ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **11** (2022), no.12, 1145–1172
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.12.2

BLOW UP FOR POSITIVE-INITIAL ENERGY AND DECAY OF A BIHARMONIC SYSTEM WITH VARIABLE-EXPONENT NONLINEARITIES

Oulia Bouhoufani[1], Mohammad M. Al-Gharabli, and Salim A. Messaoudi

ABSTRACT. This work is concerned with a coupled system of two biharmonic equations with variable exponents in the damping and source terms. Using the energy approach and for certain solution with positive initial data, we prove the blow-up theorem. Then, we establish the global existence as well as energy decay results of solutions, under appropriate conditions on the parameters of the problem, using the stable-set and the multiplier methods.

1. INTRODUCTION

Problems with biharmonic operator appear in several physical phenomena such as micro-electro-mechanical systems [26], radar imaging [3], the study of travelling waves in suspnssion bridges [17], bending behaviour of thin elastic rectangular plates [28], geometric and functional designs [8].

In the other hand, the progress of sciences and technology brought many new real-world problems such as flows of electro-rheological fluids, fluids with temperature dependent viscocity, filtration processes through a porous media, image processing and thermorheological fluids and others, which required modeling with

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 35L05, 35B40, 35L70, 93D20.
Key words and phrases. Biharmonic equations, Blow up, Coupled system, Global existence, Variable exponent, Stability.
Submitted: 13.11.2022; *Accepted:* 28.11.2022; *Published:* 30.11.2022.

non-standard mathematical functional spaces. The Lebesgue and Sobolev spaces with variable exponents have manifested to be very important and most suitable tools to tackle such models. See [4, 5, 27] for more details.

Motivated by the importance and the applications of biharmonic systems and the variable exponents nonlinearity, we are interested in the following initial-boundary-value problem:

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m(x)-2} u_t = f_1(x,u,v) & \text{in } \Omega \times (0,T), \\ v_{tt} + \Delta^2 v + |v_t|^{r(x)-2} v_t = f_2(x,u,v) & \text{in } \Omega \times (0,T), \\ u = v = 0 = \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 & \text{on } \partial\Omega \times (0,T), \\ (u(0), v(0)) = (u_0, v_0) \text{ and } (u_t(0), v_t(0)) = (u_1, v_1) & \text{in } \Omega \times \Omega, \end{cases} \tag{1.1}$$

where $T > 0, \Omega$ is a smooth and bounded domain of $\mathbb{R}^n$, $\left(n = \overline{1,6}\right)$, m and r are continuous functions on $\overline{\Omega}$ satisfying some conditions, $(\frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta})$ denotes the external normal derivative of (u,v), on the boundary $\partial\Omega$ and the terms f_1 and f_2 are defined, for all $x \in \overline{\Omega}$ and $(u,v) \in \mathbb{R}^2$, by

$$f_1(x,u,v) = \frac{\partial}{\partial u} F(x,u,v) \text{ and } f_2(x,u,v) = \frac{\partial}{\partial v} F(x,u,v), \tag{1.2}$$

with

$$F(x,u,v) = a|u+v|^{p(x)+1} + 2b|uv|^{\frac{p(x)+1}{2}}, \tag{1.3}$$

where $a, b > 0$ are two positive constants and p is a given continuous function on $\overline{\Omega}$ satisfying certain conditions.

Our aim in this work is to prove a blow-up theorem for certain solutions with positive initial data, use the stable-set approach to establish the global existence of solutions, and exploit the multiplier method to obtain the long time behavior of the energy, under suitable conditions on the variable exponents and the initial data.

2. Literature Review

For problems with biharmonic operator, Komornik [15] considered the following Petrovsky equation

$$u_{tt} + \Delta^2 u - g(\Delta u) = 0,$$

proved, by using the nonlinear semigroup theory, the well-posedness and established the energy decay estimates for weak solutions. Guesmia [13] discussed the well-posedness of a damped nonlinear coupled system of two Petrovsky equations and established various stability results. Using Komornik's Lemma, Aassila and Guesmia [7] established an exponential decay estimate for the following problem

$$\begin{cases} u_{tt} + k_1\Delta^2 u + k_2\Delta^2 u_t + \Delta g(\Delta u) = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ u = \partial_\eta u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0) = u_0 \text{ and } u_t(0) = u_1 & \text{on } \Omega. \end{cases}$$

Messaoudi [20] investigated the following Petrovsky problem with nonlinear source term

$$\begin{cases} u_{tt} + \Delta^2 u + a u_t \left|u_t\right|^{m-2} = bu\left|u\right|^{\rho-2} & \text{in } \Omega \times \mathbb{R}^+, \\ u = \partial_\eta u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0) = u_0 \text{ and } u_t(0) = u_1 & \text{on } \Omega, \end{cases}$$

where a, b are positive constants and $m > 2$. He proved an existence result and showed that the solution blows up, in finite time, if $m < p$ and exists globally otherwise.

In recent years, problems with variable-exponent nonlinearity had received a considerable amount of attention. Antontsev et al. [6] studied the following Petrovsky equation

$$u_{tt} + \Delta^2 u - \Delta u_t + \left|u_t\right|^{m(x)-2} u_t = \left|u\right|^{p(x)-2} u.$$

They proved the existence of local weak solutions by using the Banach fixed-point theorem, and gave a blow-up result for negative-initial-energy solutions, under suitable assumptions on m, p and initial data. In [19], Liao and Tan treated the following nonlinear problem

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|_2^2\right) - \Delta u_t + \left|u_t\right|^{m(x)-2} u_t = \left|u\right|^{p(x)-2} u,$$

where $M(s) = a + bs^\gamma$ is a C^1-function, $a > 0, b > 0, \gamma \geq 1$, and m, p are given measurable functions. They established some uniform decay estimates and the upper and lower bounds of the blow-up time.

Concerning coupled systems with variable-exponent nonlinearity, we have only few works. In [9], Bouhoufani and Hamchi considered the following coupled system of two nonlinear hyperbolic equations with variable-exponents

$$\begin{cases} u_{tt} - div(A\nabla u) + |u_t|^{m(x)-2}\, u_t = f_1\,(x,u,v) & \text{in } \Omega \times (0,T)\,, \\ v_{tt} - div(B\nabla v) + |v_t|^{r(x)-2}\, v_t = f_2\,(x,u,v) & \text{in } \Omega \times (0,T)\,. \end{cases}$$

They obtained, in a dounded domain, the global existence of a weak solution and established decay rates of the solution. In [23], Messaoudi et al. considered the following system

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m(x)-2} u_t + f_1(u,v) = 0 & \text{in } \Omega \times (0,T)\,, \\ v_{tt} - \Delta v + |v_t|^{r(x)-2} v_t + f_2(u,v) = 0 & \text{in } \Omega \times (0,T)\,, \end{cases}$$

with initial and Dirichlet-boundary conditions (here f_1 and f_2 are the coupling terms introduced in (1.2). The authors proved the existence of global solutions, obtained explicit decay rate estimates, under suitable assumptions on the variable exponents m, r and p and presented some numerical tests. For more results in the subject of variable-exponent nonlineaity, we refer to [12, 14, 18, 21, 24, 25].

This paper consists of three sections, in addition to the introduction and literature review. In Section 3, we define the variable-exponent Lebesgue and Sobolev spaces, and give some of their important properties. We also state (without proof) the local-existence theorem of [10]. The blow-up result in finite time and for positive-initial data, will be established in Section 4. The last section is devoted to the study of the global existence and the stability results.

3. Preliminaries

Let $q : \Omega \longrightarrow [1, \infty)$ be a measurable function. We define the Lebesgue space with a variable exponent by

$$L^{q(.)}(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \text{ measurable in } \Omega : \ \varrho_{q(.)}(\lambda f) < +\infty, \text{ for some } \lambda > 0 \right\},$$

where

$$\varrho_{q(.)}(f) = \int_\Omega |f(x)|^{q(x)} dx.$$

$L^{q(.)}(\Omega)$ is a Banach space with respect to the following Luxembourg-type norm

$$\|f\|_{q(.)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\}.$$

For $k \in \mathbb{N}$, we define the variable-exponent Sobolev space $W^{k,p(.)}(\Omega)$ as follows:

$$W^{k,q(.)}(\Omega) = \left\{u \in L^{q(.)}(\Omega) : \partial^{|\alpha|}u \in L^{q(.)}(\Omega),\ with\ |\alpha| \leq k\right\}.$$

$W^{k,q(.)}(\Omega)$ is a Banach space equipped with the following norm

$$\|u\|_{W^{k,q(.)}(\Omega)} := \sum_{0 \leq |\alpha| \leq k} \|\partial_\alpha u\|_{q(.)},$$

where $|\alpha| = \alpha_1 + ... + \alpha_n$.
In addition, we set $W_0^{k,q(.)}(\Omega)$ to be the closure of $W^{k,q(.)}(\Omega)$-functions with compact support in $W^{k,q(.)}(\Omega)$ and we denote by $H_0^{k,q(.)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,q(.)}(\Omega)$.

