Advances in Mathematics: Scientific Journal **11** (2022), no.11, 1103–1114
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.11.11

HOMMOGENOUS WEIGHTS ON THE RING $\mathfrak{R}_{5,3} = \mathbb{F}_5 + U_1\mathbb{F}_5 + U_2\mathbb{F}_5 + U_3\mathbb{F}_5$

Ouarda Haddouche, Hanifa Zekraoui, and Karima Chatouh[1]

ABSTRACT. In this paper, we investigate linear codes over the ring $\mathfrak{R}_{5,3} = \mathbb{F}_5 + u_1\mathbb{F}_5 + u_2\mathbb{F}_5 + u_3\mathbb{F}_5$, and we determine the homogeneous weight of this ring, to derive some properties corresponding to these codes.

1. INTRODUCTION

In 1990 researchers became aware of the importance of linear codes over finite rings in algebraic coding theory, and several mathematical techniques have been developed. For example, the distance function in the alphabet is not given by the usual Hamming metric but by the homogeneous weight. So the homogeneous weight was introduced in the context of coding over the finite ring and integer residues ring for the first time by Constantinescu and Heise [4]. Then generalized to arbitrary finite rings by Greferath and Schmidt [5]. However, homogeneous weight only exists over finite Frobenius rings this is what led to the work on other rings like non-chain rings, especially in [1, 2, 6].

This paper aims to develop the fundamentals of linear codes over a class of finite rings given by the following form $\mathfrak{R}_{5,3} = \mathbb{F}_5 + u_1\mathbb{F}_5 + u_2\mathbb{F}_5 + u_3\mathbb{F}_5$, with $u_i^2 = u_i$

[1] *corresponding author*
2020 *Mathematics Subject Classification.* 94B05, 94B15, 11T71, 14G50, 15Axx.
Key words and phrases. Homogeneous weight, Gray map, Linear codes, Gray Images.
Submitted: 09.11.2022; *Accepted:* 25.11.2022; *Published:* 29.11.2022.

and $u_iu_j = u_ju_i = 0$, for $1 \leq i \neq j \leq 3$. The homogeneous weight may be thought of as a generalization of the Hamming weight for finite rings, thus we give a vast space in the present paper to study this weight on $\mathcal{R}_{5,3}$, and to determine the homogeneous weight on $\mathcal{R}_{5,3}$, we use the definition of homogeneous weight on $\mathbb{F}_5$.

The rest of the paper is organized as follows. In Section 2, we give some definitions and properties of linear codes over $\mathfrak{R}_{5,3} = \mathbb{F}_5 + u_1\mathbb{F}_5 + u_2\mathbb{F}_5 + u_3\mathbb{F}_5$. In particular, the Gray map and the Gray images and generator matrix of these codes. In section 3, we introduce some results about the homogeneous weight $\mathfrak{R}_{5,3} = \mathbb{F}_5 + u_1\mathbb{F}_5 + u_2\mathbb{F}_5 + u_3\mathbb{F}_5$, we complete this section by some useful example.

2. Linear codes over $\mathfrak{R}_{5,3}$

Linear codes C of length n over $\mathfrak{R}_{5,3}$, is an $\mathfrak{R}_{5,3}$-module of $\mathfrak{R}_{5,3}^n$, where

$$\mathfrak{R}_{5,3} = \{\bar{\omega}_i = \xi_0^i + u_1\xi_1^i + u_2\xi_2^i + u_3\xi_3^i \mid \xi_0^i, \xi_1^i, \xi_2^i, \xi_3^i \in \mathbb{F}_5, 1 \leq i \leq 625\} \tag{2.1}$$

is a commutative Frobenius ring with characteristics 5, where $|\mathfrak{R}_{5,3}| = 5^4 = 625$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be any two elements of $\mathfrak{R}_{5,3}^n$, we define the inner product over $\mathfrak{R}_{5,3}$ as, $\langle x, y\rangle_{\mathfrak{R}_{5,3}} = \sum_{i=1}^{i=n} x_iy_i$. The dual code $C^\perp$ of C is defined by $C^\perp = \{x \in \mathfrak{R}_{5,3}^n | \langle x, y\rangle_{\mathcal{R}_p} = 0$ for all $y \in C\}$. If $C \subset C^\perp$, we say the code is self-orthogonal, and if $C = C^\perp$ then, the code C is self-dual.

