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CONSTRUCTION OF SOME ARCHIMEDEAN COPULAS

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ABSTRACT. We propose an approach allowing to manage the statistical data. At the same time, starting from the copulas of Gumbel,of Ali-Michael and Haq., we propose some copulas of the Archimedean family that we estimate contribute to the resolution of the problems related to the choice of the maximum or the optimal values for a group of random variables united within d random vectors of size n, independent and identically distributed, suitably normalized as n tends to infinity.

To our knowledge, there exists, in this case, the copula of Ali-Michael and Haq, Which goes in the same direction, however, the copula that we propose takes into account on the one hand the dependence parameters of the marginals and on the other hand the size of the samples including the dependence parameter of the joint distribution function

1. INTRODUCTION

The notion of copula took off in 1959, when l'Abé Sklar, wanted to solve the difficulty encountered by Maurice Fréchet, using it for the first time. In the theory of probability, when the random variables $X_1, \dots, X_n$ are independent, their joint

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distribution function is trivially calculated by simply taking the product of different marginal distribution functions $F_1, \dots, F_n$, i.e.,:

$$\forall (x_1, \dots, x_n) \in [0,1]^n, F(x_1, \cdots, x_n) = F_1(x_1) \dots F_n(x_n).$$

However, in practice, the random variables are not always independent such is often the case for the financial risks where the variables are related to each other. It is thus to circumvent this difficulty that l'Abé Sklar introduced the copula. He proposes a distribution function which, in reality, is nothing other than an entity which links the different margins of random variables.

Since then, the copulas have continued to make progress. They objectively make it possible to couple the marginal laws of random variables and to study their dependence, thus playing a fundamental role in the resolution of the problems inherent in the dependencies of random variables, particularly those encountered in the fields of finance, of hydrology, of biology. . . .

The management of statistical data is wides pread practice in everyday life. The use of copulas is one of the means which make it possible to analyze them, to put together in order to study their dependence.

Gumbel's copulas, on the one hand, and Ali-Michael and Haq.'s copulas, on the other hand, were certainly the triggers of our inspiration. It gives the ability to manage blocks of data and also highlights of dependencies between them,our contribution comes rather to reinforce them by taking into account the size chosen saample which turns out to be that of the optimal data, in fact,in our research adventure, we thought we were studying the macroeconomic data of a few countries in the CEMAC zone, the economic and monetary community of central Africa.

Having been confronted with the difficulty of finding a safe approach which uses the copulas and makes it possible to decide on the links of dependence of each country, on the one hand, and on the dependence between them, on the other hand, and especially by making the singular choice of the use the copulas, we thought to propose an approach to managing this data before moving on to a practical phase in a second article.

To do this, we will start from a function that uses the values associated with the maxima of these samples,each block constitutes a random vector.

In our study, a value will be said to be extreme, when it is quite far from the other values of the chosen population. We look for the maximum of each sample first before looking for the largest of them.

In our approach, the copula of Gumbel and that of Ali-Michael and Haq. are of great use to us. It is moreover to strengthen them that we were motivated.

In general, the notion of copula is widely used in the field of statistics, given its complexity and its scientific extent, it could be difficult if not impossible to study all the types of copulas in a single article as there would be so much to say. To attempt to do so would be to take the risk of writing a thesis.

Also, we have chosen to speak singularly of two types of copulas which, in reality, will lead us to clarify certain related sub-notions such as those of the archimedean copulas, within which, are Gumbel, Ali-Michael and Haq., copula's extreme values. There are indeed very close to reality, they have many advantages that can help us analyze certain macroeconomic data.

What could be more interesting! Because, in truth, the most important in this notion of copula is their remarkable contribution in all scientific fields.

