ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **11** (2022), no.12, 1173–1190
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.12.3

FIXED POINT THEOREMS FOR SUSUKI TYPE GENERALIZED $\mathcal{L}$-CONTRACTIONS IN BRANCIARI DISTANCE SPACES

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ABSTRACT. In this paper, the notion of Suzuki type generalized $\mathcal{L}$-contractions is introduced and a new fixed point theorem for such contractions is established.

1. INTRODUCTION AND PRELIMINARIES

Since the Banach contraction principle plays an important role and give many applications in Mathematics, many authors have generalized the contraction principle.

Suzuki [27] gave a generalization of Banach contraction principle to compact metric space by introducing the notion of a contractive map $T : X \to X$, where (X, d) is compact metric space, such that

$$\forall x, y \in X (x \neq y),\ \frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y).$$

On the other hand, Branciari [10] gave a generalization of the notion of metric spaces, which is called Branciari distance spaces, by replacing triangle inequality

2020 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. fixed point, generalized $\mathcal{L}$-contraction, Suzuki type generalized $\mathcal{L}$-contraction,metric space, Branciari distance space.

Submitted: 10.11.2022; *Accepted:* 26.11.2022; *Published:* 05.12.2022.

with trapezoidal inequality, and he gave an extension of Banach contraction principle to Branciari distance spaces. After that, the authors ([3,4,6,14,19,24–26] and references therein) obtained fixed point results in Branciari distance spaces.

It is well know that Branciari distance spaces have some topological disadvantages as follows (see [2,23,24]):

- Branciari distance does not have to be continuous in each coordinates;
- Convergent sequence in Branciari distance space does not have to be Cauchy;
- Branciari distance space (X, d) does not have a topology which is compatible with (X, d);
- The limit of convergent sequence does not have to be unique.

Consider the following conditions for a given function $\theta : (0, \infty) \to (1, \infty)$,

(θ1) θ is non-decreasing;

(θ2) for any sequence $\{z_n\}$ of points in $(0, \infty)$,

$$\lim_{n\to\infty} \theta(z_n) = 1 \Leftrightarrow \lim_{n\to\infty} z_n = 0;$$

(θ3) there is an $p \in (0, 1)$ such that

$$\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^p} = q, \text{ where } q \in (0, \infty);$$

(θ4) θ is continuous on $(0, \infty)$.

Jleli and Samet [17] obtained a generalization of the Banach contraction principle in Branciari distance spaces by introducing the concept of θ-contractions, where $\theta : (0, \infty) \to (1, \infty)$ is a function satisfying conditions (θ1), (θ2) and (θ3). Ahmad *et al.* [1] extended the result of Jleli and Samet [17] to metric spaces by applying conditions (θ1), (θ2) and (θ4).

Very recently, Cho [11] introduced the concept of $\mathcal{L}$-contractions, which is a more generalized concept than some existing notions of contractions. He proved that every $\mathcal{L}$-contraction mapping defined on complete Branciari distance spaces has a unique fixed point.

Afterward, the authors ([5, 7, 12, 13, 16, 21, 22]) gave generalizations of the result of [11].

In the paper, we introduce the new concept of Suzuki type generalized $\mathcal{L}$-contractions which is a generalization of the concept of $\mathcal{L}$-contractions, and we establish a new fixed point theorem for such contraction mappings in the setting of Branciari distance spaces. We give an example to illustrate main theorem.

Let $\xi : [1, \infty) \times [1, \infty) \to \mathbb{R}$ be a function.

Consider the following conditions:

(ξ1) $\xi(1, 1) = 1$;

(ξ2) $\xi(t, s) < \frac{s}{t}\ \ \forall s, t > 1$;

(ξ3) for any sequence $\{t_n\}, \{s_n\} \subset (1, \infty)$ with $t_n \leq s_n\ \ \forall n = 1, 2, 3, \cdots$

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1 \Rightarrow \lim_{n\to\infty} \sup \xi(t_n, s_n) < 1.$$

A function $\xi : [1, \infty) \times [1, \infty) \to \mathbb{R}$ is called *$\mathcal{L}$-simulation* [11] if and only if (ξ1), (ξ2) and (ξ3) are satisfied.