Following [3], the element x in $\mathfrak{R}_{5,3}$ can be written by

$$x = (1 + 4u_1 + 4u_2 + 4u_3)a_0 + u_1(a_1 + a_0) + u_2(a_2 + a_0) + u_3(a_3 + a_0). \tag{2.2}$$

Let C be a linear code of length n over $\mathfrak{R}_{5,3}$, there exists C_0, C_1, C_2, C_3 are linears codes of length n over $\mathbb{F}_5$, in fact the code C can be uniquely expressed as

$$C = (1 + 4u_1 + 4u_2 + 4u_3)C_0 \oplus (u_1)C_1 \oplus (u_2)C_2 \oplus (u_3)C_3, \tag{2.3}$$

where

$$\begin{aligned}
C_0 &= \{a_0 \in \mathbb{F}_5^n, \exists a_1, a_2, a_3 \in \mathbb{F}_5^n, a_0 + u_1a_1 + u_2a_2 + u_1a_3 \in C\},\\
C_1 &= \{a_0 + a_1 \in \mathbb{F}_5^n, \exists a_2, a_3 \in \mathbb{F}_5^n, a_0 + u_1a_1 + u_2a_2 + u_1a_3 \in C\},\\
C_2 &= \{a_0 + a_2 \in \mathbb{F}_5^n, \exists a_1, a_3 \in \mathbb{F}_5^n, a_0 + u_1a_1 + u_2a_2 + u_1a_3 \in C\},\\
C_3 &= \{a_0 + a_3 \in \mathbb{F}_5^n, \exists a_1, a_{32} \in \mathbb{F}_5^n, a_0 + u_1a_1 + u_2a_2 + u_1a_3 \in C\}.
\end{aligned}$$

Theorem 2.1. *Let $C = (1+4u_1+4u_2+4u_3)C_0 \oplus (u_1)C_1 \oplus (u_2)C_2 \oplus (u_3)C_3$ be a linear code of length n over $\mathfrak{R}_{5,3}$, then*

$$C^{\perp} = (1+4u_1+4u_2+4u_3)C_0^{\perp} \oplus (u_1)^{\perp}C_1 \oplus (u_2)C_2^{\perp} \oplus (u_3)C_3^{\perp}.$$

Corollary 2.1. *If G_0, G_1, G_2, G_3 are generator matrices of linear codes C_0, C_1, C_2, C_3 respectively, then the generator matrix of C is*

$$G = \begin{bmatrix} (1+4u_1+4u_2+4u_3)G_0 \\ u_1G_1 \\ u_2G_2 \\ u_3G_3 \end{bmatrix}. \tag{2.4}$$

Example 1. *Consider*

$$G_0 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix},$$
$$Gg_2 = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

are generator matrices of linear codes C_0, C_1, C_2, C_3 then the generator matrix of C can be written as

$$G = \begin{bmatrix} 1+4u_1+4u_2+4u_3 & 2+3u_1+3u_2+3u_3 & 3+4u_1+4u_2+4u_3 \\ 0 & 0 & 2+3u_1+3u_2+3u_3 \\ 0 & 2+3u_1+3u_2+3u_3 & 0 \\ 2+3u_1+3u_2+3u_3 & 0 & 2+3u_1+3u_2+3u_3 \\ 1+4u_1+4u_2+4u_3 & 4+u_1+u_2+u_3 & 0 \\ 0 & 1+4u_1+4u_2+4u_3 & 2+3u_1+3u_2+3u_3 \\ 1+4u_1+4u_2+4u_3 & 1+4u_1+4u_2+4u_3 & 0 \\ 0 & 1+4u_1+4u_2+4u_3 & 1+4u_1+4u_2+4u_3 \end{bmatrix}.$$

Next, we formulate the following definition of the Gray map

$$\begin{array}{rccc} \Psi & : & \mathcal{R}_{5,3} & \to & \mathbb{F}_5^4 \\ & & x & \mapsto & \Psi(x), \end{array} \tag{2.5}$$

with

$$\Psi(x = a_0 + u_1a_1 + u_2a_2 + u_3a_3) = (a_0, a_0+a_1, a_0+a_2, a_0+a_3).$$

This map can be extended to $\left(\mathcal{R}_{5,3}^n, d\right)$ from $(\mathbb{F}_5^{4n}, d_{Ham})$.

According to the above results, we can establish the following theorems.