2. Preliminaries

Definition 2.1. *Archimedean copulas are used in several applications due to their ease of handling. Any copula archimedean is characterized by the dependence on a generating function such that:*

$$C(u_1, \ldots, u_d) = (\phi^{-1})(\phi(u_1) + \ldots + \phi(u_{p)}))$$

if $\sum_{i=1}^{d} \phi(u_i) \leq \phi(0)$ *and* $C(u_1, \ldots, u_d) = 0$*. If not* ϕ^{-1} *is the inverse of the generator* ϕ*.*

Moreover, for ϕ *to be a generator of archimedean's copula, it must be a convex and strictly decreasing continuous function, in other words, it must be of class* C^2 *so that* $\phi(1) = 0, \phi^{'}(u) \leq 0$ *and* $\phi^{''}(u) > 0$*. Thus, for a copula to be said archimedean, it necessary and sufficient that, for* $d \geq 0$ *and* $\sum_{i=1}^{d} \phi(u_i) \leq \phi(0)$*,*

$$C(u_1, \ldots, u_d) = \phi^{-1}(\phi(u_1) + \ldots, +\phi(u_d))$$

and nowhere else. The random variables $u_i, 1 \leq i \leq n$ *can take different values between 0 and 1.*

There are actually several copulas of this type, but the best known and used are those of so-called Gumbel, de Clayton et de Frank.

In reality, what differentiates them from each other is their generator. In our study, we will be particulary interested in that of Gumbel.

Definition 2.2. *([3, 5]) The Gumbel's copula is that whose the generator is:* $\psi(t) = (-\ln(t))^{\theta}, \theta > 1)$*. Thus,* $C(u,v) = \exp\{-((-\ln u))^{\theta} + (-\ln(v))^{\theta})^{\frac{1}{\theta}}\}$*, the function thus defined satisfies the conditions of the archimedean copulas which justifies its belonging of this family. For a parameter* $\theta > 0$*, the mltivariate Gumbel copula is defined by:* $C(u_1,\ldots,u_d) = \exp\{-[\sum_{i=1}^{d}(-\ln(u_i)^{\theta}]^{\frac{1}{\theta}}\}$ *with* $d \geq 2$*. In reality, the Gumbel copula models a dependence of the upper end of a distribution.It is characterized by an asymmetry caused by greater dependence at the upper end and almost zero dependence at its lower end.*

Definition 2.3. *([3, 6, 7]) Copulas of extreme values. Copulas of this type are those which satisfy the relation:* $C(u_1^k, u_2^k) = C'^k(u_1,u_2)$ *for all positive k. To construct a distribution of two-dimensional extreme values, it suffices to couple the margins resulting from the law of the theory of extreme values with an extreme value copula.*

The gumbel copula is an example for an extreme value copula because:

$$\begin{aligned}
C(u_1^k, u_2^k) &= \exp(-[(-\ln u_1^k)^{\theta} + (-\ln u_2^k)^{\theta}]^{\frac{1}{\theta}} \\
&= \exp(-(k^{\theta}[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}])^{\frac{1}{\theta}}) \\
&= \exp(-k[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}]^{\frac{1}{\theta}}) \\
&= (\exp(-[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}])^{\frac{1}{\theta}})^{k} = C^k(u_1,u_2).
\end{aligned}$$

The following theorem, introduced by Abbé Sklar in 1959 and which bears his name, is one of the powerful tools of the theory of copulas, because it makes it possible to solve several problems related to the dependence of random variables

Theorem 2.1. *([7]) Let* $X_1,\ldots,X_n$*, n random variables whose respective distribution functions are:* $F_1(x_1),\ldots,F_n(x_n)$ *and let F be the distribute function of vector* $(X_1,\ldots,X_n)$*, there exists a function C that:* $F(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n))$ *and vice versa.*

If, moreover, the marginal functions are continuous then C is unique.

In the bivariate case, let H, be a distribution function jointed to random variables X, Y.

Let F and G be the marginal distribution functions linked respectivly to X and Y, then there exists a copula C such that: $\forall x, y \in \mathbb{R}$, $H(x,y) = C(F(x), G(y))$.

If F and G are continuous then C is unique.

