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AN ITERATIVE METHOD FOR THIRD-ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT. This article is intended to apply the reproducing kernel Hilbert space method (RKHSM) for solving a third order differential equation with multiple characteristics in a rectangular domain. The exact solution is expressed in form of series. The convergence of the iterative method to find the approximate solution is proven. Some numerical examples are studied to demonstrate the accuracy of the present method. Results obtained by the method are compared with the exact solution of each example which are found to be in good agreement with each other.

1. INTRODUCTION

Boundary value problems (BVPs) have a lot of applications, like engineering technique, control theory and optimization, the boundary layer of fluid mechanics, aero-elasticity, sandwich beamanalysis and beam deflection theory, electromagneticwaves. Moreover, boundary-value problems with boundary conditions constitute a very interesting and important class of problems.

In recent decades much attention has been paid to the study of third-order boundary value problems and numerous articles are devoted to their solvability,

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see $[4,7,10,11]$ and references therein. Third-order differential equations appear in various fields of applied mathematics and physics, for example in the deflection of a curved beam having a constant or variable section, a three-layer beam, electromagnetic waves, or flows related to gravity... For this reason, their solvability is important and thus different methods are used. In $[4,7,10]$ the authors used the spline functions to solve boundary value problems to third-order partial differential equations. In $[11]$, the authors developed a second-order method based on the Padé approximant in a recurrence relation to solve third-order three-point boundary value problems.

Yet, for the third-order three-point nonlinear BVP, it is difficult to exactly satisfy the boundary conditions, unless one designs the algorithm to exactly satisfy all the boundary conditions.

In this paper we propose new numerical methods for solving the nonlinear two dimensional third-order boundary value problems with multiple characteristics, designing the algorithms to automatically satisfy the three-point boundary conditions, which are based on a novel concept of reproducing kernel Hilbert space method.

The reproducing kernel Hilbert space method has been used recently, thereby, several works have successfully been treated the one-dimensional third-order boundary value problems, see $[8,9,12]$. However for the third-order boundary value problems in the rectangular domain, this method has been applied only in two works $[3\ ,6]$.

In $[6]$ the authors are concerned with the numerical solution of the following third-order partial differential equation with three-point boundary condition

$$\begin{aligned}
\frac{\partial^3 u(x,t)}{\partial t^3} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}u(x,t)\right) &= f(x,t) \\
\int_c^1 u(x,t) &= 0,\ t \in [0,T]\,,\, 0 \le c < 1 \\
u_{tt}(x,0) &= 0, u_t(x,0) = 0, u(x,0) = 0,
\end{aligned}$$

where $a(x,t)$ and its derivatives satisfy the condition $0 < a_0 < a(x,t) < a_1$, $|a_x(x,t)| \le b$, and $f(x,t)$ is given smooth function in $[0,1] \times [0,T]$.

In [3] the authors are concerned with the numerical solution of the following third-order partial differential equation

$$u_{xxx}(x,t)-u_{tt}(x,t)=f(x,t),(x,t)\in Q=(0,1)\times(0,1)\,,$$

subject to the boundary conditions:

$$u_t(x,0)=\varphi_1\left(x\right),u(x,1)=\varphi_2\left(x\right),$$

$$u(0,t)=\psi_1\left(t\right),u(1,t)=\psi_2\left(t\right),u_x(1,t)=\psi_3\left(t\right).$$

The rest of the paper is organized as follows. Section 2 introduces third-order three-point linear BVP with multiple characteristics in a rectangular domain. Section 3 represents several reproducing spaces required in this article. Section 4 introduces a linear operator, a complete normal orthogonal system and some essential results. Section 5 provides exact and approximate solutions to problem (2.1)-(2.3) and develops an iterative method for this kind of problems. Additional numerical examples are included in Section 6. Finally, the paper ends with a brief conclusion.

2. Third-order boundary value problem

We consider the following third order differential equation

$$\begin{aligned}(2.1)\qquad \frac{\partial^3 U}{\partial x^3}-\frac{\partial}{\partial t}\left(a(t)\frac{\partial}{\partial t}U\right)+\delta\, U(x,t) &= F(x,t),\\ (x,t) &\in Q=(0,1)\times(0,1)\,,\ \delta>0\end{aligned}$$

subject to the boundary conditions:

$$\begin{aligned}(2.2)\qquad U(0,t) &= \varphi\left(t\right),U(1,t)=\psi\left(t\right),U_x(1,t)=\xi\left(t\right),\\ U_t(x,0) &= U_t(x,1)=0\end{aligned}$$

such that the compatibility conditions are fulfilled :

$$\begin{aligned}\varphi^{'}\left(0\right) &= \psi^{'}\left(0\right)=\xi^{'}\left(0\right)=0,\\ \varphi^{'}\left(1\right) &= \psi^{'}\left(1\right)=\xi^{'}\left(1\right)=0\end{aligned}$$

where

$$\varphi\left(t\right),\psi\left(t\right)\ \in\ C^3\ [0,1]$$

$$\xi(t) \in C^2 \ [0,1], \ F(x,t) \in C^{0,2}_{x,t}(Q),$$

$a(t)$ and its derivatives satisfy the conditions $0 < a_0 \le a(t) \le a_1$, $\left|a'(t)\right| \le a_2$, $t \in (0,1)$, $a'(0) = a'(1)$.

The existence and uniqueness of the solution for Eq. (2.1)-(2.2) are discussed in [2].

In order to put boundary conditions (2.2) into the reproducing kernel space $H_{(4,3)}$constructed in the following sections, we have to homogenize these conditions. For this, let

$$\begin{aligned} u(x,t) &= U(x,t) - (x^2-2x+1)\varphi(t) - x(2-x)\psi(t) \\ &- x(x-1)\xi(t), \end{aligned}$$

then the problem (2.1)-(2.2) can be written as

$$\begin{cases} \dfrac{\partial^3 u}{\partial x^3} - \dfrac{\partial}{\partial t}\left(a(t)\dfrac{\partial}{\partial t}u\right) + \delta\, u(x,t) = f(x,t), \\ u_t(x,0) = u_t(x,1) = 0, \\ u(0,t) = 0, \ u(1,t) = 0, \ u_x(1,t) = 0. \end{cases} \tag{2.3}$$

3. The reproducing kernel Hilbert space method

The theory of reproducing kernels was used for the first time in the early 20th century [1]. Accordingly, the reproducing kernel Hilbert space method is based on generating orthonormal basis system over a compact dense interval in Sobolev space in order to construct an appropriate numerical solution.

Definition 3.1. *Let E be a nonempty abstract set. A function $K : E \times E \to \mathbb{C}$ is a reproducing kernel of the Hilbert space H if*

$$\begin{aligned} &1) \textit{ for each } t \in E, K(.,t) \in H. \\ &2) \textit{ for each } t \in E \textit{ and } \varphi \in H, \ \langle \varphi, K(.,t)\rangle = \varphi(t). \end{aligned}$$

The last condition is called "the reproducing property": the value of the function φ at the point t is reproducing by the inner product of φ with $K(.,t)$. A Hilbert space which possesses a reproducing kernel is called a RKHS. Meanwhile $K_s(t) = K(s,t)$.

