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THE TRUNCATED XLINDLEY DISTRIBUTION WITH CLASSIC AND BAYESIAN INFERENCE UNDER CENSORED DATA

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ABSTRACT. We provide a brand-new distribution based on the model of Lindley, with an emphasis on the estimation of its unknown parameters. After introducing the new distribution, we cover two approaches to estimate its parameters; in the presence of a censored scheme, we first use a traditional approach, which is The maximum likelihood technique, then we use the Bayesian approach. The BarzilaiBrown algorithm is used to derive the censored maximum likelihood estimators while a Monte Carlo Markov chains (MCMC) procedure is applied to derive the Bayesian ones. Three loss functions are used to provide the Bayesian estimators: the entropy, the generalized quadratic, and the Linex functions. Using Pitman's proximity criteria; the maximum likelihood and the Bayesian estimations are compared. All of the provided estimations techniques have been evaluated throughout simulation studies. Finally, we consider two sample Bayes predictions to predict future order statistics

1. INTRODUCTION

We use real-life applications of numerical techniques in many fields such as medicine, engineering sciences, finance, and statistics. We note that statistics

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have a critical role in our real-life applications. Often statistical analysis depends strongly on the assumed probability distributions. However, not all the problems in statistics follow the classical or standard probability distributions. for new data analysis, choosing an appropriate fundamental model is becoming more and more important in reliability and survival analysis. Even a little change from the fundamental model might cause a significant impact on the outcomes.

There are various kinds of probability distributions, In this work, we are interested in a one-parameter distribution called XL distribution which is a mixture of Exponential distribution and Lindley distribution. one of the many used continuous probability distributions is The exponential distribution. It is applied to model the interval between occurrences, also Lindley distribution is a probability distribution applied to describe the lifetime of a processor or a certain device.

Let X be a random variable following the one-parameter distribution XL mentioned above, the density function of X is given by:

$$f_{XL}(x,\alpha) = \frac{\alpha^2(2+\alpha+x)e^{-\alpha x}}{(1+\alpha)^2} \qquad x, \qquad \alpha > 0. \tag{1.1}$$

Its cumulative function is:

$$F_{XL}(x,\alpha) = 1 - \left[1 + \frac{\alpha x}{(1+\alpha)^2}\right] e^{-\alpha x}. \tag{1.2}$$

The idea of this work is based on using upper truncated data to provide a new distribution from the XL distribution. Recently there are many studies on newly founded distributions, truncated distributions are also used in a wide range of applications; Bantan et al. (2019) (introduced the Truncated Inverted Kumaraswamy-generated family of distributions). Hamedani et al. (2019) (a new family based on the exponential model called the type I general exponential class of distributions). Aldahlan et al. (2020) (introduced the truncated Cauchy power family of distributions). Mansour et al. (2020f) (a new generalization of the reciprocal exponential model with Clayton copula, Yadav et L. (2021) (The Burr-Hatke exponential distribution with some applications and a censored regression models), H. Aiachi et al.(The Bayesian inference of three parameters Burr XII).

In most cases, we apply simple distributions than more complicated ones, this work is inspired by this previous statement; we noticed that the XLindley distribution (XL) is simple and easy to apply, the XL distribution can be used quite effectively in analysing many real lifetime data set such as application to Corona, Ebola and Nipah virus; it gives adequate fits to many data sets.

The rest of this paper is arranged as follows: In section 2, we introduce the model of our interest and give its survival proprieties. In section 3, we provide the maximum likelihood estimation under type II censored data followed by the simulation study. In section 4, we cover the Bayesian estimation and its simulation using MCMC techniques. Comparison results of the estimators are presented in section 5, the Prediction problem is provided in section 6, and finally, we conclude the paper in section 7.

2. Upper truncated XL distribution (UXL distribution)

In a statistical experiment, we call the process of omitting all the values that fell outside predetermined bounds a "truncation", in which we obtain a truncated data (the remaining data points inside these bounds).

Let the random variable X, we say that X is upper (lower) truncated, at a given point level c, if only the values of X for which $X \leq c$ $(X > c)$ are considered, i.e. we exclude all the other values of X for which $X > c$ (Xc).