Vice versa if C is a copula, F and G are marginal distribution functions. If $H(x,y) = C(F(x), G(y))$ *then H is a distribution function joint whose marginal distribution functions are F and G.*

Corollary 2.1. *([1, 2, 3, 4]) Let H be the joint distribution function of dimension d between the marginal distribution functions* $F_1, \dots, F_d, d \geq 2$ *and C the copula uniting them. Further, let* $F_1^{-1}, \dots, F_d^{-1}, d \geq 2$ *the generalized inverses of the marginal distribution functions, then be for* $u = (u_1, \dots, u_d) \in [0,1]^d$,

$$C(u) = H(F_1(x_1), \dots, F_d(x_d) = H(F_1^{-1}(u_1, \dots, F_d^{-1}(u_d).$$

Theorem 2.2. *([1, 2]) Let* M_n *be a random vector whose components are the random variables* X_i $(1 \leq i \leq n)$ *such that* $M_n = max(X_1, \dots, X_n)$. *Let* $a_n \geq 0$ *and* $b_n \in \mathbb{R}$ *two suitably chosen normalization sequences, if:* $lim_{n\uparrow+\infty} P(\frac{M_n - a_n}{b_n} \leq x) = H_\theta(x)$ *and* $lim F_n(a_n x + b_n) = H_\theta(x), \forall x \in \mathbb{R}, n \to \infty$. *Then* $H_\theta(x)$ *is a function induces an extreme-value copula.*

Remark 2.1. *It should be noted that* $H_\theta(x)$ *is a non-degenerate distribution function.*

So because $H_\theta(x)$ *is a distribution function necessarily belonging to the family of extreme values. This implies, on the other hand, the induction of an extreme value copula. And this necessarily belongs to the copula family archimedean because the copulas of extreme values are part of it.*

Generally H is characterized by three parameters i.e. $H_{\theta,\mu,\rho}(x) = \exp(-(1 + \theta\frac{x-\mu}{\rho})^{\frac{-1}{\theta}}$ *with* $\theta \neq 0, 1 + \theta\frac{x-\mu}{\rho} \geq 0, \mu \in \mathbb{R}, \rho \geq 0, \theta \in \mathbb{R}$. *These parameters* μ, ρ, θ *are respectively related to the position, to the scale and to the shape.*

The parameter which is that of the shape makes it possible to distinguish three types of laws relating to the extreme values. If $\theta > 0$, *we have the law of Fréchet. If* $\theta = 0$, *we have the law of Gumbel. If* $\theta < 0$, *we have the law of Weibull.*

Another formulation of this theorem is as follows.

Theorem 2.3. *(Fisher-Tippet,* 1928*) Let* $X_1, \dots, X_n$, *n random variables whose the respective distribution functions are:* $F_1(x_1), \dots, F_n(x_n)$ *and let F be the distribution function of the vector* $(X_1, \dots, X_n)$, *if it exist two sequences* a_n *and* b_n *such that* $\frac{M_n - a_n}{b_n}$

converges in distribution, then:

$$\lim_{n\to+\infty} P(\frac{M_n - a_n}{b_n} \leq x) = H_\theta(x),$$

$H_\theta(x)$ *takes the following laws:*

- *if* $\theta > 0$ *we have the law of Frechet, i.e.* $H_\theta(x) = \exp(-(x)^{-\theta})$ *if* $x \geq 0$ *and nowhere else;*
- *if* $\theta = 0$ *we have the law of Gumbel i.e.* $H_\theta(x) = \exp(-e^{-x})\forall x \in \mathbb{R}$*;*
- *if* $\theta < 0$ *we have the law of Weibull i.e.* $\exp(-(x)^\theta$ *if* $x \geq 0$ *and nowhere else.*

3. Proposal of copulas

Before arriving at the proposal of these various copulas, we think that it is necessary to propose an appoach which allows us to built the existing which served us as a basis of reflection. It is therefore a question of first constructing the copula of Gumbel and that of Ali-Michael and Haq.

In fact, the copulas that we propose will be a generalization of these two types of copulas cited.

They will take into account the size of the samples and even the dependence parameter which is not the case for the copula of and that of Ali- Michael and Haq.

Proposition 3.1. *It proposes an approach to construct the Gumbel's copula. Let(X,Y) be a pair of positive random variables, such that:* $H(x,y) = \frac{1}{1+e^{-\theta x}+e^{-\alpha y}}$ *for all* $x, y \in \mathbb{R}; \theta \geq 0, \alpha \geq 0$.