Lemma 3.1. *A reproducing kernel function of real reproducing kernel space is symmetric.*

Reproducing Kernel Spaces

We define some useful reproducing kernel spaces.

Definition 3.2. *Let*

$$\begin{aligned} H_4[0,1] &= \{u|u(x), u'(x), u''(x), u'''(x) \text{ are absolutely continuous real value} \\ &\qquad \text{functions in } [0,1],\ u^{(4)}(x) \in L^2[0,1], u(0) = u(1) = u'(1) = 0\} \end{aligned}$$

On the other hand, let $\langle u(x), v(x) \rangle_{H_4}$ *be the inner product in the space* $H_4\,[0,1]$, *which is defined by*

$$\langle u(x), v(x) \rangle_{H_4} = u^{(2)}(0)v^{(2)}(0) + u^{(3)}(0)v^{(3)}(0) + \int_0^1 u^{(4)}(x)v^{(4)}(x)\,dx,$$

and the norm is $\|u\|_{H_4} = \sqrt{\langle u(x), u(x) \rangle_{H_4}}$, *where* $u(x),\, v(x) \in H_4[0,1]$

We have the following result.

Lemma 3.2. *The space* $H_4[0,1]$ *is a reproducing kernel Hilbert space. The reproducing kernel function* $R_x(y)$ *is given by*

$$R_x(y) = \begin{cases} \sum_{i=1}^{8} C_i\,(x)\,y^{i-1}, & y \le x, \\ \sum_{i=1}^{8} D_i\,(x)\,y^{i-1}, & y > x. \end{cases}$$

Here

$$\begin{aligned} C_1 &= -x^7/5040, C_2 = D_2 = 1741x/70272 - 181x^2/3904 + 307x^3/17568 \\ &\quad +307x^4/70272 + 181x^5/234240 + 253x^7/702720, D_1 = 0, \\ C_3 &= D_3 = -181x/3904 + 173x^2/1952 - 35x^3/976 - 35x^4/3904 \\ &\quad -173x^5/117120 - x^7/7808, \\ C_4 &= D_4 = -307x/17568 + 35x^2/976 - 29x^3/1464 - 29x^4/5856 \\ &\quad -7x^5/11712 + x^7/35136, \end{aligned}$$

$$\begin{aligned}
C_5 &= D_5 = -307x/70272 + 35x^2/3904 + 35x^3/17568 + \\
&\quad 35x^4/70272 - 7x^5/46848 + x^7/140544, \\
C_6 &= D_6 = -181x/234240 - 21x^2/7808 - 7x^3/11712 \\
&\quad -7x^4/46848 + 7x^5/156160 - x^7/468480, \\
C_7 &= -x/720,\ D_7 = 0, C_8 = 253x/702720 - x^2/7808 - \\
&\quad x^3/35136 - x^4/140544 + x^5/468480 - x^7/9838080, \\
D_8 &= -1/5040 + C_8
\end{aligned}$$

Proof. Using several integrations by parts of $\int_0^1 u^{(4)}(y)R_y^{(4)}(y)dy$, it yields

$$\begin{aligned}
\langle u(y), R_x(y)\rangle_{H_4} &= u^{^{(2)}}(0)\partial_y^2 R_x(0) + u^{^{(3)}}(0)\partial_y^3 R_x(0) \\
&+ \sum_{i=0}^{3}(-1)^{4-i}u^{(i)}(0)\partial_x^{7-i}R_y(0) \\
&+ \sum_{i=0}^{3}(-1)^{3-i}u^{(i)}(1)\partial_x^{7-i}R_y(1) \\
&- \int_0^1 u(y)\partial_y^8 R_x(y)dy.
\end{aligned}$$

Since $u(y) \in H_4[0,1]$, we get

$$u(0) = u(1) = u'(1) = 0,$$

then

$$\begin{aligned}
\langle u(y), R_x(y)\rangle_{H_4} &= u^{^{(2)}}(0)\partial_y^2 R_x(0) + u^{^{(3)}}(0)\partial_x^3 R_x(0) \\
&+ \sum_{i=1}^{3}(-1)^{4-i}u^{(i)}(0)\partial_y^{7-i}R_x(0) \\
&+ \sum_{i=2}^{3}(-1)^{3-i}u^{(i)}(1)\partial_y^{7-i}R_x(1) \\
&- \int_0^1 u(y)\partial_y^8 R_x(y)dy.
\end{aligned}$$

Taking into account the property of the reproducing kernel $\langle u(y), R_x(y)\rangle_{H_4} = u(y)$, then $R_x(y)$ is the solution of the following generalized differential equation

$$\partial_y^8 R_x(y) = \delta\,(x-y)\,, \tag{3.1}$$

with the boundary conditions

$$\begin{aligned} \partial_y^3 R_x(0) - \partial_y^4 R_x(0) &= 0, \\ \partial_y^2 R_x(0) + \partial_y^5 R_x(0) &= 0, \\ \partial_y^6 R_x(0) &= \partial_y^4 R_x(1) = \partial_y^5 R_x(1) = 0. \end{aligned} \tag{3.2}$$

Therefore

$$R_x(y) = \begin{cases} \sum_{i=1}^{8} C_i\,(x)\,y^{i-1}, & y \le x, \\ \\ \sum_{i=1}^{8} D_i\,(x)\,y^{i-1}, & y > x. \end{cases} \tag{3.3}$$

On the other hand, since $R_x(y)$ satisfies

$$\partial_y^i R_x(x+0) = \partial_y^i R_x(x-0), \qquad i = 0,1,2,3,4,5,6. \tag{3.4}$$

Then, integrating (3.4) from $(x-\epsilon)$ to $(x+\epsilon)$ with respect to y and letting $\epsilon \to 0$, we have the jump degree of $\partial_y^7 R_x(y)$ at $x = y$ given by

$$\partial_y^7 R_x(x+0) - \partial_y^7 R_x(x-0) = -1. \tag{3.5}$$

Since $R_x(y) \in H_4[0,1]$, we obtain

$$R_x(0) = 0, \quad R_x(1) = 0,\ \partial_y^1 R_x(1) = 0. \tag{3.6}$$

Through (3.4)-(3.7), the unknown coefficients of (3.3) can be obtained. □

Definition 3.3. *Define the space*

$$\begin{aligned} H_3[0,1] &= \{u(t)|u(t), u'(t), u''(t) \text{ are absolutely continuous} \\ &\qquad \text{real value functions in } [0,1],\ u^{(3)}(t) \in L^2[0,1],\ \ u'(0) = u'(1) = 0\big\}. \end{aligned}$$