Note that $\xi(t, t) < 1\ \forall t > 1$.

Denote $\mathcal{L}$ by the class of all $\mathcal{L}$-simulation functions $\xi : [1, \infty) \times [1, \infty) \to \mathbb{R}$.

Example 1. *[11] Let $\xi_b, \xi_w, \xi_c : [1, \infty) \times [1, \infty) \to \mathbb{R}$ be functions defined as follows, respectively:*

(1) $\xi_b(t, s) = \frac{s^k}{t}\ \ \forall t, s \geq 1$ *where* $k \in (0, 1)$*;*

(2) $\xi_w(t, s) = \frac{s}{t\phi(s)}\ \ \forall t, s \geq 1$ *where* $\phi : [1, \infty) \to [1, \infty)$ *is nondecreasing and lower semicontinuous such that* $\phi^{-1}(\{1\}) = 1$*;*

(2) $\xi_c(t, s) = \begin{cases} 1 & \text{if } (s, t) = (1, 1), \\ \frac{s}{2t} & \text{if } s < t, \\ \frac{s^\lambda}{t} & \text{otherwise,} \end{cases}$

$\forall s, t \geq 1$, *where* $\lambda \in (0, 1)$.

Then $\xi_b, \xi_w, \xi_c \in \mathcal{L}$.

Example 2. *[12] Let $\xi_k : [1, \infty) \times [1, \infty) \to \mathbb{R}, k = 1, 2, 3,$ be functions defined as follows:*

(1) $\xi_1(t, s) = \frac{\psi(s)}{\varphi(t)},\ \forall t, s \geq 1$ *where* $\psi, \varphi : [1, \infty) \to [1, \infty)$ *are continuous functions such that* $\psi(t) = \varphi(t) = 1$ *if and only if* $t = 1$, $\psi(t) < t \leq \varphi(t),\ \forall t > 1$ *and* φ *is an increasing function;*

(2) $\xi_2(t,s) = \frac{\eta(s)}{t}$, $\forall s, t \geq 1$, *where* $\eta : [1,\infty) \to [1,\infty)$ *is upper semi-continuous with* $\eta(t) < t$, $\forall t > 1$ *and* $\eta(t) = 1$ *if and only if* $t = 1$*;*

(3) $\xi_3(t,s) = \frac{s}{\int_0^t \phi(u)du}$, $\forall s, t \geq 1$, *where* $\phi : [0,\infty) \to [0,\infty)$ *is a function such that for each* $t \geq 1$, $\int_0^t \phi(u)du$ *exists and* $\int_0^t \phi(u)du > t$, *and* $\int_0^1 \phi(u)du = 1$.

Then $\xi_1, \xi_2, \xi_3 \in \mathcal{L}$.

Let X be a non-empty set.

A map $d : X \times X \to [0,\infty)$ is called Branciari distance [10] on X if and only if it satisfies the following conditions: for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y

(d1) $d(x,y) = 0$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$;

(d3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.

The pair (X,d) is called a Branciari distance space [10].

Note that triangle inequality implies trapezoidal inequality, and so every metric space is a Branciari distance space.

The following definitions are in [10].

Let (X,d) be a Branciari distance space. Then we say that

(1) a sequence $\{x_n\} \subset X$ is *convergent* to x (denoted by $\lim_{n\to\infty} x_n = x$) if and only if $\lim_{n\to\infty} d(x_n, x) = 0$;

(2) a sequence $\{x_n\} \subset X$ is *Cauchy* if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$;

(3) (X,d) is *complete* if and only if every Cauchy sequence in X is convergent to some point in X.

Lemma 1.1. *[18] Let* (X,d) *be a Branciari distance space,* $\{x_n\} \subset X$ *be a Cauchy sequence and* $x, y \in X$*. If there exists a positive integer* N *such that*

(1) $x_n \neq x_m$ $\forall n, m > N$*;*

(2) $x_n \neq x$ $\forall n > N$*;*

(3) $x_n \neq y$ $\forall n > N$*;*

(4) $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y)$*,*

then $x = y$.