Theorem 2.2. *If G_0, G_1, G_2, G_3 are a generator matrices of linear codes C_0, C_1, C_2, C_3 respectively, then $\Psi\left(G\right)$ is a generator matrix of $\Psi\left(C\right)$*

$$\Psi(G) = \begin{bmatrix} G_0 & 0 & 0 & 0 \\ G_0 & G_1 & 0 & 0 \\ G_0 & 0 & G_2 & 0 \\ G_0 & 0 & 0 & G_3 \end{bmatrix}. \tag{2.6}$$

Example 2. *In $\mathfrak{R}_{5,3}$, if $G_i = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, are generator matrices of C_i, for $i = \overline{0,3}$ then the code $\Phi(C)$ generated by*

$$\Phi(G) = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 3 & 0 & 0 & 0 & 0 \\ 3 & 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 & 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 1 & 3 \\ 3 & 1 & 0 & 0 & 0 & 0 & 3 & 1 \end{bmatrix}.$$

Theorem 2.3. *Let $C = (1 + 4u_1 + 4u_2 + 4u_3)C_0 \oplus (u_1)C_1 \oplus (u_2)C_2 \oplus (u_3)C_3$ be a linear code of length n over $\mathfrak{R}_{5,3}$. Then,*

$$\Psi(C) = C_0 \otimes C_1 \otimes C_2 \otimes C_3 \tag{2.7}$$

and

$$C = |C_0||C_1||C_2||C_3|. \tag{2.8}$$

Corollary 2.2. *Let C be a linear code of length n over $\mathfrak{R}_{5,3}$, then $\Phi(C^{\perp}) = [\Phi(C)]^{\perp}$. Further, C is a self-dual code if and only if $\Phi(C)$ is a self-dual code.*

Corollary 2.3. *Let $C = (1+4u_1+4u_2+4u_3)C_0 \oplus (u_1)C_1 \oplus (u_2)C_2 \oplus (u_3)C_3$ be a linear code of length n over $\mathfrak{R}_{5,3}$, where C_i are $[n, k_i, d_i]$-linear codes over $\mathbb{F}_5$, for $0 \leq i \leq 3$. Then $\Psi\left(C\right)$ is $[4n, \sum_{i=0}^{3} k_i, d = \min(d_0, d_1, d_2, d_3)]$-linear code over $\mathbb{F}_5$.*

3. HOMOGENEOUS WEIGHT ON THE RING $\mathfrak{R}_{5,3}$

Based on the simplistic representation of [8], we will compute the homogeneous weight on the ring $\mathfrak{R}_{5,3}$. These results will be used later on to define and examine several applications.

The homogeneous weight ω on $\mathfrak{R}_{5,3}$, with generating character χ is given by

$$\begin{array}{rcl} \omega : \mathfrak{R}_{5,3} & \longrightarrow & \mathbb{R} \\ x & \longrightarrow & \omega(x) = \eta.(1 - \frac{1}{|\mathfrak{R}_{5,3}^{\times}|}. \sum_{u \in \mathfrak{R}_{5,3}^{\times}} \chi(xu)), \end{array} \tag{3.1}$$

where $\mathfrak{R}_{5,3}^{\times}$ is the group of units of $\mathfrak{R}_{5,3}$.

The method in [7], leads to

1. The finite direct sum of Frobenius rings as the Frobenius ring.
2. If the Frobenius rings R_1, R_2, R_3, R_4 each have right generating characters $\chi_1, \chi_2, \chi_3, \chi_4$, then $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4$ has generating character $\chi = \prod_{i=1}^{4} \chi_i$.
3. The generating character of $\overbrace{\mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5}^{4}$ is defined as

$$\begin{array}{rcl} \chi : \overbrace{\mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5}^{4} & \longrightarrow & \mathbb{T} \\ x = (x_1, x_2, x_3, x_4) & \longrightarrow & \chi(x) = e^{\frac{2i\pi}{5} tr(x_1 + x_2 + x_3 + x_4)}, \end{array} \tag{3.2}$$

where tr is the function trace and T is the multiplicative group of unit complex numbers, which is also a one-dimensional torus.

Theorem 3.1. *If $R_1, R_2, \ldots, R_n$ are rings with identity and I an ideal in $\prod_{j=1}^{n} R_j$, then $I = \prod_{j=1}^{n} A_j$ with A_j is an ideal in R_j, where*

$$\prod_{j=1}^{n} R_j = \{(x_1, x_2, \ldots, x_n) | \; x_j \in R_j\}. \tag{3.3}$$

In this case, using similar arguments as in Theorem 3.1, we can establish the representation of the ideals of $\mathbb{F}_5^4$ by the following sets.