H as defined is absolutely continuous on $\mathbb{R}x\mathbb{R}$*, we can then speak of a copula.*

According to Sklar's Theorem 2.1, $H(x,y) = C(F(x), G(y))$ *où F and G are the respective margins of random variables X and Y.*

Vice versa, it exist the il existe the quantiles or reciprocal function F^{-1} *and* G^{-1} *such that:* $C(u,v) = H(F^{-1}(u), G^{-1}(v))$.

The marginal laws F and G associated with X and Y will be: $F(x) = \lim_{y\to+\infty} H_\theta(x,y) = \frac{1}{1+e^{-\theta x}}$ *and* $G(y) = \lim_{x\to+\infty} H_\theta(x,y) = \frac{1}{1+e^{-\alpha y}}$.

The reciprocal functions associated will be:

$$y = \frac{1}{1+e^{-\theta x}} \iff y(1+e^{-\theta x}) = 1 \iff e^{-\theta x} = \frac{1-y}{y} \iff x = -\frac{1}{\theta} log\frac{1-y}{y}.$$

Whence,

$$F^{-1}(u) = \log(\frac{u}{1-u})^{\frac{1}{\theta}} \quad \textit{and} \quad G^{-1}(v) = \log(\frac{v}{1-v})^{\frac{1}{\alpha}}.$$

Thus:

$$\begin{aligned} C(u,v) &= \frac{1}{1+e^{-\theta\log(\frac{u}{1-u})^{\frac{1}{\theta}}}+e^{-\alpha\log(\frac{v}{1-v})^{\frac{1}{\alpha}}}} = \frac{1}{1+e^{\theta\log(\frac{1-u}{u})^{\frac{1}{\theta}}}+e^{\alpha\log(\frac{1-v}{v})^{\frac{1}{\alpha}}}} \\ &= \frac{1}{1+e^{\log(\frac{1-u}{u})}+e^{\log(\frac{1-v}{v})}} = \frac{1}{1+\frac{1-u}{u}+\frac{1-v}{v}} = \frac{uv}{u+v-uv}. \end{aligned}$$

It's Gumbel's law (1961)

Proposition 3.2. *Another proposal of an approach to construct the Gumbel's copula.*

This approach consist to use the maximum values. Indeed, let $(X_1^1,\dots,X_n^1),\dots,(X_1^d,\dots,X_n^d)$ *be a sample of vectors of size n, made up of independent and identically distributed random variables. Let* $M_j = \max(X_1^j,\dots,X_n^j)$ *with* $1 \le j \le d$*.*

Our objective being to find the maximum of the random variables of these vectors, we assume that this maximum is reached at index i. We will then have:

$$M_i = \max(X_1^i,\dots,X_n^i).$$

Let F_i *the distribution function of* M_i*. We will have:*

$$\begin{aligned} F_i(x) &= P(M_i \le x) = P((X_1^i \le, x \dots, X_n^i \le x)) = P((X_1^i \le, x) \dots P(X_n^i \le x)) \\ &= [P((X_1^i \le, x)]^n = [F(x)]^n \end{aligned}$$

By normalizing M_i *with suitably chosen constant sequences, we will have:*

$$P(\frac{M_i - a_i}{b_i} \le x) = P(M_i \le b_i x + a_i) = [F(b_i x + a_i)]^n = F_i(b_i x + a_i).$$

Consider the function H: (x,y)$\longrightarrow \frac{1}{1+e^{-x}+e^{-y}}$*. Then* $F(x) = \lim_{n\to+\infty} H(x,y) = \frac{1}{1+e^{-x}}$*, which means that* $F(b_i x + a_i) = \frac{1}{1+e^{-b_i x - a_i}}$*. Since the limited development of* $\frac{1}{1+u} = 1 - u + \frac{u^2}{2} + \dots$ *when* $u \downarrow 0$ *then* $F(b_i x + a_i) = \frac{1}{1+e^{-(b_i x + a_i)}} = 1 - e^{-(b_i x + a_i)} = 1 - e^{-b_i x} e^{-a_i} = 1 - \frac{e^{-b_i x}}{e^{a_i}}$ *car* $e^{-(b_i x + a_i)} \longrightarrow 0$ *when* $x \longrightarrow +\infty$*. By choosing* $b_i = 1, a_i = \log n$ *we will have:*

$$P(\frac{M_i - a_i}{b_i} \le x) = [1 - \frac{e^{-x}}{e^{\log n}}]^n = [1 - \frac{e^{-x}}{n}]^n,$$