2. Fixed point results

Let (X, d) be a Branciari distance space.

A map $T : X \to X$ is *Suzuki type generalized $\mathcal{L}$-contraction* with respect to $\xi \in \mathcal{L}$ if and only if it satisfies the condition: for all $x, y \in X$ with $d(Tx, Ty) > 0$

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } \xi(\theta(d(Tx, Ty)), \theta(M(x, y))) \geq 1, \tag{2.1}$$

where $\theta : (0, \infty) \to (1, \infty)$ is a function, and $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$. When $M(x, y) = d(x, y)$ in (2.1), $T : X \to X$ is called Suzuki type $\mathcal{L}$-contraction with respect to $\xi \in \mathcal{L}$.

Recall that a map $T : X \to X$ is $\mathcal{L}$-contraction [11] with respect to $\xi \in \mathcal{L}$ if and only if the condition holds: for all $x, y \in X$ with $d(Tx, Ty) > 0$

$$\xi(\theta(d(Tx, Ty)), \theta(d(x, y))) \geq 1,$$

where $\theta : (0, \infty) \to (1, \infty)$ is a function.

Lemma 2.1. *Let $l > 0$, and let $\{t_n\} \subset (l, \infty)$ be a non-increasing sequence such that*

$$\lim_{n\to\infty} t_n = l.$$

If $\theta : (0, \infty) \to (1, \infty)$ is non-decreasing, then we have

$$\lim_{n\to\infty} \theta(t_n) = \lim_{n\to\infty} \theta(t_{n-1}) = \lim_{t\to l^+} \theta(t) > 1.$$

Proof. Since θ is non-decreasing and $\{t_n\}$ is non-increasing,

$$\lim_{t\to l^+} \theta(t) = \lim_{n\to\infty} \theta(t_n) \leq \lim_{n\to\infty} \theta(t_{n-1}) \leq \lim_{t\to l^+} \theta(t).$$

Thus we have

$$\lim_{n\to\infty} \theta(t_n) = \lim_{n\to\infty} \theta(t_{n-1}) = \lim_{t\to l^+} \theta(t) > \theta(l) > 1.$$

□

Now, we prove our main result.

Theorem 2.1. *Let (X, d) be a complete Branciari distance space, and let $T : X \to X$ be a Suzuki type generalized $\mathcal{L}$-contraction with respect to $\xi \in \mathcal{L}$. If θ is non-decreasing, then T has a unique fixed point, and the Picard sequence $\{x_n = T^n x_0\}$ $\forall x_0 \in X$ converges to the fixed point.*

Proof. Firstly, we show the uniqueness of fixed point whenever it exists. Assume that w and u are fixed points of T. If $u \neq w$, then $d(w,u) > 0$ and $\frac{1}{2}d(w,Tw) = 0 < d(w,u)$. Thus it follows from (2.1) that

$$\begin{aligned} 1 \leq & \xi(\theta(d(Tw,Tu)), \theta(M(w,u))) \\ = & \xi(\theta(d(w,u)), \theta(d(w,u))) \\ < & \frac{\theta(d(w,u))}{\theta(d(w,u))}. \end{aligned}$$

Thus we have

$$\theta(d(w,u)) < \theta(d(w,u))$$

which is a contradiction. Hence $w = u$, and fixed point of T is unique.

Secondly, we prove the existence of fixed point. Let $x_0 \in X$ be a point. Define a sequence $\{x_n\} \subset X$ by $x_n = Tx_{n-1} = T^n x_0 \; \forall n = 1, 2, 3 \cdots$.