1. $\overbrace{\mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5}^{4} = \langle (x_1, x_2, x_3, x_4) \rangle, x_1, x_2, x_3, x_4 \neq 0$
2. $\mathbb{F}_5 \times \{0\} \times \{0\} \times \{0\} = \langle (x_1, 0, 0, 0) \rangle, x_1 \neq 0$
3. $\{0\} \times \mathbb{F}_5 \times \{0\} \times \{0\} = \langle (0, x_2, 0, 0) \rangle, x_2 \neq 0$
4. $\{0\} \times \{0\} \times \mathbb{F}_5 \times \{0\} = \langle (0, 0, x_3, 0) \rangle, x_3 \neq 0$

5. $\{0\} \times \{0\} \times \{0\} \times \mathbb{F}_5 = \langle (0,0,0,x_4) \rangle, x_4 \neq 0$
6. $\{(0,0,0,0)\} = \langle (0,0,0,0) \rangle$

Remark 3.1. *The number of zero divisors of $\mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5$ are 16, and the number of unites of this product are $|\{\mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5\}^{\times}| = 4^4$. Moreover the ideals $I_1 = \langle (x_1,0,0,0) \rangle$, $I_2 = \langle (0,x_2,0,0) \rangle$, $I_3 = \langle (0,0,x_3,0) \rangle$, $I_4 = \langle (0,0,0,x_4) \rangle$ in $\mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5 \times \mathbb{F}_5$ are maximals.*

Finally, to summarize these pieces of information, we formulate the following theorem for calculating homogeneous weight on $\mathfrak{R}_{5,3}$.

Theorem 3.2. *The homogeneous weight on $\mathfrak{R}_{5,3}$ is given by*

$$\omega_{hom}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{65}{64}\eta_1 & \text{if x is divisor of zero} \\ \frac{255}{256}\eta_2 & \text{if x is unit .} \end{cases}$$

Proof. Let $\mathcal{H} \in \mathbb{F}_5^4$. According to Equation 3.2, the homogeneous weight of

$$x = (x_1, x_2, x_3, x_4) \in \mathcal{H}$$

is

$$\omega_{hom}(x) = \eta_j.(1 - \frac{1}{4^4} \sum_{a \in (\mathcal{H})^{\times}} e^{\frac{2i\pi}{5} tr(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)}), \ for \ j = \overline{1,2}. \tag{3.4}$$

In the following, we calculate the homogeneous weight on three cases.

In case 1: if $(x_1, x_2, x_3, x_4) = (0,0,0,0)$, we have $tr((0,0,0,0)) = 0$, and

$$\omega_{hom}((0,0,0,0)) = \eta_1.(1 - \frac{1}{4^4} \sum_{a \in (\mathcal{H})^{\times}} e^0), \tag{3.5}$$

then

$$\omega_{hom}((0,0,0,0)) = \eta_1.(1 - \frac{4^4}{4^4}),$$

so that

$$\omega_{hom}((0,0,0,0)) = 0. \tag{3.6}$$

In case 2: if $x = (x_1, x_2, x_3, x_4) \in \mathcal{H}$ is a zero divisor. Assume that $x_2, x_3, x_4 = 0$ and $x_1 \neq 0$ (that means $x_1 \in \mathbb{F}_5^*$) and $|\mathbb{F}_5^*| = 4$, we have

$$tr(\langle a, x\rangle_{\mathfrak{R}_{5,3}}) = tr(a_1x_1), a = (a_1, a_2, a_3, a_4) \in (\mathcal{H})^\times.$$

The number of elements $\rho \in \mathbb{F}_5^*$, such that $tr(\rho) = 0 \ is \ 0$, for $j = 1, 2, 3, 4$, the number of elements $\varrho \in \mathbb{F}_5^*$, such that $tr(\varrho) = j$ is 1. Using Equation 3.2, we obtain

$$\begin{aligned}\sum_{a\in(\mathcal{H})^\times} e^{\frac{2i\pi}{5}tr(ax)} &= 0 + 4\sum_{j=1}^{4} e^{\frac{2i\pi}{5}j} \\ &= -1,\end{aligned}$$

as well as

$$\omega_{hom}(x_1, x_2, x_3, x_4) = \frac{65}{64}\eta_1. \tag{3.7}$$

In case 3: If $x = (x_1, x_2, x_3, x_4) \in \mathcal{H}$ is a unit. Since the set $(\mathcal{H})^\times$ form a group under the multiplication, for $0 \leq j_i \leq 4$ and $1 \leq i \leq 4$.