Lemma 3.1. ***(Young's Inequality [5, 16])*** *Let $r, q, s \geq 1$ be measurable functions defined on Ω, such that*

$$\frac{1}{s(y)} = \frac{1}{r(y)} + \frac{1}{q(y)},\ \textit{for}\ a.e\ y \in \Omega.$$

Then, for all $a, b \geq 0$, we have

$$\frac{(ab)^{s(.)}}{s(.)} \leq \frac{(a)^{r(.)}}{r(.)} + \frac{(b)^{q(.)}}{q(.)}.$$

Lemma 3.2. ***(Hölder's Inequality [5, 16])*** *Let $r, q, s : \Omega \longrightarrow [1, \infty)$ be measurable functions, such that*

$$\frac{1}{s(y)} = \frac{1}{r(y)} + \frac{1}{q(y)},\ \textit{for}\ \ a.e.\ y \in \Omega.$$

If $f \in L^{r(.)}(\Omega)$ and $g \in L^{q(.)}(\Omega)$, then $fg \in L^{s(.)}(\Omega)$, with

$$\|fg\|_{s(.)} \leq 2\|f\|_{r(.)}\|g\|_{q(.)}.$$

Lemma 3.3. *[5, 16] If $1 < q^- \leq q(x) \leq q^+ < +\infty$ holds then, for any $f \in L^{q(.)}(\Omega)$,*

$$\textit{min}\left\{\|f\|_{q(.)}^{q^-}, \|f\|_{q(.)}^{q^+}\right\} \leq \varrho_{q(.)}(f) \leq \textit{max}\left\{\|f\|_{q(.)}^{q^-}, \|f\|_{q(.)}^{q^+}\right\},$$

where

$$q^- = ess\inf_{x\in\Omega}\ q(x)\ \textit{and}\ q^+ = ess\sup_{x\in\Omega}\ q(x).$$

Definition 3.1. *We say that a function $q : \Omega \longrightarrow \mathbb{R}$ is log-Hölder continuous on Ω, if there exists a constant $\theta > 0$ such that for all $0 < \delta < 1$ and for $a.e.\ x, y \in \Omega$, with*

$|x-y|<\delta$, *we have*

$$|q(x)-q(y)| \leq -\frac{\theta}{log|x-y|}.$$

Lemma 3.4. ***(Poincaré's Inequality [16])*** *Let Ω be a bounded domain of $\mathbb{R}^n$ and q be a log-Hölder continuous function on Ω. Then,*

$$\|f\|_{q(.)} \leq C\|\Delta f\|_{q(.)}, \ for\ all\ f \in W_0^{2,q(.)}(\Omega),$$

where C is a positive constant depending on q^-, q^+ and Ω only. In particular, the space $W_0^{2,q(.)}(\Omega)$ has an equivalent norm given by $\|.\|_{W_0^{2,q(.)}(\Omega)} = \|\Delta.\|_{q(.)}$.

Lemma 3.5. ***(Embedding Property [11])*** *Let $q:\Omega \longrightarrow [1,\infty)$ be a measurable function such that, for a.e $x\in\Omega$, we have*

$$\begin{cases} 2 \leq q^- \leq q^+ < \infty, & \textit{if } n \leq 4, \\ 2 \leq q^- \leq q(x) \leq q^+ < \frac{4n}{n-4}, & \textit{if } n > 4. \end{cases}$$

Then, there exists a continuous and compact embedding $H_0^2(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.

For the existence of the local (weak) solution of problem (1.1), we recall our resut in [[10], Theorem 3.3, p. 10], which is given as follows.

Theorem 3.1. *Let $n=\overline{1,6}$. Assume that $m,r,p \in C(\overline{\Omega})$ such that, for all $x\in\overline{\Omega}$, we have*

$$(H.1) \qquad \left| \begin{array}{ll} 2 \leq m^-, & \textit{if } \ n \leq 4, \\ 2 \leq m^- \leq m(x) \leq m^+ \leq 10, & \textit{if } \ n=5, \\ 2 \leq m^- \leq m(x) \leq m^+ \leq 6, & \textit{if } \ n=6, \end{array} \right.$$

$$(H.2) \qquad \left| \begin{array}{ll} 2 \leq r^-, & \textit{if } \ n \leq 4, \\ 2 \leq r^- \leq r(x) \leq r^+ \leq 10, & \textit{if } \ n=5, \\ 2 \leq r^- \leq r(x) \leq r^+ \leq 6, & \textit{if } \ n=6 \end{array} \right.$$

and

$$(H.3) \qquad \left| \begin{array}{ll} 3 \leq p^-, & \textit{if } \ n \leq 4, \\ 3 \leq p^- \leq p(x) \leq p^+ \leq 5, & \textit{if } \ n=5, \\ p(x)=3, & \textit{if } \ n=6. \end{array} \right.$$

Then, for any (u_0, u_1) *and* (v_0, v_1) *in* $H_0^2(\Omega) \times L^2(\Omega)$, *the problem* (1.1) *has a unique local weak solution* (u, v) *on* $[0, T)$, *for* T *small enough, with*

$$\left| \begin{array}{l} u, v \in L^{\infty}\left([0,T); H_0^2(\Omega)\right), \\ u_t \in L^{\infty}\left([0,T); L^2(\Omega)\right) \cap L^{m(.)}\left(\Omega \times (0,T)\right), \\ v_t \in L^{\infty}\left([0,T); L^2(\Omega)\right) \cap L^{r(.)}\left(\Omega \times (0,T)\right). \end{array} \right. \tag{3.1}$$

Here,

$$m^- = \inf_{x\in\overline{\Omega}} m(x) \quad m^+ = \sup_{x\in\overline{\Omega}} m(x).$$
$$r^- = \inf_{x\in\overline{\Omega}} r(x) \quad r^+ = \sup_{x\in\overline{\Omega}} r(x).$$
$$p^- = \inf_{x\in\overline{\Omega}} p(x) \text{ and } p^+ = \sup_{x\in\overline{\Omega}} p(x).$$

From the expressions (1.2) and (1.3), one can easily see that, for all $(u, v) \in \mathbb{R}^2$,

$$u\, f_1(x,u,v) + v f_2(x,u,v) = (p(x)+1)F(x,u,v). \tag{3.2}$$

We, also, have the following results.

Lemma 3.6. *[1] There exist* $C_1, C_2 > 0$ *such that, for all* $x \in \overline{\Omega}$ *and* $(u, v) \in \mathbb{R}^2$, *we have*

$$C_1\big(|u|^{p(x)+1} + |v|^{p(x)+1}\big) \le F(x,u,v) \le C_2(|u|^{p(x)+1} + |v|^{p(x)+1}). \tag{3.3}$$

Corollary 3.1. *For all* $x \in \overline{\Omega}$ *and* $(u, v) \in \mathbb{R}^2$, *we have*

$$C_1\big(\zeta(u) + \zeta(v)\big) \le \int_\Omega F(x,u,v)dx \le C_2\big(\zeta(u) + \zeta(v)\big), \tag{3.4}$$

where

$$\zeta(u) = \int_\Omega |u|^{p(x)+1} dx \text{ and } \zeta(v) = \int_\Omega |v|^{p(x)+1} dx.$$

The energy functional associated to our problem is

$$E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) - \int_\Omega F(x,u,v)\,dx, \tag{3.5}$$

for all $t \in [0, T)$. A direct computation implies, for $a.e.$ $t \in (0, T)$,

$$E'(t) = -\int_\Omega |u_t|^{m(x)} dx - \int_\Omega |v_t|^{r(x)} dx \le 0. \tag{3.6}$$

4. Blow up result

In this section, our goal is to show that any solution of problem (1.1) blows up in some finite time $T^* \in (0, T)$, if

$$\max\{m^+, r^+\} < p^- \text{ and } 0 < E(0) < E_1, \tag{4.1}$$

where

$$E_1 = \left(\frac{1}{2} - \frac{1}{p^- + 1}\right)\gamma_1^2, \quad \gamma_1 = \left(d_*\left(p^- + 1\right)\right)^{\frac{1}{1-p^-}}, \tag{4.2}$$

$$d_* = \left(\sqrt{2^{(p^-+1)}}a + 2b\right)c_*^{p^-+1}$$

and c_* is a positive constant which comes from the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^{p(.)+1}(\Omega)$.