The inner product and the norm in $H_3\,[0,1]$ *are defined respectively by*

$$\begin{aligned} \langle u(t), v(t)\rangle_{H_3} &= \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(3)}(t)v^{(3)}(t)\,dt, \\ \|u\|_{H_3} &= \sqrt{\langle u(t), u(t)\rangle_{H_3}}, \quad u(t), v(t) \in H_3[0,1]. \end{aligned}$$

Lemma 3.3. *The space $H_3[0,1]$ is a complete reproducing kernel space and its reproducing kernel $G_s(t)$ can be given by*

$$G_s(t) = \begin{cases} \frac{1}{768}\ s^4(-32\ t + 12t^2 + 4t^3 - t^4) + s^2(\frac{t^2}{16} - \frac{1}{16}t^3 + \frac{t^4}{64}) \\ \quad +s^3(\frac{t^2}{48} - \frac{1}{48}t^3 + \frac{t^4}{192}) + 1 + \frac{1}{120}s^5,\ s \leq t, \\ s^3(\frac{t^2}{48} - \frac{t^3}{48} + \frac{t^4}{192}) + \frac{1}{768}\ s^4(-32\ t + 12t^2 + 4t^3 - t^4) \\ \quad -\frac{1}{24}st^4 + s^2(\frac{t^2}{16} - \frac{1}{16}t^3 + \frac{t^4}{64}) + 1 + \frac{t^5}{120},\ t < s. \end{cases}$$

Definition 3.4. *Define*

$$H_{(4,3)}(Q) = \left\{ u(x,t) | \frac{\partial^5 u}{\partial^3 x \partial^2 t} \text{ are absolutely continuous real-valued functions} \right.$$
$$\left. \begin{array}{c} \text{in } Q,\ \frac{\partial^7 u}{\partial^4 x \partial^3 t} \in L^2(Q), \frac{\partial u(x,1)}{\partial t} = \frac{\partial u(x,0)}{\partial t} = 0, \\ u(0,t) = u(1,t) = \frac{\partial u(1,t)}{\partial x} = 0 \end{array} \right\}.$$

The inner product and the norm in $H_{(4,3)}(Q)$ are defined respectively by

$$\begin{aligned} \langle u(x,t), v(x,t) \rangle_{H_{(4,3)}} &= \sum_{i=0}^{2} \left(\int_0^1 \frac{\partial^3}{\partial t^3} \frac{\partial^i}{\partial t^i} u(0,t) \times \frac{\partial^3}{\partial t^3} \frac{\partial^i}{\partial t^i} v(0,t) dt \right. \\ &\quad + \int_0^1 \frac{\partial^4}{\partial x^4} \frac{\partial^i}{\partial t^i} u(x,0) \times \frac{\partial^4}{\partial x^4} \frac{\partial^i}{\partial t^i} v(x,0) dx \\ &\quad \left. + \frac{\partial^2}{\partial x^2} \frac{\partial^i}{\partial t^i} u(0,0) \times \frac{\partial^2}{\partial x^2} \frac{\partial^i}{\partial t^i} v(0,0) \right) \\ &\quad + \int_0^1 \int_0^1 \frac{\partial^4}{\partial x^4} \frac{\partial^3}{\partial t^3} u(x,t) \frac{\partial^4}{\partial x^4} \frac{\partial^3}{\partial t^3} v(x,t) dt dx, \end{aligned}$$

and

$$\|u\|_{H_{(4,3)(Q)}} = \sqrt{\langle u(x,t), u(x,t) \rangle_{H_{(4,3)}}},\ \ u \in H_{(4,3)}(Q).$$

Lemma 3.4. *[5]. $H_{(4,3)}(Q)$ is a reproducing kernel space and its reproducing kernel function is*

$$K_{(y,s)}(x,t) = R_y(x) G_s(t).$$

such that for any $u(x,t) \in H_{(4,3)}(Q)$

$$u(y,s) = \left\langle u(x,t), K_{(y,s)}(x,t) \right\rangle_{H_{(4,3)}}$$

Definition 3.5. *[5]. Define*

$$H_1[0,1] = \left\{ u(.) | u(.) \text{ is absolutely continuous in } [0,1],\ u'(.) \in L^2[0,1] \right\}.$$

The inner product and the norm in $H_1[0,1]$ are defined respectively by

$$\begin{aligned} \langle u(x), v(x)\rangle_{H_1} &= u(0)v(0) + \int_0^1 u'(x)v'(x)\,dx, \\ \|u\|_{H_1} &= \sqrt{\langle u(x), u(x)\rangle_{H_1}}, \quad u(x), v(x) \in H_1[0,1]. \end{aligned}$$

Definition 3.6. *[5]. Define*

$$\hat{H}\,(Q) = \left\{ u | u(x,t) \text{ is completely continuous in } Q, \frac{\partial^2 u}{\partial x \partial t} \in L^2\,(Q) \right\}.$$

The inner product and the norm in $\hat{H}\,(Q)$ are defined respectively by

$$\begin{aligned} \langle u(x,t), v(x,t)\rangle_{\hat{H}} &= \int_0^1 \frac{\partial}{\partial t}u(0,t)\frac{\partial}{\partial t}v(0,t)dt \\ &\quad + \langle u(x,0), v(x,0)\rangle_{H_1} \\ &\quad + \int_0^1\int_0^1 \frac{\partial}{\partial x}\frac{\partial}{\partial t}u(x,t)\frac{\partial}{\partial x}\frac{\partial}{\partial t}v(x,t)dtdx, \\ \|u\|_{\hat{H}} &= \sqrt{\langle u(x,t), u(x,t)\rangle_{\hat{H}}}, \;\; u \in \hat{H}\,(Q)\,. \end{aligned}$$

Lemma 3.5. *[5]. $\hat{H}\,(Q)$ is a complete reproducing kernel space, its reproducing kernel function is*

$$\check{R}_{(y,s)}(x,t) = \acute{R}_y(x)\acute{R}_s(t).$$

Here

$$\acute{R}_y(x) = \begin{cases} 1+y, \;\; y \le x, \\ 1+x, \;\; x < y. \end{cases}$$

4. A BOUNDED LINEAR OPERATOR

Define the differential operator $L : H_{(4,3)}\,(Q) \to \hat{H}\,(Q)$ by

(4.1) $$(Lu)\,(x,t) = u_{xxx}(x,t) - a(t)u_{tt}(x,t),$$

then equation (2.3) can be converted into the following equivalent form in $H_{(4,3)}\,(Q)$,

(4.2) $$(Lu)\,(x,t) = F(x,t,u(x,t),u_t(x,t)),$$

where

$$\begin{aligned} F(x,t,u(x,t),u_t(x,t)) &= f(x,t) - \delta u(x,t) + a_t(t)u_t(x,t) \\ &+ x(2-x)a(t)\psi_2^{(2)}(t) + x(x-1)a(t)\psi^{(2)}(t) \\ &+ (x^2-2x+1)a(t)\xi^{(2)}(t). \end{aligned}$$

$u(x,t) \in H_{(4,3)}(Q)$, $F(x,t,u(x,t),u_t(x,t)) \in \hat{H}(Q)$.