The probability density function (PDF) of the upper truncated XLindly distribution at the point $\beta > 0$, is given by

$$f_{UXL}(x, \alpha, \beta) = \frac{f_{XL}(x, \alpha)}{F_{XL}(x, \beta)}, \qquad x, \alpha, \beta > 0.$$

Replacing by (1.1) and (1.2):

$$f_{UXL}(x, \alpha, \beta) = \frac{\alpha^2 (2 + \alpha + x)(1 + \beta)^2 e^{-\alpha x + \beta x}}{(1 + \alpha)^2 A_\beta(x)}, \tag{2.1}$$

where $A_\beta(x) = (1 + \beta)^2 (e^{\beta x} - 1) + \beta x$. Then, using the same formula for the cumulative function (CDF), we obtain:

$$F_{UXL}(x, \alpha, \beta) = \frac{F_{XL}(x, \alpha)}{F_{XL}(x, \beta)} = \frac{1 - [1 + \dfrac{\alpha x}{(1 + \alpha)^2}] e^{-\alpha x}}{1 - [1 + \dfrac{\beta x}{(1 + \beta)^2}] e^{-\beta x}}$$

$$F_{UXL}(x, \alpha, \beta) = \frac{A_\alpha(x)(1 + \beta)^2 e^{-\alpha x + \beta x}}{(1 + \alpha)^2 A_\beta(x)}, \tag{2.2}$$

where $A_\alpha(x) = (1+\alpha)^2(e^{\alpha x} - 1) + \alpha x$.

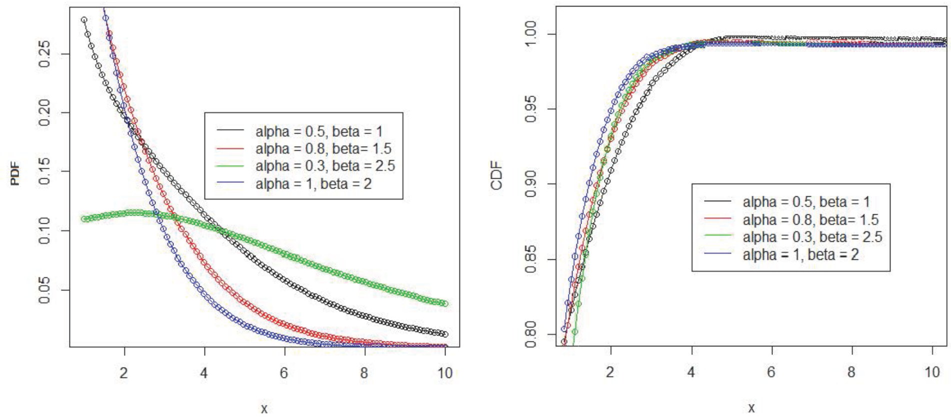

FIGURE 1. Density and cumulative functions under different values of parameters

The corresponding survival function is given by:

$$S(t) = 1 - F(t;\alpha,\beta) = \frac{(1+\alpha)^2 A_\beta(t) - A_\alpha(t)(1+\beta)^2 e^{-\alpha t+\beta t}}{(1+\alpha)^2 A_\beta(t)}. \tag{2.3}$$

The failure rate at the moment t is expressed as:

$$h(t) = \frac{f(t)}{S(t)} = \frac{\alpha^2(2+\alpha+t)(1+\beta)^2}{(1+\alpha)^2 A_\beta(t) e^{\alpha t - \beta t} - A_\alpha(t)(1+\beta)^2}. \tag{2.4}$$

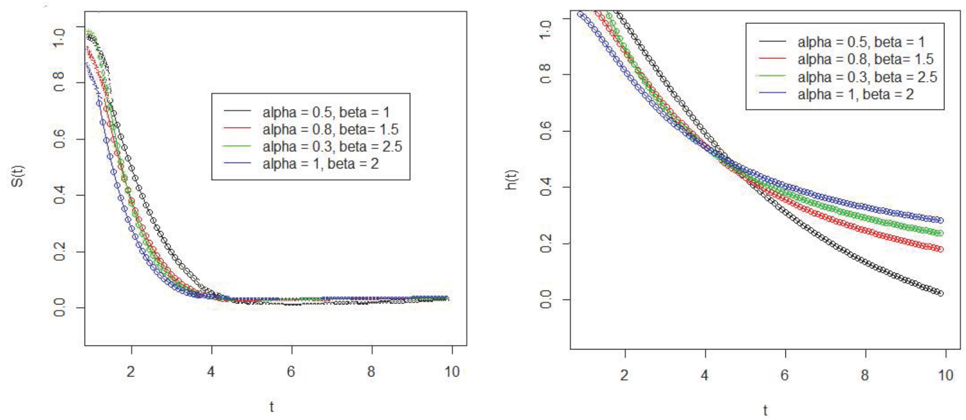

FIGURE 2. Survival function and rate failure under different values of parameters

Assuming that the random variables $X_1, X_2, \dots, X_n$ are independent and do follow the UXL distribution, the joint probability density function is:

$$f_{UXL}(x_1, x_2, \dots, x_n, \alpha, \beta) = \frac{\alpha^{2n}(1+\beta)^{2n} e^{\sum_{i=1}^{n}(-\alpha x_i + \beta x_i)}}{(1+\alpha)^2 n} \prod_{i=1}^{n} \frac{2+\alpha+x_i}{A_\beta(x_i)}. \tag{2.5}$$

3. Maximum likelihood estimation

We use one of the most popular approaches in classic statistical inferences: the Maximum likelihood estimation (MLE) method, it is mostly used since its rationale is clear and adaptable. we use this method to estimate the parameters of a probability distribution assuming that we have some observed data. for that purpose, the maximizing of the likelihood function is accomplished to the observed data as probable as possible. However, in the majority of cases, it will be essential to use numerical techniques to determine the probability function's maximum.