Remark 4.1. *The following well-known inequalities are needed for the proof of our subsequent lemmas.*

1. *For* $A, B \geq 0$ *and* $d \geq 1$*, we have*

$$(A + B)^d \leq 2^{d-1}\left(A^d + B^d\right). \tag{4.3}$$

2. *For* $z \geq 0,\ 0 < \delta \leq 1$ *and* $a > 0$*, we have*

$$z^\delta \leq z + 1 \leq \left(1 + \frac{1}{a}\right)(z + a). \tag{4.4}$$

3. *For* $X,\ Y \geq 0,\ \delta > 0$ *and* $\frac{1}{\lambda} + \frac{1}{\beta} = 1$*, Young's inequality gives*

$$XY \leq \frac{\delta^\lambda}{\lambda}X^\lambda + \frac{\delta^{-\beta}}{\beta}Y^\beta. \tag{4.5}$$

4. *The embedding result (Lemma 3.5), Hölder's and Young's inequalities and* (4.3) *imply that*

$$\|u + v\|_{p(.)+1} \leq \sqrt{2}c_*\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right)^{1/2} \tag{4.6}$$

and

$$\|uv\|_{\frac{p(.)+1}{2}} \leq c_*^2\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right). \tag{4.7}$$

Lemma 4.1. *For any solution* (u, v) *of the system* (1.1)*, with initial energy*

$$E(0) < E_1 \tag{4.8}$$

and initial data satisfying

$$\gamma_1 < \left(\|\Delta u_0\|_2^2 + \|\Delta v_0\|_2^2\right)^{1/2} \leq \frac{1}{\sqrt{2}c_*}, \tag{4.9}$$

there exists $\gamma_2 > \gamma_1$ such that

$$\gamma_2 \leq \left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right)^{1/2}, \ \forall t \in [0, T). \tag{4.10}$$

Proof. Let $\gamma = (\|\Delta u\|_2^2 + \|\Delta v\|_2^2)^{1/2}$. Then, by (3.5), we have

$$E(t) \geq \frac{1}{2}\gamma^2 - \int_\Omega F(x, u, v)\, dx. \tag{4.11}$$

The use of Lemma 3.3, (4.6) and (4.7) leads to

$$\begin{aligned}
\int_\Omega F(x,u,v)\,dx &= a\int_\Omega |u+v|^{p(x)+1}\,dx + 2b\int_\Omega |uv|^{\frac{p(x)+1}{2}}\,dx \\
&\leq a\max\left\{\|u+v\|_{p(.)+1}^{p^-+1}, \|u+v\|_{p(.)+1}^{p^++1}\right\} \\
&\quad + 2b\max\left\{\|uv\|_{\frac{p(.)+1}{2}}^{\frac{p^-+1}{2}}, \|uv\|_{\frac{p(.)+1}{2}}^{\frac{p^++1}{2}}\right\} \\
&\leq a\max\left\{\left(\sqrt{2}c_*\gamma\right)^{p^-+1}, \left(\sqrt{2}c_*\gamma\right)^{p^++1}\right\} \\
&\quad + 2b\max\left\{(c_*\gamma)^{p^-+1}, (c_*\gamma)^{p^++1}\right\}.
\end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), we obtain

$$\begin{aligned}
E(t) \geq &\frac{1}{2}\gamma^2 - a\max\left\{\left(\sqrt{2}c_*\gamma\right)^{p^-+1}, \left(\sqrt{2}c_*\gamma\right)^{p^++1}\right\} \\
&- 2b\max\left\{(c_*\gamma)^{p^-+1}, (c_*\gamma)^{p^++1}\right\}.
\end{aligned} \tag{4.13}$$

For γ in $\left[0, \frac{1}{\sqrt{2}c_*}\right]$, one can easily check that

$$c_*^2\gamma^2 \leq 2c_*^2\gamma^2 \leq 1.$$

Consequently, we have

$$\left(\sqrt{2}c_*\gamma\right)^{p^-+1} \geq \left(\sqrt{2}c_*\gamma\right)^{p^++1} \text{ and } (c_*\gamma)^{p^-+1} \geq \left(\sqrt{2}c_*\gamma\right)^{p^++1}.$$

Thus, 4.13 reduces to

$$E(t) \geq \frac{1}{2}\gamma^2 - \left(\sqrt{2^{(p^-+1)}}a + 2b\right) c_*^{p^-+1}\gamma^{p^-+1}.$$

If we set

$$h(\gamma) = \frac{1}{2}\gamma^2 - d^*\gamma^{p^-+1},$$

then

$$E(t) \geq h(\gamma), \text{ for all } \gamma \in \left[0, \frac{1}{\sqrt{2}c_*}\right]. \tag{4.14}$$

It is clear that h is strictly increasing on $[0, \gamma_1)$ and strictly decreasing on $[\gamma_1, +\infty)$. Since $E(0) < E_1$ and $E_1 = h(\gamma_1)$, then, we can find $\gamma_2 > \gamma_1$ such that $h(\gamma_2) = E(0)$. But,

$$\gamma_0 = \left(\|\Delta u_0\|_2^2 + \|\Delta v_0\|_2^2\right)^{1/2} \in \left(\gamma_1, \frac{1}{\sqrt{2}c_*}\right],$$

therefore, by (4.14), we get

$$h(\gamma_2) = E(0) \geq h(\gamma_0).$$

This implies that $\gamma_0 \geq \gamma_2$. Hence, $\gamma_2 \in \left(\gamma_1, \frac{1}{\sqrt{2}c_*}\right]$.

To prove (4.10), we assume on the contrary that there is a $t_0 \in [0, T)$ such that

$$\left(\|\Delta u(., t_0)\|_2^2 + \|\Delta v(., t_0)\|_2^2\right)^{1/2} < \gamma_2.$$

Since the function $t \longmapsto (\|\Delta u\|_2^2 + \|\Delta v\|_2^2)^{1/2}$ is continuous and $\gamma_2 > \gamma_1$, t_0 can be selected so that

$$\left[\|\Delta u(., t_0)\|_2^2 + \|\Delta v(., t_0)\|_2^2\right]^{1/2} > \gamma_1.$$

Using (4.14) and the fact that h is decreasing on $\left[\gamma_1, \frac{1}{\sqrt{2}c_*}\right]$, we obtain

$$E(t_0) \geq h\left(\left(\|\Delta u(., t_0)\|_2^2 + \|\Delta v(., t_0)\|_2^2\right)^{1/2}\right) > h(\gamma_2) = E(0),$$

which contradicts the fact that $E(t) \leq E(0)$, for all $t \in [0.T)$. Thus, (4.10) is established. □

Lemma 4.2. *Let $\mathcal{H}(t) = E_1 - E(t)$, for all $t \in [0, T)$. Then, we have*

$$0 < \mathcal{H}(0) \leq \mathcal{H}(t) \leq \int_\Omega F(x, u, v)\, dx, \text{ for all } t \in [0, T) \tag{4.15}$$

and

$$\int_\Omega F(x, u, v)\, dx \geq d_*\gamma_2^{p^-+1}. \tag{4.16}$$

Proof. Using (3.6), (4.8) and (4.11), it follows that

$$0 < E_1 - E(0) = H(0) \le H(t) \le E_1 - \frac{1}{2}\gamma^2 + \int_\Omega F(x,u,v)\,dx. \tag{4.17}$$

From the fact that $h(\gamma_1) = \frac{1}{2}\gamma_1^2 - d_*\gamma_1^{p^-+1} = E_1$, we have

$$E_1 - \frac{1}{2}\gamma_1^2 = -d_*\gamma_1^{p^-+1},$$

then since $\gamma \ge \gamma_2 > \gamma_1$, we obtain

$$\mathcal{H}(t) \le -d_*\gamma_1^{p^-+1} + \int_\Omega F(x,u,v)\,dx \le \int_\Omega F(x,u,v)\,dx.$$

Thus, (4.15) is established.

To prove (4.16), we use (4.15), to obtain

$$E(0) \ge \frac{1}{2}\gamma^2 - \int_\Omega F(x,u,v)\,dx.$$

which implies that

$$\int_\Omega F(x,u,v)\,dx \ge \frac{1}{2}\gamma^2 - E(0).$$

But $E(0) = h(\gamma_2)$ and $\gamma \ge \gamma_2$, so

$$\int_\Omega F(x,u,v)\,dx \ge \frac{1}{2}\gamma_2^2 - h(\gamma_2) = d_*\gamma_2^{p^-+1}.$$

□

Lemma 4.3. *[22] There exist $C_3, C_4, C_5 > 0$ such that any solution of* (1.1) *satisfies*

$$\|u\|_{p^-+1}^{p^-+1} + \|v\|_{p^-+1}^{p^-+1} \le C_3(\zeta(u) + \zeta(v)), \tag{4.18}$$

$$\int_\Omega |u|^{m(x)}\,dx \le C_4\left[(\zeta(u)+\zeta(v))^{\frac{m^+}{p^-+1}} + (\zeta(u)+\zeta(v))^{\frac{m^-}{p^-+1}}\right] \tag{4.19}$$

and

$$\int_\Omega |v|^{r(x)}\,dx \le C_5\left[(\zeta(u)+\zeta(v))^{\frac{r^+}{p^-+1}} + (\zeta(u)+\zeta(v))^{\frac{r^-}{p^-+1}}\right], \tag{4.20}$$

where $\zeta(u)$ and $\zeta(v)$ are defined in Corollary 3.1.