Lemma 4.1. *L is an invertible bounded linear operator from $H_{(4,3)}(Q)$ into $\hat{H}(Q)$.*

Proof. Since

$$\begin{aligned} u(x,t) &= \langle u(y,s), R_y(x)G_s(t)\rangle_{H_{(4,3)}}, \\ u_{tt}(x,t) &= \left\langle u(y,s), R_y(x)\frac{\partial^2}{\partial t^2}G_s(t)\right\rangle_{H_{(4,3)}}, \\ u_{xxx}(x,t) &= \left\langle u(y,s), G_s(t)\frac{\partial^3}{\partial x^3}R_y(x)\right\rangle_{H_{(4,3)}}, \end{aligned}$$

and

$$\begin{aligned} \|Lu\|^2_{\hat{H}_{(1,1)}} &= \left\| \frac{\partial^3 u}{\partial x^3} - a(t)\,\frac{\partial^2 u}{\partial t^2} \right\|^2_{\hat{H}} \\ &\le \left\| \frac{\partial^2 u}{\partial t^2}\right\|^2_{\hat{H}} + |a(t)|^2 \left\| \frac{\partial^3 u}{\partial x^3}\right\|^2_{\hat{H}}, \end{aligned}$$

we deduce that

$$\begin{aligned} |u_{tt}(x,t)| &\le \|u\|_{H_{(4,3)}} \left\| \frac{\partial^2}{\partial t^2}G_s(t)\right\|_{H_1} \|R_y(x)\|_{H_4} \\ |u_{xxx}(x,t)| &\le \|u\|_{H_{(4,3)}} \left\| \frac{\partial^3}{\partial x^3}R_y(x)\right\|_{H_1} \|G_s(t)\|_{H_3}. \end{aligned}$$

From the continuity of $R_y(x)$, $G_s(t)$, $\frac{\partial^2}{\partial t^2}G_s(t)$ and $\frac{\partial^3}{\partial x^3}R_y(x)$, it yields

$$\begin{aligned} \|R_y(x)\|_{H_4} &\le M_1, \ \|G_s(t)\|_{H_3'} \le M_2, \\ \left\| \frac{\partial^3}{\partial x^3}R_y(x)\right\|_{H_1} &\le M_3, \ \left\| \frac{\partial^2}{\partial t^2}G_s(t)\right\|_{H_1} \le M_4. \end{aligned}$$

Hence

$$\|Lu\|^2_{\hat{H}_{(1,1)}} \le M\,\|u\|^2_{H_{(4,3)}},$$

where $M = a_1^2 M_1^2\ M_4^2 + M_2^2\ M_3^2$.

Since (4.2) has a unique solution [2], it indicates L is invertible.The proof is complete. $\square$

Let us choose a countable subset $S = \{(x_1, t_1), (x_2, t_2), \ldots\}$ in Q and define

$$\varphi_i(x,t) = \check{R}_{(x_i,t_i)}(x,t),\ \omega_i(x,t) = L^*\varphi_i(x,t). \tag{4.3}$$

Lemma 4.2. *Assume that S is dense in Q, then, $\{\omega_i(x,t)\}_{i=1}^{\infty}$ is a complete system in $H_{(4,3)}(Q)$ and*

$$\omega_i(x,t) = \left(L_{(y,s)}K_{(y,s)}(x,t)\right)/_{(y,s)=(x_i,t_i)}.$$

By Gram-Schmidt process, we obtain an orthogonal basis$\{\varpi_i(x,t)\}_{i=1}^{\infty}$ of $H_{(4,3)}$, such that

$$\varpi_i(x,t) = \sum_{j=1}^{i} \beta_{ij}\omega_i(x,t), \tag{4.4}$$

where β_{ij} are orthogonal coefficients.

5. Representation of exact and approximate solutions

Lemma 5.1. *If S is dense in Q, then, the exact solution of equation (4.2) is*

$$u(x,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{i} \beta_{ij} F(x_j, t_j, u(x_j,t_j), u_t(x_j,t_j))\,\varpi_i(x,t). \tag{5.1}$$

Proof. The exact solution $u(x,t)$ can be expanded in the Fourier series in terms of normal orthogonal basis $\{\varpi_i(x,t)\}_{i=1}^{\infty}$ in $H_{(4,3)}(Q)$:

$$\begin{aligned}
u(x,t) &= \sum_{i=1}^{\infty} \langle u(x,t), \varpi_i(x,t)\rangle_{H_{(4,3)}} \varpi_i(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i} \beta_{ij} \langle u(x,t), \omega_j(x,t)\rangle_{H_{(4,3)}} \varpi_i(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i} \beta_{ij} \langle u(x,t), L^*\varphi_i(x,t)\rangle_{H_{(4,3)}} \varpi_i(x,t)
\end{aligned}$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}\left\langle Lu\left(x,t\right),\varphi_i\left(x,t\right)\right\rangle_{\hat{H}(Q)}\varpi_i\left(x,t\right)$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}\left\langle Lu\left(x,t\right),\check{R}_{(x_i,t_i)}(x,t)\right\rangle_{\hat{H}(Q)}\varpi_i\left(x,t\right)$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}\left(Lu\right)\left(x_j,t_j\right)\varpi_i\left(x,t\right),$$

then

$$u\left(x,t\right)=\sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}F\;\left(x_j,t_j,u(x_j,t_j),u_t(x_j,t_j)\right)\varpi_i\left(x,t\right).$$

□

Iterative method and convergence theorem.

We give initial function $u_0\left(x,t\right)$, using the form (5.1), an iterative sequence is constructed:

$$u_n\left(x,t\right)=\sum_{i=1}^{+\infty}\sum_{j=1}^{i}\beta_{ij}F\left(x_j,t_j,u_{n-1}(x_j,t_j),\left(u_{n-1}\right)_t\left(x_j,t_j\right)\right)\varpi_i\left(x,t\right).$$

Now, the approximate solution u_n can be obtained by the n-term intercept of the exact solution u and

$$u_n\left(x,t\right)=\sum_{i=1}^{n}\sum_{j=1}^{i}\beta_{ij}F\left(x_j,t_j,u_{n-1}(x_j,t_j),\left(u_{n-1}\right)_t\left(x_j,t_j\right)\right)\varpi_i\left(x,t\right). \tag{5.2}$$

Lemma 5.2. *If* $u_n\left(x,t\right)$ *is given by (5.2), then*

$$\left(Lu_n\right)\left(x,t\right)=F(x,t,u_{n-1}(x,t),\left(u_{n-1}\right)_t\left(x,t\right)).$$

Theorem 5.1. *If* $C\left\|L^{-1}\right\|\leq 1$, $\left|a_t(t)\right|\leq a_1$, *then* $u_n\overset{\|\cdot\|_{H_{(4,3)}}}{\rightarrow}u$ *as* $\left(n\rightarrow+\infty\right)$.