In the case of complete data, the likelihood function is the joint probability density function (2.5). Here we are interested in type II censored data to estimate the parameters. Considering the n-sample $(x_1, x_2, \dots, x_n)$ and a fixed constant m, we assume that the m-sample $(x_1, x_2, \dots, x_m)$ is generated from the UXL distribution. The likelihood function of this sample is: for $n, m \in \mathbb{N}$

$$L(\theta, \beta, X) = N \prod_{i=1}^{m} f_{UXL}(x, \theta, \beta)[1 - F_{UXL}(x_m, \theta, \beta)]^{n-m},$$

where $N = \dfrac{n!}{(n-m)!}$. Replacing both (2.1) and (1.2) we have:

$$L(\alpha, \beta, X) = N B^m e^{\sum_{i=1}^{m}(-\alpha x_i + \beta x_i)} C^{n-m}(x_m) \prod_{i=1}^{m} \frac{2+\alpha+x_i}{A_\beta(x_i)}, \tag{3.1}$$

where

$$\begin{cases} B = \dfrac{\alpha^2(1+\beta)^2}{(1+\alpha)^2}, \\ C(x_m) = \dfrac{(1+\alpha)^2 A_\beta(x_m) - A_\alpha(x_m)(1+\beta)^2 e^{-\alpha x_m + \beta x_m}}{(1+\alpha)^2 A_\beta(x_m)} \end{cases}.$$

Passing by the logarithm we find:

$$
\begin{aligned}
l = l(x,\alpha,\beta) &= \ln L(\alpha,\beta,X)\\
&= \ln N + m\ln B + \sum_{i=1}^{m}(-\alpha x_i + \beta x_i) + (n-m)\ln C(x_m) + \sum_{i=1}^{m}\ln(\frac{2+\alpha+x_i}{A_\beta(x_i)})\\
&= \ln N + m\ln B - \sum_{i=1}^{m}(-\alpha x_i + \sum_{i=1}^{m}\beta x_i) + (n-m)\ln C(x_m)\\
&+ \sum_{i=1}^{m}\ln(2+\alpha+x_i) - \sum_{i=1}^{m}\ln(A_\beta(x_i)).
\end{aligned}
$$

By finding:

$$
\begin{aligned}
\frac{\partial B}{\partial \alpha} &= (1+\beta)^2(\frac{2\alpha+2\alpha^2}{(1+\alpha)^3})\\
\frac{\partial B}{\partial \beta} &= 2(1+\beta)(\frac{\alpha^2}{(1+\alpha)^2})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C(x_m)}{\partial \beta} &= \frac{(1+\alpha)^2\frac{\partial A_\beta}{\partial \beta} - (A_\alpha(2+2\beta)e^{-\alpha x_m+\beta x_m} + A_\alpha(x_m)\beta e^{-\alpha x_m+\beta x_m})(1+\theta)^2A_\beta}{((1+\theta)^2A_\beta(x_m))^2}\\
&- \frac{(1+\alpha)^2\frac{\partial A_\beta}{\partial \beta}[(1+\theta)^2A_\beta + A_\alpha(1+\beta)e^{-\alpha x_m+\beta x_m}}{((1+\alpha)^2A_\beta(x_m))^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial A_\beta(x)}{\partial \beta} &= (2+2\beta)(e^{\beta x}-1) + x(1+\beta)^2e^{\beta x} + x,\\
\frac{\partial A_\alpha(x)}{\partial \alpha} &= (2+2\alpha)(e^{\alpha x}-1) + x(1+\alpha)^2e^{\alpha x} + x.
\end{aligned}
$$

We obtain the maximum likelihood estimators $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ by solving the following non-linear system:

(3.2)

$$
(S)\begin{cases}
\hat{\alpha}_{MLE} = \frac{\partial l}{\partial \alpha} = m.\frac{\frac{\partial B}{\partial \alpha}}{B} - \sum_{i=1}^{m} x_i + (n-m)\frac{\frac{\partial C_m}{\partial \alpha}}{C(x_m)} + \sum_{i=1}^{m}\frac{1}{2+\alpha+x_i} & = 0\\
\hat{\beta}_{MLE} = \frac{\partial l}{\partial \beta} = m.\frac{\frac{\partial B}{\partial \beta}}{B} + \sum_{i=1}^{m} x_i + (n-m)\frac{\frac{\partial C_m}{\partial \beta}}{C} - \frac{\frac{\partial A_\beta(x)}{\partial \beta}}{A_\beta(x)} & = 0
\end{cases}
$$

It is clearly impossible to find the expression of the estimators analytically, so we use numerical techniques to obtain approximations of the estimators, we use the $\mathcal{R}$ programming language. the results are covered in the next part.