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T, and the proof is finished. Assume that

$$x_{n-1} \neq x_n \; \forall n = 1, 2, 3 \cdots . \tag{2.2}$$

We infer that

$$\frac{1}{2}d(x_{n-1}, Tx_{n-1}) = \frac{1}{2}d(x_{n-1}, x_n) < d(x_{n-1}, x_n). \tag{2.3}$$

It follows from (2.1), (2.2) and (2.3) that $\forall n = 1, 2, 3, \cdots$

$$\begin{aligned} 1 \leq & \xi(\theta(d(Tx_{n-1}, Tx_n)), \theta(M(x_{n-1}, x_n))) \\ = & \xi(\theta(d(x_n, x_{n+1})), \theta(M(x_{n-1}, x_n))) \\ < & \frac{\theta(M(x_{n-1}, x_n))}{\theta(d(x_n, x_{n+1}))}, \end{aligned} \tag{2.4}$$

where $m(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$.

If there exists $n = 1, 2, 3, \cdots$, such that $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$, then from (2.4) we have that

$$1 < \frac{\theta(M(x_n, x_{n+1}))}{\theta(d(x_n, x_{n+1}))} = \frac{\theta(d(x_n, x_{n+1}))}{\theta(d(x_n, x_{n+1}))} = 1$$

which is a contradiction. Hence, we obtain that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \; \forall n = 1, 2, 3, \cdots .$$

It follows from (2.4) that

$$1 < \frac{\theta(M(x_{n-1}, x_n))}{\theta(d(x_n, x_{n+1}))} = \frac{\theta(d(x_{n-1}, x_n))}{\theta(d(x_n, x_{n+1}))},\ \forall n = 1, 2, 3, \cdots .$$

Consequently, we obtain that

$$\theta(d(x_n, x_{n+1})) < \theta(d(x_{n-1}, x_n)),\ \forall n = 1, 2, 3, \cdots$$

which implies

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n)\ \forall n = 1, 2, 3, \cdots .$$

Hence $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence, and so there exists $l \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = l.$$

We now show that $l = 0$. Assume that $l > 0$. Let $t_{n-1} = \theta(d(x_{n-1}, x_n))$ and $t_n = \theta(d(x_n, x_{n+1}))\ \forall n = 1, 2, 3, \cdots$. Then $t_n < t_{n-1}\ \forall n = 1, 2, 3, \cdots$. By applying Lemma 2.1,

$$\lim_{n\to\infty} t_{n-1} = \lim_{n\to\infty} t_n = \lim_{t\to l^+} \theta(t) > \theta(l) > 1.$$

It follows from (ξ3) that

$$1 \leq \lim_{n\to\infty} \sup \xi(t_n, t_{n-1}) < 1$$

which is a contradiction. Thus,

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{2.5}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. On the contrary, assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which

$$m(k) > n(k) > k,\ d(x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and } d(x_{m(k)-1}, x_{n(k)}) < \epsilon. \tag{2.6}$$

From (2.6) we have

$$\begin{aligned}
&\epsilon \\
&\leq d(x_{m(k)}, x_{n(k)}) \\
&\leq d(x_{n(k)}, x_{m(k)-2}) + d(x_{m(k)-2}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) \\
&< \epsilon + d(x_{m(k)-2}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}).
\end{aligned} \tag{2.7}$$

Letting $k \to \infty$ in (2.7), we obtain

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.8}$$

On the other hand, we obtain

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{n(k)+1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1}).$$

Thus

$$\lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon. \tag{2.9}$$

It follows from (2.5) that there exists $N \in \mathbb{N}$ such that

$$d(x_{n(k)}, x_{n(k)+1}) < \epsilon, \ \forall k > N.$$

Thus we infer that $\forall k > N$

$$\frac{1}{2}d(x_{n(k)}, Tx_{n(k)}) = \frac{1}{2}d(x_{n(k)}, x_{n(k)+1}) < \epsilon \leq d(x_{n(k)}, x_{m(k)}).$$

It follows from (2.1) that

$$\begin{aligned} 1 \leq & \xi(\theta(d(Tx_{n(k)}, Tx_{m(k)})), \theta(M(x_{n(k)}, x_{m(k)}))) \\ = & \xi(\theta(d(x_{n(k)+1}, x_{m(k)+1})), \theta(M(x_{n(k)}, x_{m(k)})) \\ < & \frac{\theta(M(x_{n(k)}, x_{m(k)}))}{\theta(d(x_{n(k)+1}, x_{m(k)+1}))} \end{aligned}$$

which implies

$$\theta(d(x_{n(k)+1}, x_{m(k)+1})) < \theta(M(x_{n(k)}, x_{m(k)})) \tag{2.10}$$

where $M(x_{n(k)}, x_{m(k)}) = \max\{d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1})\}$.