Lemma 4.4. *Let* $\mathcal{G}(t) = \mathcal{H}^{1-\sigma}(t) + \varepsilon \int_\Omega (uu_t + vv_t)\,dx, t > 0$, *where* $\varepsilon > 0$ *to be fixed later and*

$$0 < \sigma \leq \min\left\{ \frac{p^- - m^+ + 1}{(p^- + 1)(m^+ - 1)}, \frac{p^- - r^+ + 1}{(p^- + 1)(r^+ - 1)}, \frac{p^- - 1}{2(p^- + 1)} \right\}. \tag{4.21}$$

Then, there exists $\rho > 0$, *such that*

$$\mathcal{G}'(t) \geq \varepsilon\rho\left(\mathcal{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \zeta(u) + \zeta(v)\right) \tag{4.22}$$

and, hence,

$$\mathcal{G}(t) \geq \mathcal{G}(0) > 0, \text{ for all } t > 0.$$

Proof. Differentiate $\mathcal{G}$ and use (1.1) to have

$$\begin{aligned}\mathcal{G}'(t) &= (1-\sigma)\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right)\\ &\quad + \varepsilon\int_\Omega (uf_1(x,u,v) + vf_2(x,u,v))\,dx - \varepsilon\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right)\\ &\quad - \varepsilon\int_\Omega \left(|u_t|^{m(x)-2}u_t u + |v_t|^{r(x)-2}v_t v\right)dx.\end{aligned} \tag{4.23}$$

By the definition of $\mathcal{H}$ and E, we get

$$\|\Delta u\|_2^2 + \|\Delta v\|_2^2 = 2\int_\Omega F(x,u,v)\,dx - \|u_t\|_2^2 - \|v_t\|_2^2 + 2E_1 - 2\mathcal{H}(t). \tag{4.24}$$

Combining (3.2), (4.23) and (4.24), we obtain

$$\begin{aligned}\mathcal{G}'(t) &\geq (1-\sigma)\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t) + 2\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + 2\varepsilon\mathcal{H}(t)\\ &\quad - 2\varepsilon E_1 + \varepsilon\left(p^- - 1\right)\int_\Omega F(x,u,v)\,dx\\ &\quad - \varepsilon\int_\Omega \left(|u|\,|u_t|^{m(x)-1} + |v|\,|v_t|^{r(x)-1}\right)dx.\end{aligned} \tag{4.25}$$

Then, (4.16) and (4.25) lead to

$$\begin{aligned}\mathcal{G}'(t) &\geq (1-\sigma)\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t) + 2\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \varepsilon R\int_\Omega F(x,u,v)\,dx\\ &\quad + 2\varepsilon\mathcal{H}(t) - \varepsilon\int_\Omega \left(|u|\,|u_t|^{m(x)-1} + |v|\,|v_t|^{r(x)-1}\right)dx,\end{aligned} \tag{4.26}$$

where $R = p^- - 1 - 2\left(d_*\gamma_2^{p^-+1}\right)^{-1} E_1 > 0$, since $\gamma_2 > \gamma_1$.

Now, we estimate the last two terms of (4.26). Apply (4.5) with

$$X = |u|\,,\ Y = |u_t|^{m(x)-1}\,,\ \lambda = m\left(x\right),\ \beta = \frac{m\left(x\right)}{m\left(x\right)-1},$$

to get

$$\begin{aligned}\int_{\Omega} |u|\,|u_t|^{m(x)-1}\,dx &\leq \int_{\Omega} \frac{\delta^{m(x)}}{m\left(x\right)}\,|u|^{m(x)}\,dx \\ &\quad+ \int_{\Omega} \frac{m\left(x\right)-1}{m\left(x\right)} \delta^{-m(x)/(m(x)-1)}\,|u_t|^{m(x)}\,dx.\end{aligned} \tag{4.27}$$

Let $\tilde{k}$ be a positive constant, to be selected later, and take $\delta = \left[\tilde{k}\mathcal{H}^{-\sigma}\left(t\right)\right]^{\frac{1-m(x)}{m(x)}}$ to obtain

$$\begin{aligned}\int_{\Omega} |u|\,|u_t|^{m(x)-1}\,dx &\leq \frac{\tilde{k}^{1-m^-}}{m^-} \int_{\Omega} \left[\mathcal{H}\left(t\right)\right]^{\sigma(m(x)-1)}\,|u|^{m(x)}\,dx \\ &\quad+ \frac{m^+-1}{m^-}\tilde{k}\mathcal{H}^{-\sigma}\left(t\right) \int_{\Omega} |u_t|^{m(x)}\,dx.\end{aligned} \tag{4.28}$$

Similarly, one can have

$$\begin{aligned}\int_{\Omega} |v|\,|v_t|^{r(x)-1}\,dx &\leq \frac{\tilde{k}^{1-r^-}}{r^-} \int_{\Omega} \left[\mathcal{H}\left(t\right)\right]^{\sigma(r(x)-1)}\,|v|^{r(x)}\,dx \\ &\quad+ \frac{r^+-1}{r^-}\tilde{k}\mathcal{H}^{-\sigma}\left(t\right) \int_{\Omega} |v_t|^{r(x)}\,dx.\end{aligned} \tag{4.29}$$

The properties of $m(x)$ and $\mathcal{H}(t)$ give

$$\begin{aligned}\int_{\Omega} \left[\mathcal{H}\left(t\right)\right]^{\sigma(m(x)-1)}\,|u|^{m(x)}\,dx &= \int_{\Omega} \left[\frac{\mathcal{H}\left(t\right)}{\mathcal{H}\left(0\right)}\right]^{\sigma(m(x)-1)} \left[\mathcal{H}\left(0\right)\right]^{\sigma(m(x)-1)}\,|u|^{m(x)}\,dx \\ &\leq c_1 \left[\mathcal{H}\left(t\right)\right]^{\sigma\left(m^+-1\right)} \int_{\Omega} \left[\mathcal{H}\left(0\right)\right]^{\sigma(m(x)-1)}\,|u|^{m(x)}\,dx,\end{aligned}$$

where $c_1 = 1/\left[\mathcal{H}\left(0\right)\right]^{\sigma\left(m^+-1\right)}$. But $\left[\mathcal{H}\left(0\right)\right]^{\sigma(m(x)-1)} \leq c_2,$ for all $x \in \Omega$, where $c_2 > 0$. So, for some $c_3 > 0$, we get

$$\int_{\Omega} \left[\mathcal{H}\left(t\right)\right]^{\sigma(m(x)-1)}\,|u|^{m(x)}\,dx \leq c_3 \left[\mathcal{H}\left(t\right)\right]^{\sigma\left(m^+-1\right)} \int_{\Omega} |u|^{m(x)}\,dx. \tag{4.30}$$

Recalling (3.4), (4.15) and (4.19), inequality (4.30) turns into

$$\begin{aligned}\int_{\Omega}\left[\mathcal{H}(t)\right]^{\sigma(m(x)-1)}|u|^{m(x)}\,dx &\leq c_4\left(\zeta(u)+\zeta(v)\right)^{\sigma\left(m^+-1\right)+\frac{m^+}{p^-+1}}\\ &\quad + c_4\left(\zeta(u)+\zeta(v)\right)^{\sigma\left(m^+-1\right)+\frac{m^-}{p^-+1}},\end{aligned}\tag{4.31}$$

for some $c_4 > 0$. Apply (4.4) with $z=\zeta(u)+\zeta(v),\ a=\mathcal{H}(0),\ \delta=\sigma\left(m^+-1\right)+\frac{m^+}{p^-+1}$ and then with $\delta=\sigma\left(m^+-1\right)+\frac{m^-}{p^-+1}$, respectively, we get

$$\begin{aligned}(\zeta(u)+\zeta(v))^{\sigma\left(m^+-1\right)+\frac{m^+}{p^-+1}} &\leq \left[1+\frac{1}{\mathcal{H}(0)}\right]\left(\zeta(u)+\zeta(v)+\mathcal{H}(0)\right)\\ &\leq \tilde{R}\left(\zeta(u)+\zeta(v)+\mathcal{H}(t)\right)\end{aligned}\tag{4.32}$$

and

$$\left(\zeta(u)+\zeta(v)\right)^{\sigma\left(m^+-1\right)+\frac{m^-}{p^-+1}} \leq \tilde{R}\left(\zeta(u)+\zeta(v)+\mathcal{H}(t)\right)\tag{4.33}$$

where $\tilde{R}=1+\frac{1}{\mathcal{H}(0)}$. By substituting (4.32) and (4.33) into (4.31), we obtain

$$\int_{\Omega}\left[\mathcal{H}(t)\right]^{\sigma(m(x)-1)}|u|^{m(x)}\,dx \leq c_5\left(\zeta(u)+\zeta(v)+\mathcal{H}(t)\right),\tag{4.34}$$

for some $c_5 > 0$. Repeating similar calculations, we arrive at

$$\int_{\Omega}\left[\mathcal{H}(t)\right]^{\sigma(r(x)-1)}|v|^{r(x)}\,dx \leq c_6\left(\zeta(u)+\zeta(v)+\mathcal{H}(t)\right),\tag{4.35}$$

for some $c_6 > 0$. Now, inserting (4.34) and (4.35) into (4.28) and (4.29), respectively, we infer, for some $c_7, c_8 > 0$, that

$$\begin{aligned}\int_{\Omega}|u|\,|u_t|^{m(x)-1}\,dx &\leq \frac{\tilde{k}^{1-m^-}}{m^-}c_7\left(\zeta(u)+\zeta(v)+\mathcal{H}(t)\right)\\ &\quad + \frac{m^+-1}{m^-}\tilde{k}\mathcal{H}^{-\sigma}(t)\int_{\Omega}|u_t|^{m(x)}\,dx\end{aligned}\tag{4.36}$$

and

$$\begin{aligned}\int_{\Omega}|v|\,|v_t|^{r(x)-1}v\,dx &\leq \frac{\tilde{k}^{1-r^-}}{r^-}c_8\left(\zeta(u)+\zeta(v)+\mathcal{H}(t)\right)\\ &\quad + \frac{r^+-1}{r^-}\tilde{k}\mathcal{H}^{-\sigma}(t)\int_{\Omega}|v_t|^{r(x)}\,dx.\end{aligned}\tag{4.37}$$