3.1. **Simulation studies.** We intend to perform a simulation study choosing different sample sizes; $n = 20, 50, 200$, matching different effective sample sizes, $m = 12, 30, 120$. The results are obtained after $M = 10000$ sample generations, we take $\alpha = 2$ and $\beta = 1$.

The resulting values are displayed after the use of the R programming language, more specifically, we used the package BB and the command BBsolve, short for Barzilai-Brown is known for its high capacity for solving large-scale nonlinear systems, for more details we refer toVaradhan and Gilbert ([14]).

TABLE 1. The MLE of the parameters with quadratic error (in brackets).

N = 5000	n = 20	n = 50	n = 200
m	12	30	120
α	2,0502 (0,0152)	1,9234 (0,0217)	1,9872 (0,0078)
β	0,6135 (0,0078)	0,7397 (0,0054)	0,9573 (0,0045)

We notice from the table above: for α all the estimated values are close to the real value of α and the corresponding error is small. However, for β the best estimation value and the smallest quadratic error occur when n and m are large.

4. BAYESIAN ESTIMATION

In this section, we cover the Bayesian approach, in this approach the unknown parameters are considered random variables, in other words, we have a piece of prior information that we resume a prior distribution of the parameters to be estimated.

There are two types of prior distribution; informative and non-informative, and we use both of them.

For the first parameter (α) we suppose that we have an informative prior which is gamma distribution, i.e.,

$$\pi(\alpha) = \frac{a^b}{\Gamma(b)}\alpha^{b-1}e^{-a\alpha}, \qquad \alpha > 0,\ a, b > 0.$$

Gamma distribution is often used as a prior distribution thanks to its flexibility, offering conjugate prior distributions.

For the second parameter (β) we use a non informative prior distribution,

$$\pi(\beta) = \frac{1}{\beta}.$$

Noting that the parameters are independent, the joint prior distribution is:

$$\pi(\alpha, \beta) = \frac{a^b}{\beta \Gamma(b)} \alpha^{b-1} e^{-a\alpha}, \qquad \alpha, \beta > 0, a, b > 0.$$

The Bayesian estimation is also done for type II censored data, then, using (3.1) we read the posterior distribution as:

$$\pi(\alpha, \beta, x) = k\beta^m e^{\sum_{i=1}^m (-\alpha x_i + \beta x_i)} C_{x_n}^{n-m} \beta^{-1} \alpha^{b-1} e^{-a\alpha} \prod_{i=1}^m \frac{2 + \alpha + x_i}{A_\beta(x_i)}, \tag{4.1}$$

where

$$K = \int_0^{+\infty} \int_0^{+\infty} \beta^m e^{\sum_{i=1}^m (-\theta x_i + \beta x_i)} C_{x_n}^{n-m} \beta^{-1} \theta^{b-1} e^{-a\theta} \prod_{i=1}^m \frac{2 + \alpha + x_i}{A_\beta(x_i)} d\theta d\beta.$$

Estimators and their corresponding risks

The table below explain the three loss functions (Entropy, Generalized quadratic, and Linex) used to find the estimators.

Loss function Expresion	Bayes estimators	posterior risk
Entropy: $L(\theta, \delta) = \left(\frac{\delta}{\theta}\right)^p - p \log\left(\frac{\delta}{\theta}\right) - 1$	$\hat{\delta}_E = E_\pi(\theta^{-p})^{\frac{-1}{p}}$	$p[E_\pi(\log(\theta - \log(\hat{\delta}_E)))]$
Generalized quadratic: $L(\theta, \delta) = \tau(\theta)(\theta - \delta)^2$	$\hat{\delta}_{GQ} = \frac{E_\pi(\tau(\theta)\theta)}{E_\pi(\tau(\theta)}$	$E_\pi(\tau(\theta)(\theta - \delta)^2$
Linex: $L(\theta, \delta) =$ $\exp(r(\delta - \theta)) - r(\delta - \theta) - 1$	$\hat{\delta}_L = \frac{-1}{r} \log(E_\pi(\exp(-r\theta))$	$r(\hat{\delta}_{GQ} - \hat{\delta}_L)$