Proof. From the preceding theorem, we have

$$\left(Lu_n\right)\left(x,t\right)=F(x,t,u_{n-1}(x,t),\left(u_{n-1}\right)_t\left(x,t\right)),$$

so

$$
\begin{aligned}
&\left\|u_n(x,t)-u(x,t)\right\|_{H_{(4,3)}} \\
=\;& \left\|L^{-1}\left(F(x,t,u_{n-1},(u_{n-1})_t)-F(x,t,u,u_t)\right)\right\|_{H_{(4,3)}} \\
\le\;& \left\|L^{-1}\right\| \left\|(F(x,t,u_{n-1},(u_{n-1})_t)-F(x,t,u,u_t))\right\|_{\hat{H}_{(1,1)}} \\
\le\;& \left\|L^{-1}\right\| \left\|-\delta u_{n-1}+a_t(t)\,(u_{n-1})_t+\delta u-a_t(t)u_t\right\|_{H_{(4,3)}} \\
\le\;& \left\|L^{-1}\right\| \left(\delta \left\|u-u_{n-1}\right\|_{H_{(4,3)}}+|a_t(t)|\left\|(u_{n-1})_t-u_t\right\|_{H_{(4,3)}}\right) \\
\le\;& C\left\|L^{-1}\right\| \left(\left\|u-u_{n-1}\right\|_{H_{(4,3)}}+\left\|(u_{n-1})_t-u_t\right\|_{H_{(4,3)}}\right)
\end{aligned}
\tag{5.1}
$$

where $C=\max(\delta,\ a_2)$. From $C\left\|L^{-1}\right\|\le 1$ and (5.3), and by recurrent we deduce that

$$\left\|u_n(x,t)-u(x,t)\right\|_{H_{(4,3)}}\to 0.$$

□

Proposition 5.1. *Suppose that* $u_n \overset{\|.\|}{\to} u$ *in* $H_{(4,3)}(Q)$ *as* $(n\to\infty)$*, then*

$$\left|\frac{\partial^{i+j}}{\partial x^i\partial t^j}u(x,t)-\frac{\partial^{i+j}}{\partial x^i\partial t^j}u_n(x,t)\right|\to 0,\qquad n\to\infty, i,j=0,1,2,3.$$

6. Numerical examples

In order to calculate the approximate solution we put

$$
\begin{aligned}
(x_i,t_i) &= (j/N,k/M)\,, j=0,1,..N, k=0,1,..M, i=0,1,..m=N\times M,\\
e_{n,m}(x,t) &= \left|u(x,t)-u_{n,m}(x,t)\right|.
\end{aligned}
$$

Example 1. *Consider*

$$
\begin{aligned}
U_{xxx}(x,t)-\partial_t\left(a(t)\,U_t(x,t)\right)+U(x,t)&=-\frac{1}{6}\left(-3+2t\right)t^2(e^x+1)\\
+\left(e^x-xe\right)e^{t^2(1-t)^2}&\left(-1+2t-2t^2+8t^3-10t^4+4t^5\right),
\end{aligned}
$$

for all $(x,t)\in Q=[0,1]\times[0,1]$, *subject to the boundary conditions*

$$U(0,t)=\frac{t^3}{3}-\frac{t^2}{2}, U(1,t)=0, U_x(1,t)=0,$$

and

$$U_t(x,0)=0, U_t(x,1)=0,$$

where $a(t)=e^{t^2(1-t)^2}$. *The exact solution is given by*

$$U(x,t)=\left(\frac{-t^3}{3}+\frac{t^2}{2}\right)(-e^x+xe).$$

After homogenizing the boundary conditions, we obtain

$$\begin{aligned} u_{xxx}(x,t)-\partial_t\left(a(t)\ u_t(x,t)\right)+u(x,t) &= f(x,t),\\ u(0,t) &= 0, u(1,t)=0, u_x(1,t)=0,\\ u_t(x,0) &= 0, u_t(x,1)=0, \end{aligned}$$

where

$$\begin{aligned} f(x,t) &= -\frac{1}{6}\left(2e^x-xe-(1-x)^2\right)(-3+2t)\,t^2\\ &\quad +e^{t^2(1-t)^2}\left(e^x-xe-(1-x)^2\right)(-1+2t)\\ &\quad \left(1+2t^2-4t^3+2t^4\right). \end{aligned}$$

The numerical results are presented in Tables 6.1, Figure 6.1, Figure 6.2, Figure 6.3.

Table 6.1. Comparison of results for Example 6.1: (n=5, 17), N=4, M=4, m=25

(x_i,t_i)	$u(x_i,t_i)$	$(u_{5,25})(x_i,t_i)$	$e_{5,25}(x_i,t_i)$	$(u_{17,25})(x_i,t_i)$	$e_{17,25}(x_i,t_i)$
(0.01,0.00)	0.0000000	-4.5993310^{-6}	4.5993310^{-6}	4.0334810^{-17}	4.0334810^{-17}
(0.10,0.15)	0.000236345	0.000112732	0.000123613	0.000184297	0.0000520485
(0.15,0.10)	0.000147429	0.0000678994	0.0000795298	0.0000851855	0.0000622437
(0.35,0.20)	0.000782928	0.000353514	0.000429414	0.000674087	0.000108841
(0.25,0.45)	0.00297356	0.00139026	0.00158329	0.002767	0.000206554
(0.50,0.45)	0.00280526	0.00122581	0.00157945	0.00259351	0.000211749
(0.60,0.65	0.00372888	0.00158153	0.00214735	0.00343299	0.000295891
(0.75,0.7)	0.00206305	0.00122232	0.000840732	0.00186995	0.000193098
(0.80,0.75)	0.00153499	0.000617517	0.00091747	0.00138034	0.000154652
(0.99,0.99)	5.908610^{-6}	3.6511910^{-6}	2.2574110^{-6}	5.1217510^{-6}	7.8685210^{-7}
(0.75,0.7)	-0.0204119	-0.00520635	0.0152055	-0.0204191	$7.24279*10^{-6}$
(0.80,0.75)	-0.0143628	-0.00354061	0.0108222	-0.0143717	8.8946710^{-6}
(0.99,0.99)	-0.0000491748	-0.0000106722	0.0000385026	-0.0000491967	2.18707^{-8}

.