Adding (4.36) and (4.37), it yields

$$\begin{aligned}\int_\Omega \left(|u|\,|u_t|^{m(x)-1} + |v|\,|v_t|^{r(x)-1}\right) dx \le{}& \frac{\tilde{k}^{1-m^-}}{m^-} c_7 \left(\zeta(u) + \zeta(v) + \mathcal{H}(t)\right)\\ &+ \frac{\tilde{k}^{1-r^-}}{r^-} c_8 \left(\zeta(u) + \zeta(v) + \mathcal{H}(t)\right)\\ &+ M\mathcal{H}^{-\sigma}(t)\, H'(t),\end{aligned} \tag{4.38}$$

for $M = \tilde{k} \max\{\frac{m^+-1}{m^-}, \frac{r^+-1}{r^-}\}$, since

$$H'(t) = \int_\Omega |u_t|^{m(x)}\, dx + \int_\Omega |v_t|^{r(x)}\, dx.$$

Substituting (4.38) into (4.26), we obtain, for some $c_9 > 0$,

$$\begin{aligned}\mathcal{G}'(t) \ge{}& (1-\sigma-\varepsilon M)\,\mathcal{H}^{-\sigma}(t)\,\mathcal{H}'(t) + 2\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right)\\ &+ \varepsilon\left(2 - \frac{\tilde{k}^{1-m^-}}{m^-}c_7 - \frac{\tilde{k}^{1-r^-}}{r^-}c_8\right)\mathcal{H}(t)\\ &+ \varepsilon\left(c_9 - \frac{\tilde{k}^{1-m^-}}{m^-}c_7 - \frac{\tilde{k}^{1-r^-}}{r^-}c_8\right)\left(\zeta(u) + \zeta(v)\right).\end{aligned}$$

Now, we select $\tilde{k}$ large enough so that

$$\begin{aligned}\mathcal{G}'(t) \ge{}& (1-\sigma-\varepsilon M)\,\mathcal{H}^{-\sigma}(t)\,\mathcal{H}'(t)\\ &+ \varepsilon c_{10}\left(\mathcal{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \zeta(u) + \zeta(v)\right),\end{aligned}$$

for some $c_{10} > 0$. Once $\tilde{k}$ is fixed, we select ε small enough so that

$$1-\sigma-\varepsilon M \ge 0 \text{ and } \mathcal{G}(0) = \mathcal{H}^{1-\sigma}(0) + \varepsilon\int_\Omega (u_0u_1 + v_0v_1)\, dx > 0.$$

Thus, using the fact that $\mathcal{H}$ is non-decreasing function, we establish (4.22). □

Now, we present our blow-up result.

Theorem 4.1. *Under the assumptions* (4.1) *and* (4.9)*, any solution of the system* (1.1)*) blows up in finite time.*

Proof. Using (4.3) and the definition of $\mathcal{G}$, we have

$$
\begin{aligned}
\mathcal{G}^{1/(1-\sigma)}(t) &\leq \left(\mathcal{H}^{1-\sigma}(t)+\varepsilon\int_{\Omega}|uu_t+vv_t|\,dx\right)^{1/(1-\sigma)}\\
&\leq 2^{\sigma/(1-\sigma)}\left(\mathcal{H}(t)+\left(\varepsilon\int_{\Omega}\left(|uu_t|+|vv_t|\right)dx\right)^{1/(1-\sigma)}\right)\\
&\leq c_{11}\left(\mathcal{H}(t)+\left(\int_{\Omega}\left(|u|\,|u_t|+|v|\,|v_t|\right)dx\right)^{1/(1-\sigma)}\right),
\end{aligned}
\tag{4.39}
$$

where $c_{11}=2^{\sigma/(1-\sigma)}\max\left\{1,\varepsilon^{1/(1-\sigma)}\right\}$. The Sobolev embedding, Lemma 4.3 and Hölder's and Young's inequalities give

$$
\begin{aligned}
&\left(\int_{\Omega}\left(|u|\,|u_t|+|v|\,|v_t|\right)dx\right)^{1/(1-\sigma)}\\
&\leq 2^{\sigma/(1-\sigma)}\left(\int_{\Omega}|u|\,|u_t|\,dx\right)^{1/(1-\sigma)}+2^{\sigma/(1-\sigma)}\left(\int_{\Omega}|v|\,|v_t|\,dx\right)^{1/(1-\sigma)}\\
&\leq 2^{\sigma/(1-\sigma)}\left(\|u\|_2^{1/(1-\sigma)}\|u_t\|_2^{1/(1-\sigma)}+\|v\|_2^{1/(1-\sigma)}\|v_t\|_2^{1/(1-\sigma)}\right)\\
&\leq c_{12}\left(\|u\|_{p^-+1}^{1/(1-\sigma)}\|u_t\|_2^{1/(1-\sigma)}+\|v\|_{p^-+1}^{1/(1-\sigma)}\|v_t\|_2^{1/(1-\sigma)}\right)\\
&\leq c_{13}\left(\|u\|_{p^-+1}^{2/(1-2\sigma)}+\|u_t\|_2^2+\|v\|_{p^-+1}^{2/(1-2\sigma)}+\|v_t\|_2^2\right)\\
&\leq c_{13}\left(\left(\zeta(u)+\zeta(v)\right)^{\tau}+\|u_t\|_2^2+\|v_t\|_2^2\right),
\end{aligned}
\tag{4.40}
$$

where $\tau=2/\left(p^-+1\right)(1-2\sigma)\,,c_{12},c_{13}>0$. Using (4.15), (3.4) and since $\tau\leq 1$, we get, for some $c_{14}>0$,

$$
\left(\int_{\Omega}\left(|u|\,|u_t|+|v|\,|v_t|\right)dx\right)^{1/(1-\sigma)}\leq c_{14}\left(\zeta(u)+\zeta(v)+\|u_t\|_2^2+\|v_t\|_2^2+\mathcal{H}(t)\right).
$$

Inserting the last estimate into (4.39), we obtain, for some $c_{15}>0$,

$$
\mathcal{G}^{1/(1-\sigma)}(t)\leq c_{15}\left(\mathcal{H}(t)+\|u_t\|_2^2+\|v_t\|_2^2+\zeta(u)+\zeta(v)\right).
\tag{4.41}
$$

Combining (4.22) and (4.41), we deduce that

$$
\mathcal{G}'(t)\geq\tilde{c}\mathcal{G}^{1/(1-\sigma)}(t)\,,\ \text{for all } t>0,
$$

where $\tilde{c} = \frac{\varepsilon\rho}{c_{15}}$. A simple integration over $(0, t)$ yields

$$\mathcal{G}^{\sigma/(1-\sigma)}(t) \geq \frac{1}{\mathcal{G}^{\frac{-\sigma}{1-\sigma}}(0) - \frac{\sigma\tilde{c}t}{1-\sigma}},$$

consequently, $\mathcal{G}(t) \longrightarrow +\infty$, as $t \longrightarrow T^* \leq \frac{1-\sigma}{\sigma\tilde{c}\left[\mathcal{G}^{\frac{\sigma}{(1-\sigma)}}(0)\right]}$.

This shows that the solution of problem (P) blows up in finite time. □

5. Global Existence and Decay-rate Etimates

In this section, we establish the existence of global solutions, for initial data in a certain stable set. Then, we show that the decay estimates of the solution energy are exponential or polynomial, depending on the exponents m and r.

5.1. **Global Existence.** To state and prove our first result, we introduce the two functionals defined for all $t \in (0, T)$ by

$$I(t) = I(u(t)) = \|\Delta u\|_2^2 + \|\Delta v\|_2^2 - \left(p^+ + 1\right)\int_\Omega F(x, u, v)\, dx, \tag{5.1}$$

$$J(t) = J(u(t)) = \frac{1}{2}\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) - \int_\Omega F(x, u, v)\, dx \tag{5.2}$$

and give the following Lemma.