Then, finally, $\lim_{n\to+\infty} P(\frac{M_i - a_i}{b_i} \le x) = \lim \left[1 - \frac{e^{-x}}{n}\right]^n = \exp(-e^{-x})$ *because* $\lim[1 - \frac{a}{n}]^n = \exp(-a)$ *when* $n \to +\infty$*.*

Similarly, $\lim_{n\to+\infty} F_n(b_n x + a_n) = \lim_{n\to+\infty} F_n(x + \log n) = \lim(\frac{1}{1+e^{x+\log n}}) = \lim(1 - e^{-x-\log n}) = \lim(1 - \frac{e^{-x}}{n}) = \exp(-e^{-x})$.

According to the Theorem 2.2, there is a copula of the extreme values and this copula is that of Gumbel.

Proposition 3.3. *Proposal of a new copula: generalized Gumbel copula. According from the above, the observation is that* θ *and* α *does not influence the results, consider then that* H_n *is an application which* $(x,y) \longrightarrow [\frac{1}{1+e^{-x}+e^{-y}}]^n$. *Always according from the above, H is absolutely continuous so we can talk about a copula we can then write:*

$$H_n(x,y) = C_n(F_n(x), G_n(y)) \quad \textit{and} \quad C_n(u,v) = H_n(F_n^{-1}(u), G_n^{-1}(v)).$$

We determine F_n *and* G_n *and their reciprocal functions:* $F_n(x) = \lim\limits_{y\to+\infty} H_n(x,y) = (\dfrac{1}{1+e^{-x}})^n$ *and* $G_n(y) = \lim\limits_{x\to+\infty} H_n(x,y) = (\dfrac{1}{1+e^{-y}})^n$, $F_n^{-1}(u) = \log \frac{u^{\frac{1}{n}}}{1-u^{\frac{1}{n}}}$ *and* $G_n^{-1}(v) = \log \frac{v^{\frac{1}{n}}}{1-v^{\frac{1}{n}}}$. *Similarly:* $C_n(u,v) = H_n(F^{-1}(x),$

$$G^{-1}(y)) = \left[\frac{1}{1+e^{-\log\frac{u^{\frac{1}{n}}}{1-u^{\frac{1}{n}}}} + e^{-\log\frac{v^{\frac{1}{n}}}{1-v^{\frac{1}{n}}}}}\right]^n = \left[\frac{1}{1+\frac{1-u^{\frac{1}{n}}}{u^{\frac{1}{n}}} + \frac{1-v^{\frac{1}{n}}}{v^{\frac{1}{n}}}}\right]^n$$
$$= \left[\frac{(uv)^{\frac{1}{n}}}{u^{\frac{1}{n}} + v^{\frac{1}{n}} - (uv)^{\frac{1}{n}}}\right]^n.$$

Then finally: $C_n(u,v) = \frac{uv}{[u^{\frac{1}{n}}+v^{\frac{1}{n}}-(uv)^{\frac{1}{n}}]^n}$. *This expression represents the generalized Gumbel's law since it takes into account the size of the sample. For proof, if* $n = 1$ *we find the usual Gumbel's law as established by the German researcher Gumbel in 1961.*

In this momentum, we will build Ali-Michael and Haq.'s law before proposing his generalized law.