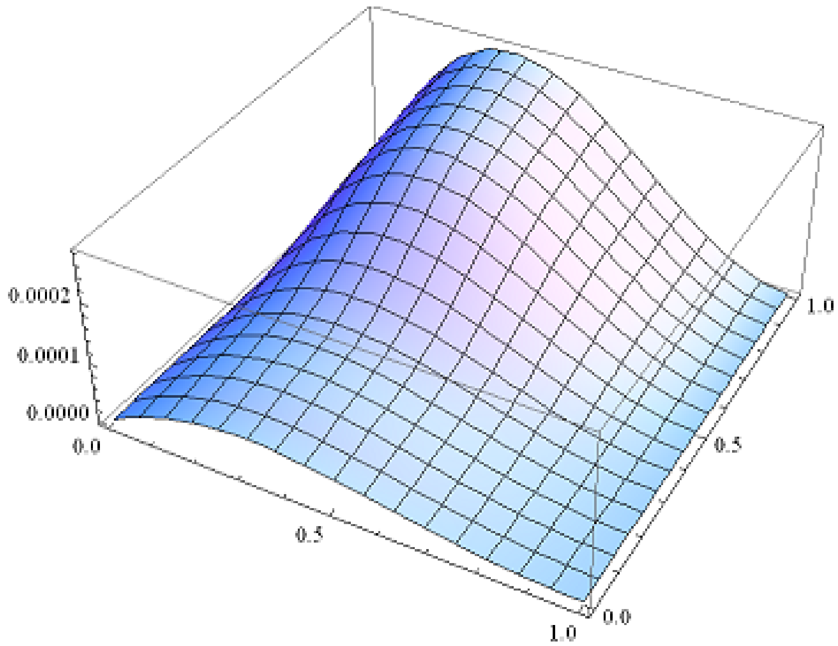

Figure 6.1. Absolute error between exact and approximate solution (n=5, m=25) for example 6.1

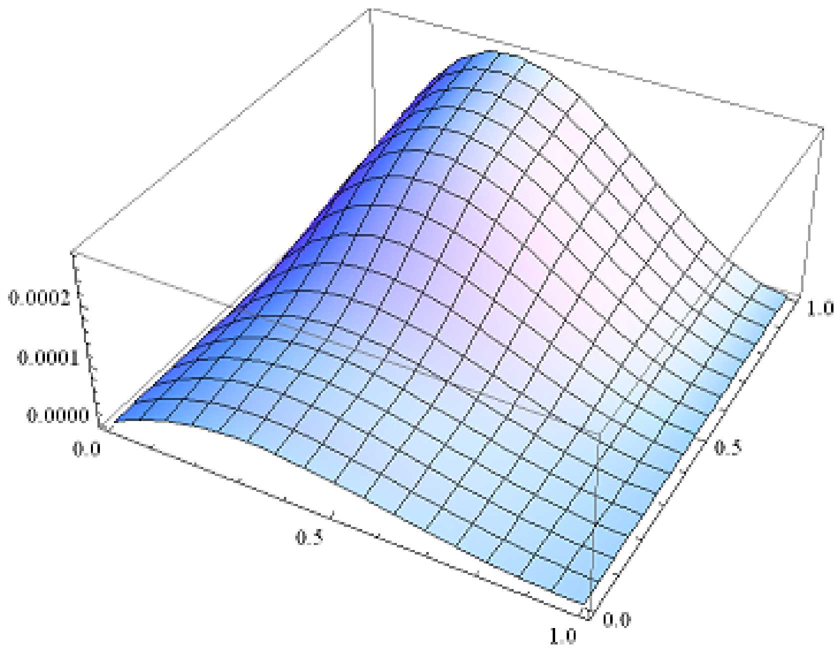

Figure 6.2. Absolute error between exact and approximate solution (n=25, m=25) for example 6.1

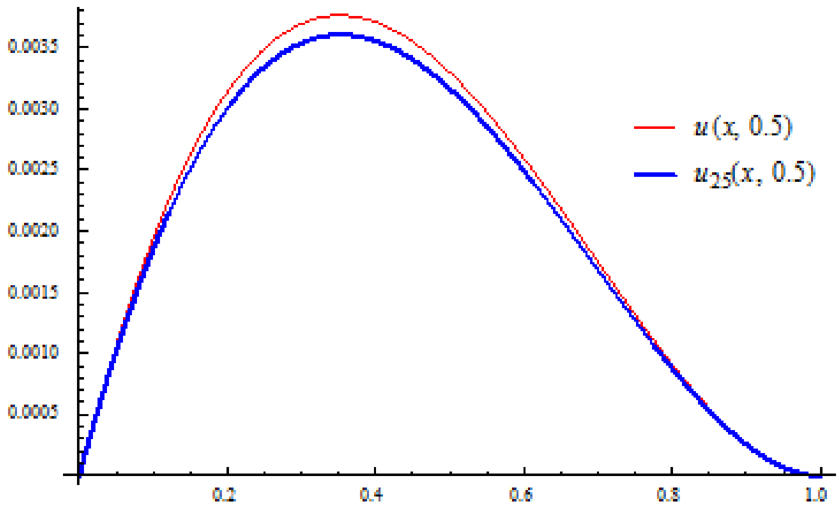

Figure 6.3. Comparison between the approximate solution and the exact solution for Example 6.1

Example 2. *Consider the following equation:*

$$\begin{cases} U_{xxx}(x,t) - t\,U_{tt}(x,t) - U_t(x,t) - U(x,t) = F(x,t) \\ U(0,t) = 0, U(1,t) = 1/6\; e^{t^2(1-t)^2}, U_x(1,t) = 1/6\; e^{t^2(1-t)^2}, \\ \qquad U_t(x,0) = 0, U_t(x,1) = 0, \end{cases}$$

where

$$\begin{aligned} F(x,t) \quad &= 1/6\; e^{t^2(1-t)^2}\left((-1+4t-18t^2+20t^3-24t^4+52t^5-48t^6+16t^7)\right. \\ &\left.\times\left(2\,x^4-3x^3\right)+18-48\,x\right) \end{aligned}$$

The exact solution is given by $U(x,t) = (x^3/2 - x^4/3)\; e^{t^2(1-t)^2}$. *After homogenizing the initial and boundary conditions we obtain*

$$u_{xxx}(x,t) - t\,u_{tt}(x,t) - u_t(x,t) - u(x,t) = f(x,t),$$

where

$$\begin{aligned} f(x,t) &= 1/6\left[18 + (-49+4t-18t^2+20t^3-24t^4+52t^5-48t^6+16t^7)\,x\right. \\ &\left.+ (-1+4t-18t^2+20t^3-24t^4+52t^5-48t^6+16t^7)\left(2\,x^4-3x^3\right)\right] e^{t^2(1-t)^2} \end{aligned}$$

The numerical results are presented in Tables 6.2, Figure 6.4 *, Figure* 6.5*, Figure* 6.6*.*

Table 6.2. Comparison Absolute error for Ex. 6.2 (n=11, 25), N=4, M=4,m=25

(x_i, t_i)	$u(x_i, t_i)$	$(u_{11,25})(x_i, t_i)$	$e_{11,25}(x_i, t_i)$	$(u_{25,25})(x_i, t_i)$	$e_{25,25}(x_i, t_i)$
(0.01,0.00)	-0.00166617	-0.000528164	0.00113801	-0.00166617	$2.41936 * 10^{-13}$
(0.10,0.15)	-0.0164655	-0.00564694	0.0108186	-0.016423	0.0000425001
(0.15,0.10)	-0.0236722	-0.00891318	0.014759	-0.0236368	0.0000354556
(0.35,0.20	-0.0429844	-0.0165564	0.0264279	-0.042873	0.000111365
(0.25,0.45)	-0.0373771	-0.0116428	0.0257343	-0.0372975	0.0000796333
(0.50,0.45)	-0.0442988	-0.0142367	0.0300621	-0.044234	0.0000648266
(0.60,0.65)	-0.0370698	-0.0103444	0.0267254	-0.0370634	$6.34791 * 10^{-6}$
(0.75,0.7)	-0.0204119	-0.00520635	0.0152055	-0.0204191	$7.24279 * 10^{-6}$
(0.80,0.75)	-0.0143628	-0.00354061	0.0108222	-0.0143717	$8.89467 * 10^{-6}$
(0.99,0.99)	-0.0000491748	-0.0000106722	0.0000385026	-0.0000491967	$2.18707 * 10^{-8}$