Lemma 5.1. *Under the assumptions of Theorem 3.1 and if*

$$I(0) > 0 \text{ and } \beta < 1,$$

where

$$\beta = C_2(p^+ + 1)max\left\{ c_*^{p^-+1}\left(\frac{2(p^+ + 1)}{p^+ - 1}E(0)\right)^{\frac{p^- - 1}{2}}, c_*^{p^+ + 1}\left(\frac{2(p^+ + 1)}{p^+ - 1}E(0)\right)^{\frac{p^+ - 1}{2}}\right\}.$$

Then,

$$I(t) > 0, \textit{ for all } t \in (0, T). \tag{5.3}$$

Proof. From the continuity of I and the fact that $I(0) > 0$, there exists t_k in $(0, T)$ such that

$$I(t) \geq 0, \ \forall t \in (0, t_k). \tag{5.4}$$

Recalling (5.1) and (5.2), we have

$$J(t) = \frac{p^+ - 1}{2(p^+ + 1)} \left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \right) + \frac{1}{p^+ + 1} I(t).$$

Combining with (5.4), we get

$$J(t) \geq \frac{p^+ - 1}{2(p^+ + 1)} \left(\|\Delta u\|_2^2 + \|\Delta u\|_2^2 \right), \forall t \in (0, t_k). \tag{5.5}$$

From the definition of the energy, we easily check that

$$E(t) = J(t) + \frac{1}{2} \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right), \tag{5.6}$$

for all $t \in (0, T)$. Consequently,

$$\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \leq \frac{2(p^+ + 1)}{(p^+ - 1)} E(t).$$

Thus, the decreasingness of E leads to

$$\max \left\{ \|\Delta u\|_2^2, \|\Delta v\|_2^2 \right\} \leq \frac{2(p^+ + 1)}{(p^+ - 1)} E(0), \forall t \in (0, t_k). \tag{5.7}$$

On the other hand, from Lemma 3.3 and the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^{p(\cdot)+1}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega |u|^{p(x)+1} dx &\leq max\{c_*^{p^-+1} \|\Delta u\|_2^{p^-+1}, c_*^{p^++1} \|\Delta u\|_2^{p^++1}\} \\ &\leq max\, \{c_*^{p^-+1} \|\Delta u\|_2^{p^--1}, c_*^{p^++1} \|\Delta u\|_2^{p^+-1}\} \|\Delta u\|_2^2.\end{aligned}$$

Combining with (5.7), this yields, for all $t \in (0, t_k)$,

$$\begin{aligned}&\int_\Omega |u|^{p(x)+1} dx \\ &\leq max \left\{ c_*^{p^-+1} \left(\frac{2(p^+ + 1)}{(p^+ - 1)} E(0) \right)^{\frac{p^- - 1}{2}}, c_*^{p^++1} \left(\frac{2(p^+ + 1)}{(p^+ - 1)} E(0) \right)^{\frac{p^+ - 1}{2}} \right\} \|\Delta u\|_2^2.\end{aligned}$$

Therefore,

$$\int_\Omega |u|^{p(x)+1} dx \leq \frac{\beta}{C_2(p^+ + 1)} \|\Delta u\|_2^2. \tag{5.8}$$

In a similar way, one can show that

$$\int_{\Omega} |v|^{p(x)+1}\, dx \leq \frac{\beta}{C_2\left(p^{+}+1\right)} \|\Delta v\|_2^2 . \tag{5.9}$$

The addition of (5.8) and (5.9)) gives

$$\int_{\Omega} \left(|u|^{p(x)+1} + |v|^{p(x)+1} \right) dx \leq \frac{\beta}{C_2\left(p^{+}+1\right)} \left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \right) . \tag{5.10}$$

Combining (5.10) with (3.4), we infer that

$$\begin{aligned} \int_{\Omega} F\left(x, u, v\right) dx &\leq \frac{\beta}{p^{+}+1} \left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \right) \\ &< \frac{1}{p^{+}+1} \left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \right) , \end{aligned} \tag{5.11}$$

for all $t \in (0, t_k]$. From the definition of I, this leads to

$$I\left(t\right) > 0.\ \forall t \in \left(0, t_k\right] .$$

By repeating the above procedure and using the decreasingness of $E,$ we can extend t_k to T and obtain (5.3). □

Theorem 5.1. *Assume that all assumptions of Lemma* 5.1 *are fulfilling. Then, the local solution* (u, v) *of system* (1.1) *is global.*

Proof. Substituting (5.5) into (5.6) and thanks to (5.3), it yields

$$E\left(t\right) \geq \frac{p^{+}-1}{2\left(p^{+}+1\right)} \left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \right) + \frac{1}{2} \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) ,$$

for all $t \in (0, T)$. Thus, we have

$$\|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2 \leq C_3 E\left(t\right) \leq C_3 E(0), \tag{5.12}$$

for $C_3 = \max\{2, \frac{2(p^{+}+1)}{p^{+}-1}\}$, which means that the norm in (5.12) is bounded independently of t. Therefore, the solution (u, v) exists globally. □

5.2. **Decay-rate Etimates.** To prove the decay result, we need the following Lemma.

Lemma 5.2. *Let the assumptions of Lemma 5.1 hold. Then, there exists a positive constant* C_4*, such that the global solution* (u, v) *satisfies*

$$\int_{\Omega} \left(|u\left(t\right)|^{m(x)} + |v(t)|^{r(x)} \right) dx \leq C_4 E(t) \textit{ for all } \ t \geq 0. \tag{5.13}$$

Proof. The result is immediate by replacing p by m and r in (5.8) and (5.9), respectively, and by recalling (5.12). ☐

Theorem 5.2. *Under the assumptions of Lemma 5.1, the solution of* (1.1) *satisfies the following decay estimates, for all* $t \geq 0$,

$$E(t) \leq \begin{cases} \frac{k}{(1+t)^{2/(\alpha-2)}}, & if\ \alpha > 2, \\ ke^{-\omega t}, & if\ \alpha = 2, \end{cases} \tag{5.14}$$

where $\alpha = max\,\{m^+, r^+\}$ *and* $k, w > 0$ *are two positive constants.*

Proof. Multiplying $(1.1)_1$ by $u(t)\,E^\eta(t)$ and $(1.1)_2$ by $v(t)\,E^\eta(t)$ and, then, integrating each result over $\Omega \times (s,T)$, for $0 < s < T$ and $\eta \geq 0$ to be specified later, we get

$$\int_s^T \int_\Omega E^\eta(t)\left[u(t)\,u_{tt}(t) + u(t)\Delta^2 u(t) + u(t)\,|u_t|^{m(x)-2}\,u_t(t)\right] dxdt$$
$$= \int_s^T \int_\Omega E^\eta(t)\,u(t)\,f_1(x,u,v)\,dxdt$$

and

$$\int_s^T \int_\Omega E^\eta(t)\left[v(t)\,v_{tt}(t) + v(t)\Delta^2 v(t) + v(t)\,|v_t(t)|^{r(x)-2}\,v_t(t)\right] dxdt$$
$$= \int_s^T \int_\Omega E^\eta(t)\,v(t)\,f_2(x,u,v)\,dxdt.$$

Green's formula and the boundary conditions lead to

$$\int_s^T \int_\Omega E^\eta(t)\left[(u(t)\,u_t(t))_t - |u_t(t)|^2 + |\Delta u(t)|^2 \right.$$
$$\left. + u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}\right] dxdt = \int_s^T \int_\Omega E^\eta(t)\,u(t)\,f_1(x,u,v)\,dxdt \tag{5.15}$$

and

$$\int_s^T \int_\Omega E^\eta(t)\left[(v(t)\,v_t(t))_t - |v_t(t)|^2 + |\Delta v(t)|^2 + v(t)\,v_t(t)\,|v_t(t)|^{r(x)-2}\right] dxdt$$
$$= \int_s^T \int_\Omega E^\eta(t)\,v(t)\,f_2(x,u,v)\,dxdt. \tag{5.16}$$

Adding and subtracting the following two terms

$$\int_s^T \int_\Omega E^\eta(t)\left[\beta\left|\Delta u(t)\right|^2+(1+\beta)\left|u_t(t)\right|^2\right]dxdt$$

and

$$\int_s^T \int_\Omega E^\eta(t)\left[\beta\left|\Delta v(t)\right|^2+(1+\beta)\left|v_t(t)\right|^2\right]dxdt,$$

to (5.15) and (5.16), respectively, and recalling (5.11), we arrive at

$$\begin{aligned}
&(1-\beta)\int_s^T E^\eta(t)\int_\Omega\left(\left|\Delta u(t)\right|^2+\left|\Delta v(t)\right|^2+\left|u_t(t)\right|^2+\left|v_t(t)\right|^2\right)dxdt\\
&+\int_s^T E^\eta(t)\int_\Omega\left[\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)_t\right.\\
&\left.-(2-\beta)\left(\left|u_t(t)\right|^2+\left|v_t(t)\right|^2\right)\right]dxdt\\
&+\int_s^T E^\eta(t)\int_\Omega\Big(u(t)\,u_t(t)\left|u_t(t)\right|^{m(x)-2}\\
&+v(t)\,v_t(t)\left|v_t(t)\right|^{r(x)-2}\Big)\,dxdt\\
=&-\int_s^T E^\eta(t)\int_\Omega\left[\beta\left(\left|\Delta u(t)\right|^2+\left|\Delta v(t)\right|^2\right)\right.\\
&\left.-(p(x)+1)\,F(x,u,v)\right]dxdt\le 0.
\end{aligned}\tag{5.17}$$

Now, by exploiting the formula:

$$\begin{aligned}
&E^\eta(t)\int_\Omega\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)_t dx\\
=&\frac{d}{dt}\left(E^\eta(t)\int_\Omega\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dx\right)\\
&-\eta E^{\eta-1}(t)\,E'(t)\int_\Omega\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dx,
\end{aligned}$$

estimate (5.17) yields

$$\begin{aligned}
2(1-\beta)\int_s^T E^{\eta+1}(t)\,dt\le{}&\eta\int_s^T E^{\eta-1}(t)\,E'(t)\int_\Omega\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dxdt\\
&-\int_s^T\frac{d}{dt}\left(E^\eta(t)\int_\Omega\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dx\right)dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_s^T E^\eta(t)\int_\Omega\left(u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}+v(t)\,v_t(t)\,|v_t(t)|^{r(x)-2}\right)dxdt\\
&+(2-\beta)\int_s^T E^\eta(t)\int_\Omega\left(|u_t(t)|^2+|v_t(t)|^2\right)dxdt\\
&=I_1+I_2+I_3+I_4. \qquad (5.18)
\end{aligned}
$$

Next, we handle the terms $I_i, i=\overline{1,4}$ and denote by C a positive generic constant.