If $M(x_{n(k)}, x_{m(k)}) = d(x_{m(k)}, x_{m(k)+1})$, then from (2.10) we have

$$\theta(d(x_{n(k)+1}, x_{m(k)+1})) < \theta(d(x_{m(k)}, x_{m(k)+1}))$$

and so

$$d(x_{n(k)+1}, x_{m(k)+1}) < d(x_{m(k)}, x_{m(k)+1}). \tag{2.11}$$

Letting $n \to \infty$ in (2.11) and using (2.5) and (2.9), we obtain that

$$\epsilon = \lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = 0$$

which leads to a contradiction.

Assume that $M(x_{n(k)}, x_{m(k)}) = d(x_{n(k)}, x_{n(k)+1})$. Then from (2.10) we have that

$$\theta(d(x_{n(k)+1}, x_{m(k)+1})) < \theta(d(x_{n(k)}, x_{n(k)+1}))$$

and so

$$d(x_{n(k)+1}, x_{m(k)+1}) < d(x_{n(k)}, x_{n(k)+1}). \tag{2.12}$$

Letting $n \to \infty$ in (2.12) and using (2.5) and (2.9), we infer that

$$\epsilon = \lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = 0$$

which is a contradiction. Thus, we have that

$$M(x_{n(k)}, x_{m(k)}) = d(x_{n(k)}, x_{m(k)}).$$

It follows from (2.10) that

$$\theta(d(x_{n(k)+1}, x_{m(k)+1})) < \theta(d(x_{n(k)}, x_{m(k)})).$$

Let

$$t_k = \theta(d(x_{n(k)+1}, x_{m(k)+1})) \text{ and } t_{k-1} = \theta(d(x_{n(k)}, x_{m(k)})).$$

Then

$$t_k < t_{k-1} \ \forall k = 1, 2, 3, \cdots .$$

Applying Lemma 2.1,

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} t_{k-1} = \lim_{t\to\epsilon^+} \theta(t) > \theta(\epsilon) > 1.$$

From ($\xi 3$) we have

$$1 \le \lim_{k\to\infty} \sup \xi(t_k, t_{k-1}) < 1,$$

which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{2.13}$$

From (2.5) and (2.13) we may assume that

$$d(x_n, x_{n+1}) < d(x_n, z), \ \forall n \ge n_0$$

for sufiiciently lage n_0. Hence

$$\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) < d(x_n, z),\ \forall n \geq n_0.$$

Thus, it follows from (2.1) that

$$1 \leq \xi(\theta(d(Tx_n, Tz)), \theta(d(x_n, z))) < \frac{\theta(d(x_n, z))}{\theta(d(Tx_n, Tz))},\ \forall n \geq n_0$$

which implies

$$\theta(d(Tx_n, Tz)) < \theta(d(x_n, z)),\ \forall n \geq n_0.$$

Hence,

$$d(Tx_n, Tz) < d(x_n, z),\ \forall n \geq n_0.$$

and hence

$$\lim_{n\to\infty} d(x_{n+1}, Tz) = 0. \tag{2.14}$$

Applying Lemma 1.1 with (2.13) and (2.14), we have $z = Tz$. □

Corollary 2.1. *Let (X, d) be a complete Branciari distance space, and let $T : X \to X$ be a Suzuki type $\mathcal{L}$-contraction with respect to $\xi \in \mathcal{L}$. If θ is non-decreasing, then T has a unique fixed point.*

Corollary 2.2. *Let (X, d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$*

$$\xi(\theta(d(Tx, Ty)), \theta(M(x, y))) \geq 1,$$

where θ is non-decreasing and $\xi \in \mathcal{L}$. Then T has a unique fixed point.