(1) Under the entropy loss function, we derive the estimators and their corresponding risks (p is an integer):

$$\hat{\alpha}_E = \tag{4.2}$$
$$\left[K \int\int B^m e^{\sum_{i=1}^m (-\alpha x_i + \beta x_i) - a\alpha} C^{n-m}(x_m) \beta^{-1} \alpha^{b-1-p} \prod_{i=1}^m \frac{2+\alpha+x_i}{A_\beta(x_i)} d\alpha d\beta\right]^{-\frac{1}{p}}$$

$$\hat{\beta}_E =$$
$$\left[K \int\int B^m e^{\sum_{i=1}^m (-\alpha x_i + \beta x_i) - a\alpha} C^{n-m}(x_m) \beta^{-p-1} \alpha^{b-1} \prod_{i=1}^m \frac{2+\alpha+x_i}{A_\beta(x_i)} d\alpha d\beta\right]^{-\frac{1}{p}}$$

$$PR(\hat{\alpha}_{GQ}) = pE_\pi(\ln \alpha - \ln \hat{\alpha}_E)$$
$$PR(\hat{\beta}_{GQ}) = pE_\pi(\ln \beta - \ln \hat{\beta}_E)$$

(2) Under the generalized quadratic loss function we derive the estimators and their corresponding risks ($\tau(\theta) = \theta^{\gamma-1}$, γ is an integer):

$$\hat{\alpha}_{GQ} = \frac{\int_0^{+\infty}\int_0^{+\infty} \beta^m e^{\sum_{i=1}^m (-\alpha x_i + \beta x_i)} C^{n-m}(x_m) \beta^{-1} \alpha^{\gamma+b-1} \prod_{i=1}^m \frac{2+\alpha+x_i}{A_\beta(x_i)} d\alpha d\beta}{\int_0^{+\infty}\int_0^{+\infty} \beta^m e^{\sum_{i=1}^m (-\alpha x_i + \beta x_i)} C^{n-m}(x_m) \beta^{-1} \alpha^{\gamma+b-2} \prod_{i=1}^m \frac{2+\alpha+x_i}{A_\beta(x_i)} d\alpha d\beta} \tag{4.3}$$

$$\hat{\beta}_{GQ} = \frac{\int_0^{+\infty}\int_0^{+\infty} B^m e^{\sum_{i=1}^m (-\alpha x_i + \beta x_i) - a\alpha} C^{n-m}(x_m) \beta^{\gamma-1} \alpha^{b-1} \prod_{i=1}^m \frac{2+\alpha+x_i}{A_\beta(x_i)} d\alpha d\beta}{\int_0^{+\infty}\int_0^{+\infty} B^m e^{\sum_{i=1}^m (-\alpha x_i + \beta x_i) - a\alpha} C^{n-m}(x_m) \beta^{\gamma-2} \theta^{b-1} \prod_{i=1}^m \frac{2+\alpha+x_i}{A_\beta(x_i)} d\alpha d\beta}$$

$$PR(\hat{\theta}_{GQ}) = E_\pi(\alpha^{\gamma+1}) - 2\hat{\alpha}_{GQ} E_\pi(\alpha^\gamma) + \hat{\alpha}_{GQ} E_\pi(\alpha^{\gamma-1}),$$
$$PR(\hat{\beta}_{GQ}) = E_\pi(\beta^{\gamma+1}) - 2\hat{\beta}_{GQ} E_\pi(\beta^\gamma) + \hat{\beta}_{GQ} E_\pi(\beta^{\gamma-1}).$$

(3) Under the Linex loss function we derive the estimators and their corresponding risks (r is an integer):

$$\hat{\alpha}_L =$$

$$\frac{-k}{r}\ln[\int\int B^m e^{\sum_{i=1}^{m}(-\alpha x_i+\beta x_i)-a\alpha-r\alpha} C^{n-m}(x_m)\beta^{-1}\alpha^{b-1}\prod_{i=1}^{m}\frac{2+\alpha+x_i}{A_\beta(x_i)}d\alpha d\beta], \tag{4.4}$$

$$\hat{\beta}_L =$$

$$\frac{-k}{r}\ln[\int\int B^m e^{\sum_{i=1}^{m}(-\alpha x_i+\beta x_i)-a\alpha-r\beta} C^{n-m}(x_m)\beta^{-1}\alpha^{b-1}\prod_{i=1}^{m}\frac{2+\alpha+x_i}{A_\beta(x_i)}d\alpha d\beta].$$

$$PR(\hat{\alpha}_L) = r(\hat{\alpha}_{GQ} - \hat{\alpha}_L),$$
$$PR(\hat{\beta}_L) = r(\hat{\beta}_{GQ} - \hat{\beta}_L).$$

We can see clearly that the Bayesian estimators in (4.2), (4.3), and (4.4) cannot be computed analytically, to obtain the estimation values we propose an MCMC procedure to approximate them.

4.1. **Simulation studies.**

4.1.1. *MCMC methods.* We are going to use the Metropolis-Hastings algorithm to generate a sample following the posterior distribution (4.1).

First, we denote $x := (\alpha, \theta, \beta)$, then, starting from a value $x_0 = (\alpha_0, \theta_0, \beta_0)$ chosen arbitrarily we propose a Chi two distribution as the instrumental (or proposal) distribution $p(x, .)$.