First, applying Young's and Poincaré's inequalities, we obtain

$$
\begin{aligned}
I_1&=\eta\int_s^T E^{\eta-1}(t)\,E'(t)\int_\Omega(u(t)\,u_t(t)+v(t)\,v_t(t))\,dxdt\\
&\le\frac{\eta}{2}\int_s^T E^{\eta-1}(t)\left(-E'(t)\right)\left[\|u(t)\|_2^2+\|u_t(t)\|_2^2+\|v(t)\|_2^2+\|v_t(t)\|_2^2\right]dt\\
&\le C\int_s^T E^{\eta-1}(t)\left(-E'(t)\right)\left[\|\Delta u(t)\|_2^2+\|\Delta v(t)\|_2^2+\|u_t(t)\|_2^2+\|v_t(t)\|_2^2\right]dt.
\end{aligned}
$$

By (5.12), this gives

$$
\begin{aligned}
I_1&\le C\int_s^T E^\eta(t)\left(-E'(t)\right)dt\\
&\le CE^{\eta+1}(s)-CE^{\eta+1}(T)\le CE^\eta(0)\,E(s)\le CE(s). \qquad (5.19)
\end{aligned}
$$

Concerning the second term, we have

$$
\begin{aligned}
I_2&=-\int_s^T\frac{d}{dt}\left(E^\eta(t)\int_\Omega(u(t)\,u_t(t)+v(t)\,v_t(t))\,dx\right)dt\\
&=E^\eta(s)\left(\int_\Omega(u(x,s)\,u_t(x,s)+v(x,s)\,v_t(x,s))\,dx\right)\\
&\quad-E^\eta(T)\left(\int_\Omega(u(x,T)\,u_t(x,T)+v(x,T)\,v_t(x,T))\,dx\right).
\end{aligned}
$$

Again, by (5.12) and Young's and Poincaré's inequalities, we get

$$
\begin{aligned}
\left|\int_\Omega u(x,s)\,u_t(x,s)\,dx\right|&\le C\left(\|\Delta u(s)\|_2^2+\|u_t(s)\|_2^2\right)\le CE(s),\\
\left|\int_\Omega u(x,T)\,u_t(x,T)\,dx\right|&\le C\left(\|\Delta u(T)\|_2^2+\|u_t(T)\|_2^2\right)\le CE(T)
\end{aligned}
$$

and, likewise,

$$\left|\int_{\Omega} v(x,s)\, v_t(x,s)\, dx\right| \le C\left(\|\Delta v(s)\|_2^2 + \|v_t(s)\|_2^2\right) \le CE(s)$$

$$\left|\int_{\Omega} v(x,T)\, v_t(x,T)\, dx\right| \le C\left(\|\Delta v(T)\|_2^2 + \|v_t(T)\|_2^2\right) \le CE(T).$$

Therefore,

$$I_2 \le CE^{\eta+1}(s) \le CE^{\eta}(0)\, E(s) \le CE(s). \tag{5.20}$$

For the third term, we apply Young's inequality (as in (4.27)) to obtain, for some $\varepsilon > 0$,

$$\begin{aligned} I_3 &= -\int_s^T E^{\eta}(t) \int_{\Omega} \left(u(t)\, u_t(t)\, |u_t(t)|^{m(x)-2} + v(t)\, v_t(t)\, |v_t(t)|^{r(x)-2}\right) dxdt \\ &\le \int_s^T E^{\eta}(t) \left(\frac{\varepsilon}{2}\int_{\Omega} |u(t)|^{m(x)}\, dx + \frac{1}{\varepsilon}\int_{\Omega} |u_t(t)|^{m(x)}\, dx\right) dt \\ &+ \int_s^T E^{\eta}(t) \left(\frac{\varepsilon}{2}\int_{\Omega} |v(t)|^{r(x)}\, dx + \frac{1}{\varepsilon}\int_{\Omega} |v_t(t)|^{r(x)}\, dx\right) dt. \end{aligned}$$

Invoking Lemma 5.2 and recalling (3.6), it yields

$$\begin{aligned} I_3 &\le \frac{\varepsilon}{2}\int_s^T E^{\eta}(t) \int_{\Omega} \left(|u(t)|^{m(x)} + |v(t)|^{r(x)}\right) dxdt \\ &\quad + \frac{1}{\varepsilon}\int_s^T E^{\eta}(t)\left(-E'(t)\right) dt \end{aligned} \tag{5.21}$$

$$\le \varepsilon C\int_s^T E^{\eta+1}(t)\, dt + C_{\varepsilon}E(s). \tag{5.22}$$

Finally, we handle I_4, as follows:

$$\begin{aligned} I_4 &= (2-\beta)\int_s^T E^{\eta}(t) \int_{\Omega} \left(|u_t(t)|^2 + |v_t(t)|^2\right) dxdt \\ &= (2-\beta)\left[\int_s^T E^{\eta}(t)\int_{\Omega} |u_t(t)|^2\, dxdt + \int_s^T E^{\eta}(t)\int_{\Omega} |v_t(t)|^2\, dxdt\right] \\ &= (2-\beta)(J_1 + J_2). \end{aligned}$$

We claim that

$$J_1, J_2 \le \varepsilon C\int_s^T E^{\eta+1}(t)\, dt + C_{\varepsilon}E(s). \tag{5.23}$$

Indeed, by taking

$$\alpha = \ max\left\{m^{+}, r^{+}\right\}, \ \tilde{\alpha} = \ min\left\{m^{-}, r^{-}\right\},$$
$$\Omega_{+} = \left\{x \in \Omega \ / \ \left|u\left(x,t\right)\right| \geq 1\right\} \text{ and } \Omega_{-} = \left\{x \in \Omega \ / \ \left|u\left(x,t\right)\right| < 1\right\},$$

we obtain

$$\begin{aligned}
J_1 &= \int_s^T E^{\eta}\left(t\right) \int_{\Omega} \left|u_t\left(t\right)\right|^2 dxdt \\
&= \int_s^T E^{\eta}\left(t\right) \left[\int_{\Omega_{-}} \left|u_t\left(t\right)\right|^2 dx + \int_{\Omega_{+}} \left|u_t\left(t\right)\right|^2 dx\right] dt \\
&\leq C \int_s^T E^{\eta}\left(t\right) \left[\left(\int_{\Omega_{-}} \left|u_t\left(t\right)\right|^{\alpha} dx\right)^{2/\alpha} + \left(\int_{\Omega_{+}} \left|u_t\left(t\right)\right|^{\tilde{\alpha}} dx\right)^{2/\tilde{\alpha}}\right] dt \\
&\leq C \int_s^T E^{\eta}\left(t\right) \left[\left(\int_{\Omega_{-}} \left|u_t\left(t\right)\right|^{m(x)} dx\right)^{2/\alpha} + \left(\int_{\Omega_{+}} \left|u_t\left(t\right)\right|^{m(x)} dx\right)^{2/\tilde{\alpha}}\right] dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_1 &\leq C \int_s^T E^{\eta}\left(t\right)\left(-E'\left(t\right)\right)^{2/\alpha} dt + C \int_s^T E^{\eta}\left(t\right)\left(-E'\left(t\right)\right)^{2/\tilde{\alpha}} dt \\
&= C(J_{\alpha} + J_{\tilde{\alpha}}).
\end{aligned} \tag{5.24}$$

Three cases are possible:

(1) if $\alpha = \tilde{\alpha} = 2$ ($m(x) = r(x) = 2$, on Ω), then

$$J_1, J_2 \leq C \int_s^T E^{\eta}\left(t\right)\left(-E^{'}\left(t\right)\right) dt \leq CE^{\eta+1}\left(s\right) - CE^{\eta+1}\left(T\right) \leq CE\left(s\right).$$

Therefore, inequality (5.23) is satisfied, for any $\varepsilon > 0$.

(2) if $\alpha > 2$ and $\tilde{\alpha} = 2$, we exploit Young's inequality with

$$\delta = \left(\eta + 1\right)/\eta \text{ and } \delta' = \eta + 1$$

to find

$$\begin{aligned}
J_{\alpha} &= \int_s^T E^{\eta}\left(t\right)\left(-E'\left(t\right)\right)^{2/\alpha} dt \\
&\leq \varepsilon C \int_s^T E^{\eta+1}\left(t\right) dt + C_{\varepsilon} \int_s^T \left(-E'\left(t\right)\right)^{2(\eta+1)/\alpha} dt.
\end{aligned}$$

So, for $\eta = \frac{\alpha}{2} - 1$, we get

$$J_\alpha \leq \varepsilon C \int_s^T E^{\eta+1}(t)\,dt + C_\varepsilon \int_s^T \left(-E'(t)\right) dt$$

$$\leq \varepsilon C \int_s^T E^{\eta+1}(t)\,dt + C_\varepsilon E(s)\,. \tag{5.25}$$

Also, in this case, we have

$$J_{\tilde{\alpha}} = \int_s^T E^{\eta}(t)\left(-E'(t)\right) dt \leq CE(s). \tag{5.26}$$

By inserting (5.25) and (5.26) into (5.24), we infer that J_1 (and similarly J_2) satisfies (5.23).