Corollary 2.3. *[11] Let (X, d) be a complete Branciari distance space, and let $T : X \to X$ be a $\mathcal{L}$-contraction mapping with respect to $\xi \in \mathcal{L}$. If θ is non-decreasing, then T has a unique fixed point.*

Corollary 2.4. *Let (X, d) be a complete metric space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } \xi(\theta(d(Tx, Ty)), \theta(M(x, y))) \geq 1$$

where θ is non-decreasing and $\xi \in \mathcal{L}$. Then T has a unique fixed point.

Corollary 2.5. *Let (X,d) be a complete metric space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$*

$$\xi(\theta(d(Tx,Ty)), \theta(M(x,y))) \geq 1,$$

where θ is non-decreasing and $\xi \in \mathcal{L}$. Then T has a unique fixed point.

We give an example to illustrate Theorem 2.1.

Example 3. *Let $X = \{1,2,3,4\}$ and define $d : X \times X \to [0,\infty)$ as follows:*

$$\begin{aligned}
&d(1,2) = d(2,1) = 3,\\
&d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1,\\
&d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = 4,\\
&d(x,x) = 0 \ \forall x \in X.
\end{aligned}$$

Then (X,d) is a complete Branciari distance space, but not a metric space (see [4]). Define a map $T : X \to X$ by

$$Tx = \begin{cases} 1 & (x = 2),\\ 3 & (x \neq 2) \end{cases}$$

Let $\theta : (0,\infty) \to (1,\infty)$ be a function defined by

$$\theta(t) = \begin{cases} e^t & (0 < t \leq 3),\\ e^6 & (t > 3). \end{cases}$$

Then θ is non-decreasing, and is not continuous.

We now show that T is a Suzuki type $\mathcal{L}$-contraction with respect to ξ_b, where $\xi_b(t,s) = \frac{s^k}{t}$ $\forall t, s \geq 1$, $k = \frac{1}{3}$. We have

$$d(Tx,Ty) = \begin{cases} d(1,3) = 1 & (x = 2, y \neq 2),\\ d(1,1) = 0 & (x = 2, y = 2),\\ d(3,3) = 0 & (x \neq 2, y \neq 2) \end{cases}$$

so

$$d(Tx,Ty) > 0 \Leftrightarrow x = 2, y \neq 2.$$

We have, for $x = 2$ and $y \neq 2$,

$$d(x,y) = \begin{cases} 3 & (x=2, y=1), \\ 1 & (x=2, y=3), \\ 4 & (x=2, y=4) \end{cases}$$

and

$$d(Tx,Ty) = d(1,3) = 1.$$

We deduce that for all $x, y \in X$ with $d(Tx,Ty) > 0$,

$$\begin{aligned} \frac{1}{2}d(x,Tx) = \frac{1}{2}d(2,T2) = \frac{1}{2}d(2,1) = \frac{3}{2} \\ < d(x,y) = \begin{cases} 3 & (x=2, y=1), \\ 4 & (x=2, y=4). \end{cases} \end{aligned}$$

Thus we have

$$\begin{aligned} &\xi_b(\theta(d(Tx,Ty)), \theta(M(x,y))) \\ =&\frac{[\theta(M(x,y))]^k}{\theta(d(Tx,Ty))} \\ =&\begin{cases} \frac{[\theta(3)]^k}{\theta(1)} & (x=2, y=1), \\ \frac{[\theta(4)]^k}{\theta(1)} & (x=2, y=4) \end{cases} \\ =&\begin{cases} 1 & (x=2, y=1), \\ e & (x=2, y=4) \end{cases} \\ \geq&1. \end{aligned}$$

Thus all hypotheses of Theorem 2.1 are satisfied, and T has a fixed point $z = 3$.