X^i is the value retained in step i, and $\tilde{x} \to p(x, .)$ is the proposed candidate,

(1) calculate
$$r = \min\left(1, \frac{\pi(\tilde{x})p(\tilde{x}, x)}{\pi(x)p(x, \tilde{x})}\right);$$
(2) generate $U_{[0,1]}$ noting $U^i = u$;
(3) verify if $r \geq u$: $X^{i+1} \leftarrow \tilde{x}$, else $X^{i+1} \leftarrow x$.

We repeat these steps for $i = \overline{1, M}$, where M is the number of iterations.

The resulting random sequence $(Xi)_{i\geq 0}$ is a Markov chain following the distribution (4.1).

Convergence diagnosis

One of the widely used methods for convergence diagnosis is a graphical method, which is the trace plot. a trace is a time series plot used in order to show the realizations of the Markov chain at each iteration as opposed to the iteration number, If the trace plot shows flat bits it does indicate having a slow convergence, i.e., the MCMC chain is stuck in some part. However, if the trace plot resembles a hairy caterpillar, it is an indication of a strong convergence which means an efficient MCMC algorithm.

We run the Markov chain for 10000 iterations initialized. Figure 3 shows the trace plots of the Markov chain samples.

We notice from the trace plot that there is no need for burn-in.

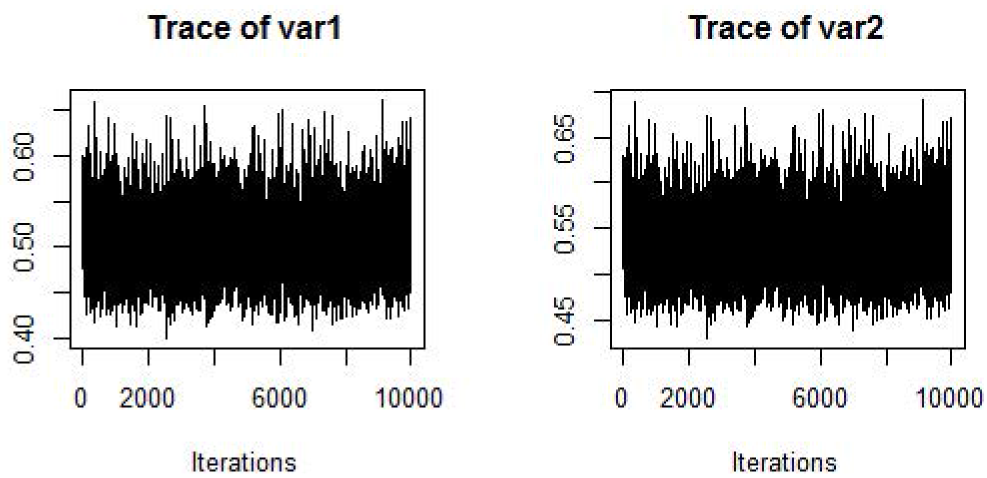

FIGURE 3. trace of the Metropolis Hastings algorithm

We display in the next tables the Bayesian estimation under the entropy, generalized quadratic and Linex loss functions, using the MCMC procedure that we explained, we choose $a = 2$ and $b = 1$ for the hyper-parameters of the prior distribution, they were chosen in this way so the prior mean becomes the excepted value of the parameter.

We picked different choices for the integers: $p, \gamma, r \in \{-2, -1.5, -1, -0.5, 2, 1.5, 1, 0.5\}$.

TABLE 2. Bayes estimators and PR (in brackets) under the entropy loss function.

	N = 5000	n = 20	n = 50	n = 200
p	m	12	30	120
-2	α	2.0942 (0.0008)	2.3990 (0.1644)	2.2144 (0.0019)
	β	1.3188 (0.0699)	1.2839 (0.0090)	0,.7034 (0.0110)
-1.5	α	2.1067 (0.0091)	1.7188 (0.1443)	2.2179 (0.0017)
	β	0.4407 (0.0611)	0.4077 (0.0661)	0.7060 (0.0012)
-1	α	2.1041 (0.0009)	1.6205 (0.0171)	2.2167 (0.0001)
	β	1.4177 (0.0072)	1.3633 (0.0073)	0.7051 (0.0003)
-0.5	α	1.7981 (0.0038)	1.7830 (0.0733)	2.2148 (0.0009)
	β	0.6493 (0.0308)	0.8755 (0.319)	0.7037 (0.0009)
0.5	α	1.8998 (0.0008)	1.8895 (0.0729)	1.9814 (0.0001)
	β	0.7638 (0.0071)	0.9856 (0.0065)	1.0024 (0.0002)
1	α	1.6981 (0.0038)	2.4830 (0.0733)	1.2148 (0.0009)
	β	0.5491 (0.0308)	1.3055 (0.0319)	0.6037 (0.0009)
1.5	α	1.7053 (0.0035)	1.6701 (0.0667)	2.2169 (0.0009)
	β	1.4239 (0.0199)	1.3881 (0.0303)	0.7059 (0.0003)
2	α	1.7697 (0.0099)	1.7644 (0.1173)	2.2188 (0.0031)
	β	1.4579 (0.0997)	1.4259 (0.0944)	0.7071 (0.0014)