Consider the following disjoint subsets of $(\mathcal{H})^\times$

$\mathcal{B}_{0000} = \langle(x_1, x_2, x_3, x_4)\rangle, tr(x_i) = 0, i \in \{1, 2, 3, 4\}$,
$\mathcal{B}_{j_1000} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = j_1$, $tr(x_i) = 0$, $i \in \{2, 3, 4\}$,
$\mathcal{B}_{0j_200} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_2) = j_2$, $tr(x_i) = 0$, $i \in \{1, 3, 4\}$,
$\mathcal{B}_{00j_30} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_3) = j_3$, $tr(x_i) = 0$, $i \in \{1, 2, 4\}$,
$\mathcal{B}_{000j_4} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_4) = j_4$, $tr(x_i) = 0$, $i \in \{1, 2, 3\}$,
$\mathcal{B}_{j_1j_200} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = j_1$, $tr(x_i) = 0$, $i \in \{3, 4\}$,
$\mathcal{B}_{j_10j_30} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = j_1$, $tr(x_i) = 0$, $i \in \{2, 4\}$,
$\mathcal{B}_{j_100j_4} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = j_1$, $tr(x_i) = 0$, $i \in \{2, 3\}$,
$\mathcal{B}_{0j_2j_30} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = j_1$, $tr(x_i) = 0$, $i \in \{1, 4\}$,
$\mathcal{B}_{0j_20j_4} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = j_1$, $tr(x_i) = 0$, $i \in \{1, 3\}$,
$\mathcal{B}_{00j_3j_4} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = j_1$, $tr(x_i) = 0$, $i \in \{1, 2\}$,
$\mathcal{B}_{j_1j_2j_30} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = j_1$, $tr(x_2) = j_2$, $tr(x_3) = j_3$, $tr(x_4) = 0$,
$\mathcal{B}_{j_1j_20j_4} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = j_1$, $tr(x_2) = j_2$, $tr(x_3) = 0$, $tr(x_4) = j_4$,
$\mathcal{B}_{j_10j_3j_4} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = j_1$, $tr(x_2) = 0$, $tr(x_3) = j_3$, $tr(x_4) = j_4$,
$\mathcal{B}_{0j_2j_3j_4} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = 0$, $tr(x_2) = j_2$, $tr(x_3) = j_3$, $tr(x_4) = j_4$,
$\mathcal{B}_{j_1j_2j_3j_4} = \langle(x_1, x_2, x_3, x_4)\rangle$, $tr(x_1) = j_1$, $tr(x_2) = j_2$, $tr(x_3) = j_3$, $tr(x_4) = j_4$,

we can obtain the following relationship

$$|\mathcal{B}_{0000}| = 0, |\mathcal{B}_{j_1000}| = 0, |\mathcal{B}_{j_1j_200}| = 0, |\mathcal{B}_{j_1j_2j_30}| = 0 \ and \ |\mathcal{B}_{j_1j_2j_3j_4}| = 1. \tag{3.8}$$

We calculate,

$$\begin{aligned}
&\sum_{a\in(\mathcal{H})^\times} e^{\frac{2i\pi}{5}tr(\langle a,x\rangle_{\mathfrak{R}_{5,3}})} \\
&= |\mathcal{B}_{0000}|e^{\frac{2i\pi}{5}0} + \sum_{j_1=1}^{4} |\mathcal{B}_{j_1000}|e^{\frac{2i\pi}{5}j_1} + \cdots + \sum_{j_4=1}^{4} |\mathcal{B}_{000j_4}|e^{\frac{2i\pi}{5}j_4} \\
&\quad + \sum_{j_1=1}^{4}\sum_{j_2=1}^{4} |\mathcal{B}_{j_1j_200}|e^{\frac{2i\pi}{5}(j_1+j_2)} + \cdots + \sum_{j_3=1}^{4}\sum_{j_4=1}^{4} |\mathcal{B}_{0j_3j_4}|e^{\frac{2i\pi}{5}(j_3+j_4)} \\
&\quad + \sum_{j_1=1}^{4}\sum_{j_2=1}^{4}\sum_{j_3=1}^{4} |\mathcal{B}_{j_1j_2j_30}|e^{\frac{2i\pi}{5}(j_1+j_2+j_3)} + \sum_{j_1=1}^{4}\sum_{j_3=1}^{4}\sum_{j_4=1}^{4} |\mathcal{B}_{j_1j_2j_30}|e^{\frac{2i\pi}{5}(j_1+j_3+j_4)} \\
&\quad + \sum_{j_1=1}^{4}\sum_{j_2=1}^{4}\sum_{j_4=1}^{4} |\mathcal{B}_{j_1j_20j_4}|e^{\frac{2i\pi}{5}(j_1+j_2+j_4)} + \sum_{j_2=1}^{4}\sum_{j_3=1}^{4}\sum_{j_4=1}^{4} |\mathcal{B}_{0j_2j_3j_4}|e^{\frac{2i\pi}{5}(j_2+j_3+j_4)} \\
&\quad + \sum_{j_1=1}^{4}\sum_{j_2=1}^{4}\sum_{j_3=1}^{4}\sum_{j_4=1}^{4} |\mathcal{B}_{j_1j_2j_3j_4}|e^{\frac{2i\pi}{5}(j_1+j_2+j_3+j_4)} \\
&= 0 + 0.. + \sum_{j_1=1}^{4} e^{\frac{2i\pi}{5}j_1} \sum_{j_2=1}^{4} e^{\frac{2i\pi}{5}j_2} \sum_{j_3=1}^{4} e^{\frac{2i\pi}{5}j_3} \sum_{j_4=1}^{4} e^{\frac{2i\pi}{5}j_4} \\
&= (-1)(-1)(-1)(-1) = 1
\end{aligned}$$

