{0|i-1}^1}_{li \times mi} | \underbrace{L_i^2}_{li \times mi} | \underbrace{L_i^3}_{li \times li}) \left(\frac{U_{0|i-1}}{U_{i|2i-1}} \right)$$

$$Z_{i+1} = Y_{i+1|2i-1} / \left(\frac{U_{0|i}}{\frac{U_{i+1|2i-1}}{Y_{0|i}}} \right).$$

Phase 2: Compute the (SVD) Decomposition: The (SVD) algebraic decomposition method is used to simplify the calculation of the preceding and succeeding states $\hat{X}_i$ and $\hat{X}_{i+1}$ after getting the projections by the earlier stage, we write:

$$\left(L_i^1 \mid L_i^3\right) \left(\frac{U_{0i-1}}{Y_{0|i-1}} \right) = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} V^t,$$

with the non-singular value number is provided by:

$$\Gamma_i = U_1 \Sigma_1^{1/2} \text{ et } \Gamma_{i-1} = \underline{U_1} \Sigma_1^{1/2}.$$

Phase 3: identify the States $\hat{X}_i$ and $\hat{X}_{i+1}$: Using the Moore-Penrose pseudo inverse, the states $\hat{X}_i$ and $\hat{X}_{i+1}$ are computed in this stage as follow:

$$\tilde{X}_i = \Gamma_i^{\dagger} \left(L_i^1 \mid L_i^3\right) \left(\frac{U_{0|i-1}}{Y_{0|i-1}}\right),$$
$$\tilde{X}_{i+1} = \Gamma_{i-1}^{\dagger} \left(L_{i+1}^1 \mid L_{i+1}^3\right) \left(\frac{U_{0|i}}{Y_{0|i}}\right).$$

The parameters $P^{\perp}$ and $\Gamma^{\dagger}$ denote a matrix whose row space is orthogonal to the row space of P and the Moore-Penrose pseudo inverse, respectively.

Phase 4: Get the least-squares solutions: (ρ_1 and ρ_2 are residues): Finding the least squares solution λ_1, λ_2 ,λ_3 and λ_4 is the last stage to determine the matrices of the system approximately

$$\left(\frac{\tilde{X}_{i+1}}{Y_{i|i}}\right) = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \cdot \left(\frac{\tilde{X}_i}{U_{i|i}}\right) + \left(\frac{\rho_1}{\rho_2}\right)$$
$$\rho = \begin{pmatrix} U_{0,2i-1} \\ Z_i \\ \hat{X}_i \end{pmatrix}^{\perp}$$

Calculating the system matrices: The system matrices A, B, C, and D are approximately calculated as follows:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \leftarrow \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \tag{3.1}$$

Alternatively, this algorithm is outlined in the following section, taking into account:

- The indices P and S, respectively, denote the previous or earlier state and the subsequent or later state.
- The inputs are activated by a persistence of order $2i$ $(rank(U_{0,2i-1}) = 2mi)$,and scale matrices $W_1 \in \mathbb{R}^{li\times li}$ and $W_2 \in \mathbb{R}^{li\times li}$ realise:

$$\mathrm{rank}(Z_p) = \mathrm{rank}(Z_p W_2),$$

and o_i such as the oblique projection:

$$o_i = Y_S/U_S Z_P,$$

and

$$W_1 o_i W_2 = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{3.2}$$
$$W_1 o_i W_2 = U_1 S_1 V_1^T$$

We obtain:

- Step 01: determine the projections:

$$o_i = \Gamma_i X_S.$$

- Step 02: Determine the (SVD) decomposition of system of equations (3.2), The quantity of singular values that differ from zero, as indicated by the matrix S_1, determines the model's order.
- Step 03: Following are the formulas for the extended Observability matrix, Γ_i:

$$\Gamma_i = W_1^{-1} U_1 S_1^{\frac{1}{2}} T.$$

- Step 04: Determine the state matrices:

$$X_S W_2 = T^{-1} U_1 S_1^{\frac{1}{2}} V_1^T,$$
$$X_S = \Gamma_i o_i.$$

The similarity transformation matrix $T \in \mathbb{R}^{n \times n}$ is non-singular, the parameter $S_1^{\frac{1}{2}}$ is given for symmetry purposes.

System Order:

We create the Hankel matrices $(Y_P, Y_S)^T$ and $(U_P, U_S)^T$ to identify the system's order, from a finite number q of input-output data (u_k, y_k) after instrumental variable matrix deduction, and find the equation's oblique projection (3.3), by multiplying o_i left and right by the corresponding weight matrices, W_1 and W_2 , that are applied to enhance approximation of $\Gamma_i X_S$, we get the equation (3.2), then apply the SVD of $W_1 o_i W_2$, The order of the system is n when S_1 is a diagonal matrix made up of n singular values that are not zero. System's matrices identification: The matrices A and C are identified from the Observability matrix Γ_i column space; after estimating Γ_i, the matrix C is actually derived from the first l lines of Γ_i. The matrix A is calculated from the following equation:

$$A = \underline{\Gamma}_i^{\dagger} \bar{\Gamma}_i,$$

with: Γ_i is Γ_i without the primary l lines, Γ_i is Γ_i without the primary l lines, $\Gamma^\dagger$ symbolizes the pseudo inverse of Moore-Penrose. We can get the matrices B and D by:

$$\Gamma_i^\perp \left[Y_S/U_s\right] U_s^\dagger = \Gamma_i^\perp H_i. \tag{3.3}$$

By multiplying equation (3.1) on the left by $\Gamma_i^\perp$ an on the right by $U_{i|2i-1}^\dagger$ we obtain:

$$\Gamma_i^\perp Y_S U_S^\dagger = \Gamma_i^\perp \Gamma_i X_S U_S^\dagger + \Gamma_i^\perp H_i U_S U_S^\dagger,$$

and by considering the product $\Gamma_i^\perp\Gamma$ is zero we get:

$$\Gamma_i^\perp Y_S U_S^\dagger = \Gamma_i^\perp H_i.$$

By making a change of variable as follows:

$$M = LH,$$

with: $L = \Gamma_i^\perp = \begin{pmatrix} L_1 & L_2 & \cdots & L_i \end{pmatrix}$ and $M = \Gamma_i^\perp Y_S U_S^\dagger = \begin{pmatrix} M_1 & M_2 & \cdots & M_i \end{pmatrix}$. B and D are found by solving a system of equations by the linear regression algorithm:

$$\begin{pmatrix} X_{i+1} \\ Y_{i|\mathrm{i}} \end{pmatrix} = \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{pmatrix} \begin{pmatrix} X_i \\ U_{i|\mathrm{i}} \end{pmatrix},$$

with a single row of the input blocks and their corresponding outputs, $U_{i|\mathrm{i}}$ and $Y_{i|\mathrm{i}}$ are the Hankel block matrices.

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