TABLE 3. Bayes estimators and PR (in brackets) under generalized quadratic loss

	N = 5000	n = 20	n = 50	n = 200
γ	m	12	30	120
-2	α	1.6490 (0.0089)	1.6825 (0.0041)	1.6432 (0.0016)
	β	0.6657 (0.1491)	0.5033 (0.0611)	0.8113 (0.0008)
-1.5	α	1.7990 (0.0087)	2.0825 (0.0061)	2.2127 (0.0016)
	β	0.8657 (0.7091)	0.7039 (0.0633)	0.7120 (0.0008)
-1	α	1.9282 (0.0004)	1.9841 (0.0001)	2.0015 (0.0001)
	β	0.9296 (0.0003)	0.9789 (0.0009)	0.9998 (0.0001)
-0.5	α	2.0994 (0.0089)	2.0888 (0.0070)	2.2138 (0.0018)
	β	1.2999 (0.0825)	1.2701 (0.711)	0.7131 (0.0012)
0.5	α	1.7510 (0.0095)	1.7926 (0.0077)	2.1839 (0.0020)
	β	0.6891 (0.0909)	1.3591 (0.995)	1.7139 (0.0019)
1	α	1.7575 (0.0091)	2.0977 (0.0078)	2.1841 (0.0031)
	β	1.4228 (0.1094)	1.3803 (0.1071)	1.7149 (0.0025)
1.5	α	1.6743 (0.0098)	1.5632 (0.0081)	2.1232 (0.0042)
	β	1.4768 (0.1241)	0.6754 (0.1181)	0.7903 (0.0033)
2	α	2.1099 (0.0098)	2.0990 (0.0081)	2.1841 (0.0042)
	β	1.4768 (0.1241)	1.4191 (0.1181)	0.7158 (0.0033)

TABLE 4. Bayes estimators and PR (in brackets) under Linex loss function.

	N = 5000	n = 10	n = 50	n = 200
r	m	12	30	120
-2	α	1.7021 (0.0039)	1.6549 (0.0066)	2.1015 (0.0006)
	β	1.4647 (0.1041)	1.1058 (0.0147)	1.4315 (0.0481)
-1.5	α	1.5409 (0.1666)	1.5861 (0.0009)	1.7174 (0.0003)
	β	1.4721 (0.1884)	1.4183 (0.0131)	0.7145 (0.0004)
-1	α	1.7201 (0.0039)	1.6815 (0.0038)	1.8179 (0.0012)
	β	1.4806 (0.0411)	1.4455 (0.0519)	0.7054 (0.0013)
-0.5	α	1.5555 (0.0519)	1.5815 (0.0183)	2.0070 (0.0057)
	β	0.2191 (0.0049)	0.1251 (0.0195)	0.7094 (0.0057)
0.5	α	2.2041 (0.0107)	1.9813 (0.0031)	2.0019 (0.0015)
	β	0.8070 (0.0020)	1.1619 (0.0105)	0.8021 (0.0012)
1	α	1.7228 (0.0105)	1.7819 (0.0081)	1.8153 (0.0004)
	β	0.7117 (0.1033)	1.4639 (0.0581)	0.7059 (0.0013)
1.5	α	2.0182 (0.0013)	2.0501 (0.0007)	1.9731 (0.0003)
	β	0.9692 (0.0014)	0.9462 (0.0099)	0.9859 (0.0004)
2	α	1.6991 (0.0007)	1.8058 (0.0147)	2.2061 (0.0015)
	β	1.3815 (0.0183)	1.1251 (0.0195)	0.7091(0.0032)

We notice that:

- Under the entropy loss function, the value $p = 0.5$ gives us the best posterior risk.
- Under the generalized loss function, the value $\gamma = -1$ gives us the best posterior risk.
- Under the Linex loss function, the value $r = 1.5$ a gives us the best posterior risk.
- We have the smallest posterior risk when n and m are large.

Then, in order to compare the three loss functions based on the corresponding posterior risk, the next table presents the three loss functions at their best performance; when $p = 0.5$, $\gamma = -1$, and $r = 1.5$.

TABLE 5. Bayes estimators and PR (in brackets) under the three loss functions.

	N = 5000	n = 20	n = 50	n = 200
Loss functions	m	12	30	120
Entropy (p=0.5)	α	1.8998 (0.0008)	1.8895 (0.0729)	1.9814 (0.0001)
	β	0.7638 (0.0071)	0.9856 (0.0065)	1.0024 (0.0002)
(GQ)γ= -1	α	1.9282 (0.0004)	1.9841 (0.0001)	2.0015 (0.0001)
	β	0.9296 (0.0003)	0.9789 (0.0009)	0.9998 (0.0001)
Linex (r = 1.5)	α	2.0182 (0.0013)	2.0501 (0.0007)	1.9731 (0.0003)
	β	0.9692 (0.0014)	0.9462 (0.0099)	0.9859 (0.0004)

We see clearly that the generalized quadratic loss function gives us the smallest posterior risk among the three loss functions.