Note that $\mathcal{L}$-contraction condition is not satisfied. In fact, if $x = 2, y = 3$, then

$$1 \leq \xi_b(\theta(d(T2,T3)), \theta(d(2,3))) = \frac{[\theta(1)]^k}{\theta(1)}$$

which implies

$$k \geq 1.$$

Also, note that θ-contraction condition [1] does not hold.

Let $\theta : (0,\infty) \to (1,\infty)$ be a function such that (θ1),(θ2) and (θ4) are satisfied. If

$$\theta(d(T2,T3)) \leq [\theta(d(2,3))]^k, \text{ where } k \in (0,1),$$

then

$$\theta(1) \leq [\theta(1)]^k.$$

Thus $k \geq 1$. Hence T is not a θ-contraction map.

Let

$$d(T2,T3) \leq kd(2,3), \text{ where } k \in (0,1).$$

Then $k \geq 1$, and so T is not a contraction map.

Remark 2.1. *Conditions ($\theta2$), ($\theta3$) and ($\theta4$) are not required in the proof of Theorem 2.1, and hence Corollary 2.3.*

3. CONSEQUENCE

Applying simulation functions given in Example 1.1, we have some fixed point results.

By taking $\xi = \xi_b$ in Theorem 2.1, we obtain Corollary 3.1.

Corollary 3.1. *Let (X,d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } \theta(d(Tx,Ty)) \leq [\theta(M(x,y))]^k,$$

where θ is non-decreasing and $k \in (0,1)$. Then T has a unique fixed point.

Corollary 3.2. *Let (X,d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } \theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k,$$

where θ is non-decreasing and $k \in (0,1)$. Then T has a unique fixed point.

Remark 3.1. *Corollary 3.2 is an extension of Theorem 3.3 of [1] to Branciari distance space without continuity of function θ.*

Corollary 3.3. *Let (X,d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\theta(d(Tx,Ty)) \leq [\theta(M(x,y))]^k,$$

where θ is non-decreasing and $k \in (0,1)$. Then T has a unique fixed point.

Corollary 3.4. *Let (X,d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k,$$

where θ is non-decreasing and $k \in (0,1)$. Then T has a unique fixed point.

Remark 3.2.

(1) *Corollary 3.4 is an extension of Theorem 2.2 of [1] to Branciari distance spaces without conditions $(\theta 2)$ and $(\theta 4)$.*
(2) *(2) Corollary 3.4 is a generalization of Theorem 2.1 of [17] without condition $(\theta 2)$ and $(\theta 3)$, and hence it is an answer of open question of [20].*

By taking $\xi = \xi_w$ in Theorem 2.1, we obtain the following result.

Corollary 3.5. *Let (X,d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } \theta((Tx,Ty)) \leq \frac{\theta(M(x,y))}{\phi(\theta(M(x,y)))},$$

where θ is non-decreasing and $\phi : [1,\infty) \to [1,\infty)$ is nondecreasing and lower semi-continuous such that $\phi^{-1}(\{1\}) = 1$. Then T has a unique fixed point.

Corollary 3.6. *Let (X,d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } \theta(d(Tx,Ty)) \leq \frac{\theta(d(x,y))}{\phi(\theta(d(x,y)))},$$

where θ is non-decreasing and $\phi : [1,\infty) \to [1,\infty)$ is nondecreasing and lower semi-continuous such that $\phi^{-1}(\{1\}) = 1$. Then T has a unique fixed point.

Corollary 3.7. *Let (X,d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } d(Tx,Ty) \leq M(x,y) - \varphi(M(x,y)), \tag{3.1}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is nondecreasing and lower semicontinuous such that $\varphi^{-1}(\{0\}) = 0$. Then T has a unique fixed point.

The proof of Corollary 3.5 is similar with proof of Corollary 9 of [11].

Corollary 3.8. *Let (X,d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)), \tag{3.1}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is nondecreasing and lower semicontinuous such that $\varphi^{-1}(\{0\}) = 0$. Then T has a unique fixed point.

Remark 3.3. *Corollary 3.6 (resp. Corollary 3.8) is a generalization of Corollary 8 (resp. Corollary 9) of [11].*

By taking $\xi = \xi_2$ in Theorem 2.1, we obtain the following result.