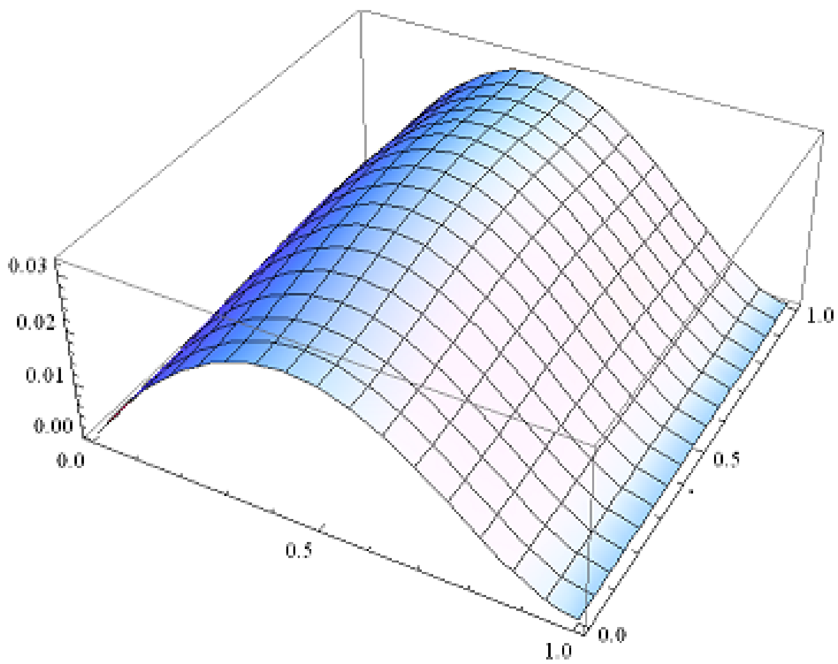

Figure 6.4. Absolute error between exact and approximate solution (n=11, m=25) for example 6.2

Example 3. *Consider*

$$\begin{aligned} u_{xxx}(x,t) - (t+1)\, u_{tt}(x,t) - u_t(x,t) + u(x,t) &= f(x,t), \\ u(0,t) &= 0, u(1,t) = 0, u_x(1,t) = 0, \\ u_t(x,0) &= 0, u_t(x,1) = 0, \end{aligned}$$

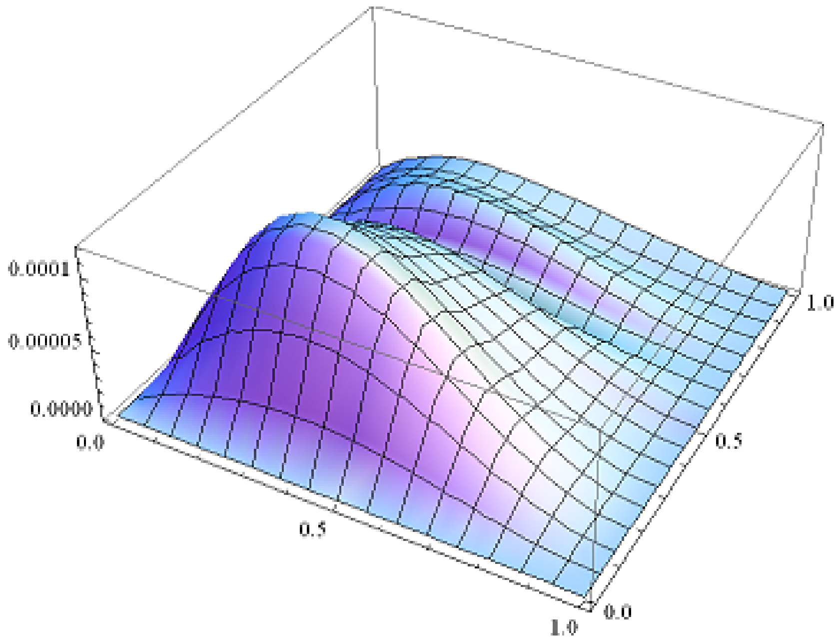

Figure 6.5. Absolute error between exact and approximate solution (n=25, m=25) for example 6.2

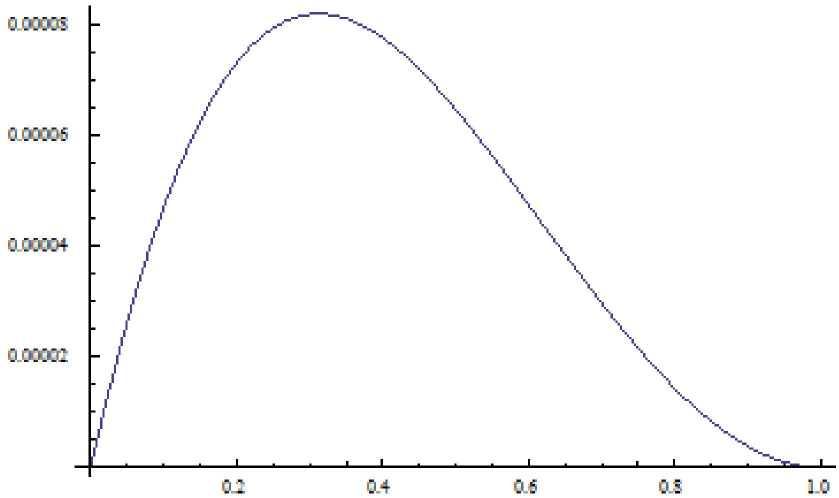

Figure 6.6. Absolute error between exact and approximate solution (n=20, m=25,t=0.45) for example 6.2

for all $(x, t) \in Q = [0, 1] \times [0, 1]$, *where*

$$\begin{aligned} f(x,t) &= \frac{1}{2}(-e^x + (1-x)^2 + xe)2(1+t)e^t \\ & -1 + 6t - t^2 + e(-2 - 4t + t^2) - \frac{1}{2}(-2e^t + (1-t)^2 + et^2)e^x. \end{aligned}$$

The exact solution is given by $u(x,t) = \frac{1}{2}\,(-2e^t + (1-t)^2 + e\,t^2)(-e^x + (1-x)^2 + x\,e)$. *The numerical results are presented in Tables 6.3, Figure* 6.7 *, Figure* 6.8*, Figure* 6.9.