5. COMPARISON OF THE ESTIMATION METHODS

There are many procedures to compare between different methods of estimation, in order to compare two estimators the pitman criterion is a simple and logical procedure that is defined as: see Jozani [11].

According to Pitman closeness criterion, an estimator δ_1 of a parameter θ preforms better than another estimator called δ_2 if

$$P_\theta[|\delta_1 - \theta| < |\delta_2 - \theta|] > \frac{1}{2}.$$

	N = 5000	n = 20	n = 50	n = 200
Loss functions	m	12	30	120
Entropy (p=0.5)	α	0.579	0.682	0.625
	β	0.543	0.542	0.599
(GQ)γ= -1	α	0.789	0.602	0.668
	β	0.753	0.559	0.643
Linex (r = 1.5)	α	0.697	0.634	0.5779
	β	0.632	0.579	0.5623

TABLE 6. Pitman comparison of the estimators.

In Table 6, we presented the values of the Pitman probabilities which allows us to compare the Bayesian estimators with the MLE estimator which is done under the three loss functions when $p = 0.5, \gamma = -1, andr = 1.5$. According to the definition above, when the probability is greater than 0.5, the Bayesian estimators are better than the MLE estimators. Then we notice that, according to this criterion, the Bayesian estimators of the two parameters are better than the MLE. Also, the generalized quadratic loss function has the best values in comparison with the other two loss functions with $0.789_{|n=20,m=12}$ and $0.753_{|n=20,m=12}$.

6. Bayesian prediction for future order statistics

We assume that $X_{1,m,n}, \cdots, X_{m,m,n}$ is a type II censored lifetime sample of size m, drawn from a two parameters UX Lindley distribution, this sample represents the past (informative) sample, otherwise, we denote $Y_1, \cdots, Y_N$, a random sample of size N of unobserved observations (future) from the same distribution, we suppose that both samples are independent, our aim is to make a Bayesian prediction for the $k^{th}, 1 \leq k \leq N$, ordered lifetime in a future sample of size N

The density function of the k^{th}, ordered lifetime Y_k in the future sample (of size N) is given by:

$$H_{(k)}(y_k; \alpha, \beta) = k\binom{N}{k}\left[S(y_k)\right]^{N-k}\left[F_{UXL}(y_k, \alpha, \beta)\right]^{k-1} f_{UXL}(y_k, \alpha, \beta).$$

Using the the functions and the survival function given in (2.1), (2.2) and (2.3), we obtain

$$H_{(k)}(y_k; \alpha, \beta) = \alpha^2(1+y_k)(1+\beta)^2 e^{-\alpha y_k + \beta y_k} \frac{D^{N-k}E^{k-1}}{(1+\alpha)^{2N-1}A_\beta(y_k)^N}, \tag{6.1}$$

where

$$D(y_k) = (1+\theta)^2 A_\beta(y_k) - A_\alpha(x)(1+\beta)^2 e^{\alpha y_k + \beta y_k} F(y_k) = A_\alpha(y_k)(1+\beta)^2 e^{-\alpha y_k + \beta y_k}.$$

Thus, the Bayesian predictive density function of Y_k is:

$$H_{(k)}(y_k|X) = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} H_{(k)}(y_k; \alpha, , \beta)\pi(\alpha, \beta|X)\mathrm{d}\alpha\mathrm{d}\beta,$$

where $\pi(\alpha, \beta|X)$ is the joint posterior density (4.1), it is clear that this Bayesian predictive density function can not be calculated or expressed analytically.

We propose the use of the natural predictor $\hat{Y}_k = E(Y_k)$ by using the MCMC sampling procedure described in the sections above, so we can obtain a simulation-based estimation.

7. Conclusion

In this paper, we introduced and studied a new version of the Lindley model called the two parameters Upper truncated XL (UXLE) model. We introduce the model, provide its survival function and failure rate, we find the maximum likelihood, under a censored scheme. Also developed and explored the Bayesian estimation under three loss functions, a thorough comparison of the two methods is conducted; using the Pitman criterion. The following results can be specifically highlighted: amongst the three loss functions utilized the generalized quadratic provides the best estimators. When the sample size is great both methods give the smallest error(quadratic error for the maximum likelihood approach and the posterior risk for the Bayesian approach). Comparing the two approaches indicate that Bayesian estimation performs better than the maximum likelihood estimation particularly when the sample size is small, it is generally emphasized that the Bayesian technique and the Maximum Likelihood methods are advised.

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