Then, we find the following result.

$$\omega_{hom}(x_1, x_2, x_3, x_4) = \frac{255}{256}\eta_2. \tag{3.9}$$

□

Example 3. *Let* C_0, C_1, C_2, C_3 *are* $[4,2]$*-linear codes over* $\mathfrak{R}_{5,3}$*, with generator matrices*

$$G_0 = \begin{bmatrix} 2131 \\ 1322 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0221 \\ 1120 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1031 \\ 0103 \end{bmatrix},$$

and

$$G_3 = \begin{bmatrix} 1011 \\ 0112 \end{bmatrix}.$$

Assume that $\varepsilon = 1 + 4u_1 + 4u_2 + 4u_3$*, the generator matrix* G *is computed as follows*

$$G = \begin{bmatrix} 2\varepsilon & \varepsilon & 3\varepsilon & \varepsilon \\ \varepsilon & 3\varepsilon & 2\varepsilon & 2\varepsilon \\ 0 & 2u_1 & 2u_1 & u_1 \\ u_1 & u_1 & 2u_1 & 0 \\ u_2 & 0 & 3u_2 & u_2 \\ 0 & u_2 & 0 & 3u_2 \\ u_3 & 0 & u_3 & u_3 \\ 0 & u_3 & u_3 & 2u_3 \end{bmatrix}.$$

If $\eta_1 = 0$ *and* $\eta_2 = \dfrac{512}{255}$*, for all* $c \in C$*, then we have*

$$w_{hom}(c) \in \{6, 8, 10, 14, 16\}.$$

Furthermore

$$\Psi(G) = \begin{bmatrix} 2131 & 0000 & 0000 & 0000 \\ 1322 & 0000 & 0000 & 0000 \\ 2131 & 0221 & 0000 & 0000 \\ 1322 & 1120 & 0000 & 0000 \\ 2131 & 0000 & 1031 & 0000 \\ 1322 & 0000 & 0103 & 0000 \\ 2131 & 0000 & 0000 & 1011 \\ 1322 & 0000 & 0000 & 0112 \end{bmatrix}.$$

Example 4. *Let* C_0, C_1, C_2, C_3 *are* $[26, 4]$*-linear codes over* $\mathfrak{R}_{5,3}$*, with generator matrices*

$$G_i = \begin{bmatrix} 00142323230023014140231414 \\ 00002233112344122334001144 \\ 10111111222222333333444444 \\ 01111111111111111111111111 \end{bmatrix}, \textit{ for } 0 \leq i \leq 3.$$

Assume that $\varepsilon = 1 + 4u_1 + 4u_2 + 4u_3$*, then the generator matrix of C, is given as follows*

$$G = \begin{bmatrix} \overline{\mathcal{G}} & \overline{\overline{\mathcal{G}}} \end{bmatrix},$$

with

$$\overline{\mathcal{G}} = \begin{bmatrix}
0 & 0 & \varepsilon & 4\varepsilon & 2\varepsilon & 3\varepsilon & 2\varepsilon & 3\varepsilon & 2\varepsilon & 3\varepsilon & 0 & 0 & 2\varepsilon \\
0 & 0 & 0 & 0 & 2\varepsilon & 2\varepsilon & 3\varepsilon & 3\varepsilon & \varepsilon & \varepsilon & 2\varepsilon & 3\varepsilon & 4\varepsilon \\
\varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2\varepsilon & 2\varepsilon & 2\varepsilon & 2\varepsilon & 2\varepsilon \\
0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
0 & 0 & u_1 & 4u_1 & 2u_1 & 3u_1 & 2u_1 & 3u_1 & 2u_1 & 3u_1 & 0 & 0 & 2u_1 \\
0 & 0 & 0 & 0 & 2u_1 & 2u_1 & 3u_1 & 3u_1 & u_1 & u_1 & 2u_1 & 3u_1 & 4u_1 \\
u_1 & 0 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & 2u_1 & 2u_1 & 2u_1 & 2u_1 & 2u_1 \\
0 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 \\
0 & 0 & u_2 & 4u_2 & 2u_2 & 3u_2 & 2u_2 & 3u_2 & 2u_2 & 3u_2 & 0 & 0 & 2u_2 \\
0 & 0 & 0 & 0 & 2u_2 & 2u_2 & 3u_2 & 3u_2 & u_2 & u_2 & 2u_2 & 3u_2 & 4u_2 \\
u_2 & 0 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & 2u_2 & 2u_2 & 2u_2 & 2u_2 & 2u_2 \\
0 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 \\
0 & 0 & u_3 & 4u_3 & 2u_3 & 3u_3 & 2u_3 & 3u_3 & 2u_3 & 3u_3 & 0 & 0 & 2u_3 \\
0 & 0 & 0 & 0 & 2u_3 & 2u_3 & 3u_3 & 3u_3 & u_3 & u_3 & 2u_3 & 3u_3 & 4u_3 \\
u_3 & 0 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & 2u_3 & 2u_3 & 2u_3 & 2u_3 & 2u_3 \\
0 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3
\end{bmatrix}$$