(3) if $\alpha \geq \tilde{\alpha} > 2$, we apply Young's inequality with

$$\delta = \tilde{\alpha}/\left(\tilde{\alpha}-2\right) \text{ and } \delta' = \tilde{\alpha}/2$$

to obtain

$$J_{\tilde{\alpha}} = \int_s^T E^{\eta}(t)\left(-E'(t)\right)^{2/\tilde{\alpha}} dt$$

$$\leq \varepsilon C \int_s^T E(t)^{\eta\tilde{\alpha}/(\tilde{\alpha}-2)}\,dt + C_\varepsilon E(s)\,.$$

But $\eta\tilde{\alpha}/\left(\tilde{\alpha}-2\right) = \eta + 1 + \left(\alpha - \tilde{\alpha}\right)/\left(\tilde{\alpha}-2\right)$, since $\eta + 1 = \frac{\alpha}{2}$. Therefore,

$$J_{\tilde{\alpha}} \leq \varepsilon C \left(E(s)\right)^{(\alpha-\tilde{\alpha})/(\alpha-2)} \int_s^T E^{\eta+1}(t)\,dt + C_\varepsilon E(s)$$

$$\leq \varepsilon C \int_s^T E^{\eta+1}(t)\,dt + C_\varepsilon E(s)\,. \tag{5.27}$$

The addition of (5.25) with (5.27) leads to (5.23).

We conclude that the claim is true for any $\alpha \geq \tilde{\alpha} \geq 2$. Therefore,

$$I_4 \leq \varepsilon C \int_s^T E^{\eta+1}(t)\,dt + C_\varepsilon E(s)\,. \tag{5.28}$$

Now, substituting (5.19), (5.20), (5.22) and (5.28) into (5.18), we get

$$2(1-\beta)\int_s^T E^{\eta+1}(t)\,dt \leq \varepsilon C \int_s^T E^{\eta+1}(t)\,dt + C_\varepsilon E(s)\,,$$

with $\eta = \frac{\alpha}{2} - 1$. So,

$$2\left(1-\beta\right)\int_{s}^{T} E^{\frac{\alpha}{2}}\left(t\right)dt \leq \varepsilon C \int_{s}^{T} E^{\frac{\alpha}{2}}\left(t\right)dt + C_{\varepsilon}E\left(s\right).$$

Choosing ε small enough, we obtain

$$\int_{s}^{T} E^{\frac{\alpha}{2}}\left(t\right)dt \leq CE\left(s\right).$$

Letting $T \longrightarrow \infty$, it yields

$$\int_{s}^{\infty} E^{\frac{\alpha}{2}}\left(t\right)dt \leq CE\left(s\right), \forall s > 0.$$

Applying Komornik's lemma, we get the desired decay estimates. □

ACKNOWLEDGMENT

The authors thank University Batna 2, King Fahd University of Petroleum and Minerals and University of Sharjah. The second and third authors are supported by KFUPM, project # INCB2205.

REFERENCES

[1] K. AGRE, M. RAMMAHA: *Systems of nonlinear wave equations with damping and source terms*, Differential Integral Equations. **19** (2006), 1235–1270.

[2] C. ALVES, M. CAVALCANTI, V. CAVALCANTY, M. RAMMAHA, D. TOUNDYKOV: *On existence, uniform decay rates and blow up for solutions of systems of nonlinear wave equations with damping and source terms*, Discrete and Continuous Dynamical Systems Series S. **2**(3) (2009), 583–608.

[3] L.E. ANDERSSON, T. ELFVING, G. H. GOLUB: *Solution of biharmonic equations with application to radar imaging*, Computational and Applied Mathematics. **94**(2) (1998), 153–180.

[4] S. ANTONTSEV, S. SHMAREV: *Blow-up of solutions to parabolic equations with nonstandard growth conditions*, Computational and Applied Mathematics. **234**(9) (2010), 2633–2645.

[5] S. ANTONTSEV, S. SHMAREV: *Evolution PDEs with Nonstandard Growth Conditions*, Atlantis Studies in Differential Equations. Vol. **4**, Atlantis Press, Paris, 2015.

[6] S. ANTONTSEV, J. FERREIRA, E. PISKIN: *Existence and blow up of solutions for a strongly damped Petrovsky equation with variable-exponent nonlinearities*, Electronic Journal of Differential Equations. 2021 (2021), 1–18.

[7] M. AASSILA, A. GUESMIA: *Energy decay for a damped nonlinear hyperbolic equation*, Applied Mathematics Letters. **12**(1999), 49–52.

[8] M.I.G. BLOOR, M.J. WILSON: *An approximate analytic solution method for the biharmonic problem*, Proccedings of the Royal Society A: Mathiematical, Physical and Engineering Science. **462**(2068) (2006), 1107–1121.

[9] O. BOUHOUFANI, I. HAMCHI: *Coupled system of nonlinear hyperbolic equations with variable-exponents: Global existence and Stability*, Mediterranean Journal of Mathematics. **17**(166) (2020), 1–15.

[10] O. BOUHOUFANI, S.A. MESSAOUDI, M. ALAHYANE: *Existence, Blow up and Numerical approximations of Solutions for a Biharmonic Coupled System with Variable exponents*, Authorea. August 10, 2022. DOI: 10.22541/au. 166010582.26966044/v1 (Preprint)

[11] D.V. CRUZ-URIB, A. FIORENZA: *Variable Lebesgue space: Foundations and Harmonic Analysis*, Springer Heidelberg, New York Dordrecht London, 2013.

[12] S. GHEGAL, I. HAMCHI, S.A. MESSAOUDI: *Global existence and stability of a nonlinear wave equation with variable-exponent nonlinearities*, Applicable Analysis. **99**(8) (2020), 1333–1343.

[13] A. GUESMIA: *Energy decay for a damped nonlinear coupled system*, Mathematical Analysis and Applications. **239**(1999), 38–48.

[14] B. GUO, W. GAO: *Blow up of solutions to quasilinear hyperbolic equations with $p(x,t)$-Laplacian and positive initial energy*, Comptes Rendus. Mécanique. **342**(2014), 513–519.

[15] V. KOMORNIK: *Well-posedness and decay estimates for a Petrovesky system by a semigroup approach*, Acta Scientiarum Mathematicarum (Szeged), **60**(1995), 451–466.

[16] D. LARS, P. HASTO, M. RUZICKA: *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Vol. 2017, 2011.

[17] A.C. LAZER, P.J. MCKENNA: *Large-amplitude periodique oscillations in suspension bridges: some new connections with nonlinear analysis*, Siam Review. **32**(4) (1990), 537–578.

[18] X. LI, B. GUO, M. LIAO: *Asymptotic stabilty of solutions to quasilinear hyperbolic equations with variable sources*, Computers and Mathematics with Applications. **79**(4) (2000), 1012–1022.

[19] M. LIAO, Z. TAN: *On behavior of solutions to a Petrovsky equation with damping and variable-exponent source*, Mathematics and Applications. (2021), 1–21.

[20] S.A. MESSAOUDI: *Global Existence and Nonexistence in a System of Petrovsky*, Mathematical Analysis and Applications. **265**(2002), 296–308.

[21] S.A. MESSAOUDI, J.H. AL-SMAIL, A.A. TALAHMEH: *Decay for solutions of a nonlinear damped wave equation with variable-exponent nonlinearities*, Computers and Mathematics with applications. **76**(8) (2018), 1863–1875.

[22] S.A. MESSAOUDI, O. BOUHOUFANI, I. HAMCHI, M. ALAHYANE: *Existence and blow up in a system of wave equations with nonstandard nonlinearities*, Electronic Journal of Differential Equations. **91** 2021 (2021), 1–33.

[23] S. MESSAOUDI, A. TALAHMEH, M. AL-GHARABLI: *On the existence and stability of a nonlinear wave system with variable exponents*, Asymptotic Analysis. **1**(2021), 1–28.

[24] S. Messaoudi, A.Talahmeh, J. Al-Smail: *Nonlinear damped wave equation: Existence and blow-up*. Computers and Mathematics with applications. **74**(12) (2017), 3024–3041.

[25] S.H. Park, J.R. Kang: *Blow-up of solutions for a viscoelastic wave equation with variable Exponents*, Mathematical Methods in the Applied Sciences. **42**(2019), 2083–2097.

[26] J.A. Pelesko, D.H. Bernstein: *Modeling Mems and Nems*, CRC press, 2002.

[27] M. Ružička: *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics. Vol. **1748**, Springer, Berling, Heidelberg, 2000.

[28] S. Timoshenko, W.K. Sergius: *Theory of plates and shells*, McGraw-hill, 2^{nd} Edition, New York, 1995, 2006.

Department of Mathematics
University Batna 2
Batna,
Algeria.
Email address: `o.bouhoufani@univ-batna2.dz`

Department of Mathematics
University King Fahd of Petroleum and Minerals
Dhahran,
KSA.
Email address: `mahfouz@kfupm.edu.sa`

Department of Mathematics
University of Sharjah
P. O. Box 27272 Sharjah,
UAE.
Email address: `smessaoudi@sharjah.ac.ae`