Table 6.3. Comparison of results for Ex. 6.3 (n=9, 22), N=4, M=4,m=25

(x_i, t_i)	$u(x_i, t_i)$	$(u_{9,25})(x_i, t_i)$	$e_{9,25}(x_i, t_i)$	$(u_{22,25})(x_i, t_i)$	$e_{22,25}(x_i, t_i)$
(0.01,0.00)	0.00415102	0.00302248	0.00112855	0.00416952	0.0000184985
(0.10,0.15)	0.0348391	0.0249128	0.00992634	0.0350123	0.00017316
(0.15,0.10)	0.0472799	0.0334371	0.0138428	0.0475083	0.000228419
(0.35,0.20	0.0671678	0.0452082	0.0219597	0.0675208	0.000352949
(0.25,0.45)	0.0605971	0.0417747	0.0188223	0.0610012	0.000404116
(0.50,0.45)	0.0571673	0.0260705	0.0310968	0.0575495	0.000382122
(0.60,0.65)	0.0436891	0.0274247	0.0162644	0.0440447	0.000355572
(0.75,0.7)	0.02199	0.0132544	0.00873562	0.0221793	0.00018928
(0.80,0.75)	0.0151041	0.00897844	0.00612566	0.0152408	0.0001367
(0.99,0.99)	0.0000481998	0.0000272136	0.0000209863	0.0000486821	4.8227210^{-8}

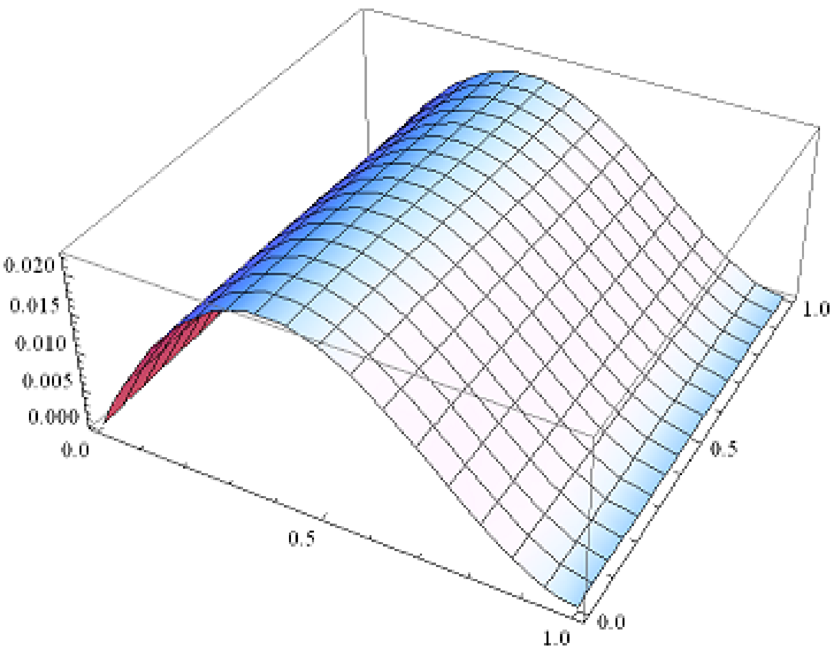

Figure 6.7. Absolute error between exact and approximate solution (n=9, m=25) for example 6.3

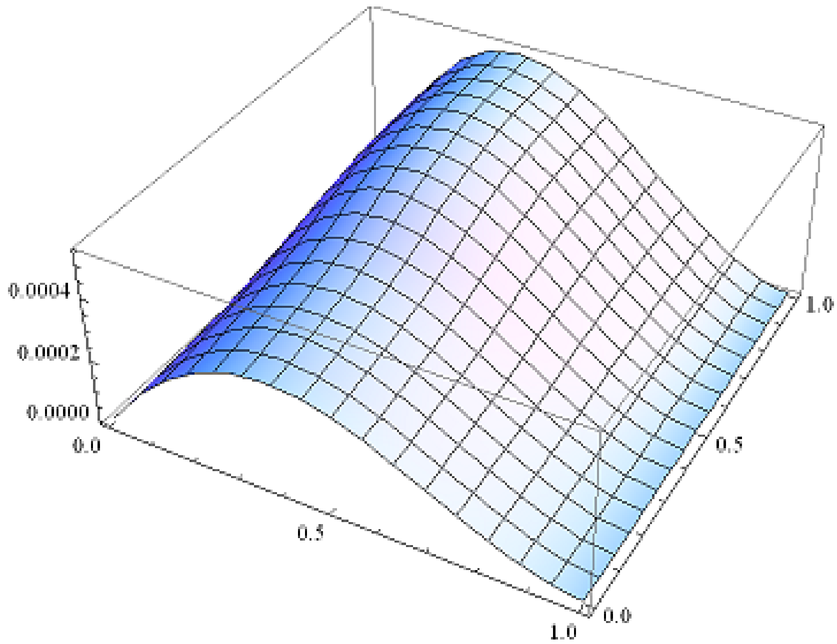

Figure 6.8. Absolute error between exact and approximate solution (n=25, m=25) for example 6.3

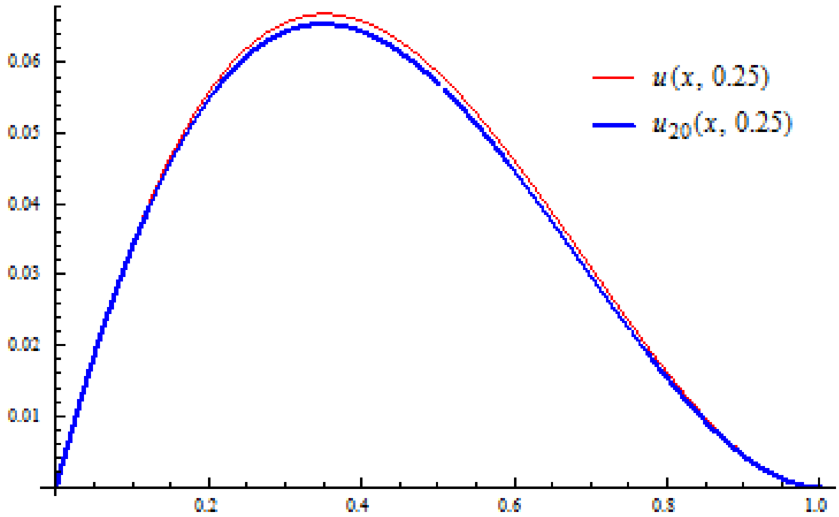

Figure 6.9. Comparison between the approximate solution and the exact solution for Example 6.3

7. Conclusion

In this paper, we applied the reproducing kernel Hilbert space method to solve the third order differential equation with the multiple characteristics. From this work, we can conclude that the absolute errors decrease monotonically if n increases. The results are accompanied by numerical examples indicating that the approximate solution converge to the exact solution. Therefore, the effectiveness of our proposed method is confirmed for this class of third order boundary value problems.

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