and

$$\overline{\overline{\mathcal{G}}} = \begin{bmatrix}
3\varepsilon & 0 & \varepsilon & 4\varepsilon & \varepsilon & 4\varepsilon & 0 & 2\varepsilon & 3\varepsilon & \varepsilon & 4\varepsilon & \varepsilon & 4\varepsilon \\
4\varepsilon & \varepsilon & 2\varepsilon & 2\varepsilon & 3\varepsilon & 3\varepsilon & 4\varepsilon & 0 & 0 & \varepsilon & \varepsilon & 4\varepsilon & 4\varepsilon \\
2\varepsilon & 3\varepsilon & 3\varepsilon & 3\varepsilon & 3\varepsilon & 3\varepsilon & 3\varepsilon & 4\varepsilon & 4\varepsilon & 4\varepsilon & 4\varepsilon & 4\varepsilon & 4\varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
3u_1 & 0 & u_1 & 4u_1 & u_1 & 4u_1 & 0 & 2u_1 & 3u_1 & u_1 & 4u_1 & u_1 & 4u_1 \\
4u_1 & u_1 & 2u_1 & 2u_1 & 3u_1 & 3u_1 & 4u_1 & 0 & 0 & u_1 & u_1 & 4u_1 & 4u_1 \\
2u_1 & 3u_1 & 3u_1 & 3u_1 & 3u_1 & 3u_1 & 3u_1 & 4u_1 & 4u_1 & 4u_1 & 4u_1 & 4u_1 & 4u_1 \\
u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 & u_1 \\
3u_2 & 0 & u_2 & 4u_2 & u_2 & 4u_2 & 0 & 2u_2 & 3u_2 & u_2 & 4u_2 & u_2 & 4u_2 \\
4u_2 & u_2 & 2u_2 & 2u_2 & 3u_2 & 3u_2 & 4u_2 & 0 & 0 & u_2 & u_2 & 4u_2 & 4u_2 \\
2u_2 & 3u_2 & 3u_2 & 3u_2 & 3u_2 & 3u_2 & 3u_2 & 4u_2 & 4u_2 & 4u_2 & 4u_2 & 4u_2 & 4u_2 \\
u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 & u_2 \\
3u_3 & 0 & u_3 & 4u_3 & u_3 & 4u_3 & 0 & 2u_3 & 3u_3 & u_3 & 4u_3 & u_3 & 4u_3 \\
4u_3 & u_3 & 2u_3 & 2u_3 & 3u_3 & 3u_3 & 4u_3 & 0 & 0 & u_3 & u_3 & 4u_3 & 4u_3 \\
2u_3 & 3u_3 & 3u_3 & 3u_3 & 3u_3 & 3u_3 & 3u_3 & 4u_3 & 4u_3 & 4u_3 & 4u_3 & 4u_3 & 4u_3 \\
u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3 & u_3
\end{bmatrix}.$$

If $\eta_1 = \frac{64}{65}$ *and* $\eta_2 = \frac{768}{255}$*, for all* $c \in C$*, we then have* $w_{hom}(c) \in \{75, 150, 183\}$.