Corollary 3.9. *Let (X,d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } \theta(d(Tx,Ty)) \leq \eta(\theta(M(x,y))),$$

where θ is non-decreasing and $\eta : [1,\infty) \to [1,\infty)$ is upper semi-continuous with $\eta(t) < t,\ \forall t > 1$ and $\eta(t) = 1$ if and only if $t = 1$. Then T has a unique fixed point.

Corollary 3.10. *Let (X,d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } \theta(d(Tx,Ty)) \leq \eta(\theta(d(x,y))),$$

where θ is non-decreasing and $\eta : [1,\infty) \to [1,\infty)$ is upper semi-continuous with $\eta(t) < t,\ \forall t > 1$ and $\eta(t) = 1$ if and only if $t = 1$. Then T has a unique fixed point.

Corollary 3.11. *Let (X,d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } d(Tx,Ty) \leq \psi(M(x,y)), \tag{3.2}$$

where $\psi : [0,\infty) \to [0,\infty)$ is upper semi-continuous with $\psi(t) < t,\ \forall t > 0$ and $\psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof. Let $\theta(t) = e^t, \ \forall t > 0$. From (3.2) we have that, for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\theta(d(Tx, Ty)) = e^{d(Tx,Ty)} \leq e^{\psi(M(x,y))}. \tag{3.3}$$

Let $\psi(t) = \ln(\eta(\theta(t))), \ \forall t \geq 0$, where $\eta : [1, \infty) \to [1, \infty)$ is upper semi-continuous with $\eta(t) < t, \ \forall t > 0$ and $\eta(t) = 1$ if and only if $t = 1$. Then ψ is upper semi-continuous with $\psi(t) < t, \ \forall t > 0$ and

$$\psi(t) = 0 \Leftrightarrow \eta(\theta(t)) = 1 \Leftrightarrow \theta(t) = e^t = 1 \Leftrightarrow t = 0.$$

It follows from (3.3) that, for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\theta(d(Tx, Ty)) \leq e^{\ln(\eta(\theta(M(x,y))))} = \eta(\theta(M(x, y))).$$

By Corollary 3.9, T has a unique fixed point. □

Corollary 3.12. *Let (X, d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \textit{ implies } d(Tx, Ty) \leq \psi(d(x, y)), \tag{3.2}$$

where $\psi : [0, \infty) \to [0, \infty)$ is upper semi-continuous with $\psi(t) < t, \ \forall t > 0$ and $\psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Remark 3.4. *Corollary 3.11 is a generalization and an extention of Theorem 1 of [9].*

By taking $\theta(t) = 2 - \frac{2}{\pi}\arctan(\frac{1}{t^\alpha}) \ \forall t > 0$, where $\alpha \in (0, 1)$ in Corollary 3.1 and Corollary 3.5, we have the following two results, respectively.

Corollary 3.13. *Let (X, d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \textit{ implies}$$

$$2 - \frac{2}{\pi}\arctan(\frac{1}{d(Tx, Ty)^\alpha}) \leq \left[2 - \frac{2}{\pi}\arctan(\frac{1}{M(x, y)^\alpha})\right]^k,$$

where $\alpha \in (0, 1)$ and $k \in (0, 1)$. Then T has a unique fixed point.

Corollary 3.14. *Let (X, d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \textit{ implies}$$

$$2-\frac{2}{\pi}\arctan(\frac{1}{d(Tx,Ty)^{\alpha}}) \leq \frac{2-\frac{2}{\pi}\arctan(\frac{1}{M(x,y)^{\alpha}})}{\phi(2-\frac{2}{\pi}\arctan(\frac{1}{M(x,y)^{\alpha}}))},$$

where $\alpha \in (0,1)$ and $\phi : [1,\infty) \to [1,\infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$. Then T has a unique fixed point.

Remark 3.5. *Corollary 3.12 (resp. Corollary 3.13) is a generalization of Theorem 2.3 of [1] (resp. Corollary 10 of [11]).*

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