Moreover, The generator matrix of $\Psi(C)$ *is defined by* $\Psi(G) = \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 \end{bmatrix}$, *where*

$$\mathcal{D}_1 = \begin{bmatrix}
00142323230023014140231414 & 00000000000000000000000000 \\
00002233112344122334001144 & 00000000000000000000000000 \\
10111111222222333333444444 & 00000000000000000000000000 \\
01111111111111111111111111 & 00000000000000000000000000 \\
00142323230023014140231414 & 00142323230023014140231414 \\
00002233112344122334001144 & 00002233112344122334001144 \\
10111111222222333333444444 & 10111111222222333333444444 \\
01111111111111111111111111 & 01111111111111111111111111 \\
00142323230023014140231414 & 00000000000000000000000000 \\
00002233112344122334001144 & 00000000000000000000000000 \\
10111111222222333333444444 & 00000000000000000000000000 \\
01111111111111111111111111 & 00000000000000000000000000 \\
00142323230023014140231414 & 00000000000000000000000000 \\
00002233112344122334001144 & 00000000000000000000000000 \\
10111111222222333333444444 & 00000000000000000000000000 \\
01111111111111111111111111 & 00000000000000000000000000
\end{bmatrix}$$

and

$$\mathcal{D}_2 = \begin{bmatrix}
00000000000000000000000000 & 00000000000000000000000000 \\
00000000000000000000000000 & 00000000000000000000000000 \\
00000000000000000000000000 & 00000000000000000000000000 \\
00000000000000000000000000 & 00000000000000000000000000 \\
00000000000000000000000000 & 00000000000000000000000000 \\
00000000000000000000000000 & 00000000000000000000000000 \\
00000000000000000000000000 & 00000000000000000000000000 \\
00000000000000000000000000 & 00000000000000000000000000 \\
00142323230023014140231414 & 00000000000000000000000000 \\
00002233112344122334001144 & 00000000000000000000000000 \\
10111111222222333333444444 & 00000000000000000000000000 \\
01111111111111111111111111 & 00000000000000000000000000 \\
00000000000000000000000000 & 00142323230023014140231414 \\
00000000000000000000000000 & 00002233112344122334001144 \\
00000000000000000000000000 & 10111111222222333333444444 \\
00000000000000000000000000 & 01111111111111111111111111
\end{bmatrix}$$

4. Conclusion

In this paper, we considered linear codes over a specific Frobenius ring $\mathfrak{R}_{5,3} = \mathbb{F}_5 + u_1\mathbb{F}_5 + u_2\mathbb{F}_5 + u_3\mathbb{F}_5$, with $u_i^2 = u_i$ and $1 \leq i \leq 3$, that is endowed with duality and distance preserving Gray map. Also, we mainly investigate the structural properties of the constructed homogeneous weight on this ring.

References

[1] K. Chatouh, K. Guenda, T.A. Gulliver, L. Noui: *Simplex and MacDonald codes over R_q*, J. Appl. Math. Comput, **55**(1-2) (2017), 455-478.

[2] K. Chatouh, K. Guenda, T.A. Gulliver: *New Classes of Codes Over $R_{q,p,m} = \mathbb{Z}_{p^m}[u_1, u_2, \cdots, u_q]/\langle u_i^2 = 0, u_iu_j - u_ju_i \rangle$ and Their Applications*, Computational and Applied Mathematics, **39** (2020), 1-39.

[3] A. Dertli, Y. Cengellenmis, S. Eren: *Quantum Codes Over $F_2 + uF_2 + vF_2$*, Palest. J. Math, **4** (2015) , 547-552.

[4] I. Constantinescu, W. Heise: *A metric for codes over residue class rings*, Probl. Peredachi Informatsii, **33** (1997), 22-28.

[5] M. Greferath, S.E. Schmidt: *Gray isometries for finite chain rings and a nonlinear ternary $(36, 3^{12}, 15)$ code*, IEEE Trans. Inform. Theory, **45** (1999), 2522-2524.

[6] A. Dertli, Y. Cengellenmis, S. Eren: *Some results on the linear codes over the finite ring $F_2 + v_1F_2 + \ldots + v_rF_2$*, International Journal of Quantum Information, **14**(1) (2016), (12 pages).

[7] T.W. Hungerford: Algebra (Graduate Texts in Mathemaics 73), New York, Springer-Verlag, 1974.

[8] M.L. John, S. Virgilio: *Bounds on the p^r-ary image of linear block codes over the finite semi-local Frobenius ring $F_{p^r} + vF_{p^r}$*, Southeast-Asian J. of Sciences, **1**(1) (2012), 76-98.

Department of Mathematics, University of Mostefa Ben Boulaïd, Batna 2, Batna, Algiers, Algeria.

Email address: haddoucheouarda840@gmail.com

Department of Mathematics, University of Larbi Ben M´hidi, Oum el bouaghi, Algiers, Algeria.

Email address: hanifazekraoui@yahoo.fr

Department of Economic, Commercial and Management Sciences, University of Batna 1 , Batna, Algiers, Algeria.

Email address: karima.chatouh@gmail.com