Proposition 3.4. *Proposal of an approach to construct Ali-Michael et Haq.'s law. Let (X,Y) be a pair of positive random variables, such that:* $H_\theta(x,y) = \frac{1}{1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}}$ *for all* $x, y \in \mathbb{R}, 1-\theta > 0$ *and then* $\theta < 1$. *H is such that definited is absolutely continuous on* $\mathbb{R}x\mathbb{R}$, *we can then speak of a copula.*

According Sklar's theorem, $H(x,y) = C(F_n(x), G_n(y))$, *knowing that* F_n *and* G_n *are the respective marginal laws of random variables X and Y.*

Vice versa, it exist the reciprocal functions F_n^{-1} *et* G_n^{-1} *such that:* $C(u,v) = H(F_n^{-1}(u), G_n^{-1}(v))$. *The marginal laws F and G associated with X and Y will be:* $F(x) = \lim_{y\to+\infty} H_\theta(x,y) = \dfrac{1}{1+e^{-x}}$ *and* $G(y) = \lim_{x\to+\infty} H_\theta(x,y) = \frac{1}{1+e^{-y}}$
The reciprocal functions associated will be:

$$y = \frac{1}{1+e^{-x}} \iff y(1+e^{-x} = 1 \iff e^{-x} = \frac{1-y}{y} \iff x = \log\frac{y}{1-y},$$

from where: $F^{-1}(u) = \log\frac{u}{1-u}$ *and* $G^{-1}(v) = \log\frac{v}{1-v}$. *Thus:*

$$\begin{aligned} C(u,v) &= \frac{1}{1+e^{-\log\frac{u}{1-u}} + e^{-\log\frac{v}{1-v}} + (1-\theta)e^{-\log\frac{u}{1-u}-\log\frac{v}{1-v}}} \\ &= \frac{1}{1+e^{\log\frac{1-u}{u}} + e^{\log(\frac{1-v}{v}} + (1-\theta)e^{\log\frac{1-u}{u}+\log\frac{1-v}{v}}} \\ &= \frac{1}{1+\frac{1-u}{u}+\frac{1-v}{v}) + (1-\theta)(\frac{1-u}{u}\frac{1-v}{v}} = \frac{uv}{1+\theta(u+v-uv-1)}. \end{aligned}$$

This is the law of Ali-Michael et Haq.

If $\theta = 1$*, we obtain Gumbel's law, and if* $\theta = 0, C(u,v) = uv$ *the independent law.*

Proposition 3.5. *Proposal of a new law: generalized Ali-Michael-Haq. Consider the map which to every* H_n*, à tout* $(x,y) \longrightarrow [\frac{1}{1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}}]^n$*, so:* $H_n(x,y) = [\frac{1}{1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}}]^n$ *for all* $x, y \in \mathbb{R}, 1-\theta > 0$ *and therefore* $\theta < 1$. *According to the above,* H_n *is absolutely continuous so we can speak of a copula. We can then write:* $H_n(x,y) = C_n(F(x), G(y))$ *and* $C_n(u,v) = H_n(F^{-1}(u), G^{-1}(v))$.

Let's determine F_n *and* G_n *and their reciprocal functions:* $F_n(x) = \lim_{y\to+\infty} H_n(x,y) = (\dfrac{1}{1+e^{-x}})^n$ *and* $G_n(x) = \lim_{x\to+\infty} H_n(x,y) = (\frac{1}{1+e^{-y}})^n$, $F_n^{-1}(u) = \log\frac{u^{\frac{1}{n}}}{1-u^{\frac{1}{n}}}$ *and* $G_n^{-1}(v) = \log\frac{v^{\frac{1}{n}}}{1-v^{\frac{1}{n}}}$*, from which:*

$$\begin{aligned} C_n(u,v) &= H_n(F^{-1}(x), G^{-1}(y)) \\ &= \left[\frac{1}{1+e^{\log\frac{1-u^{\frac{1}{n}}}{u^{\frac{1}{n}}}} + e^{\log\frac{1-v^{\frac{1}{n}}}{v^{\frac{1}{n}}}} + (1-\theta)e^{\log\frac{1-u^{\frac{1}{n}}}{u^{\frac{1}{n}}}+\log\frac{1-v^{\frac{1}{n}}}{v^{\frac{1}{n}}}}}\right]^n \\ &= \left[\frac{1}{1+\frac{1-u^{\frac{1}{n}}}{u^{\frac{1}{n}}}+\frac{1-v^{\frac{1}{n}}}{v^{\frac{1}{n}}}+(1-\theta)\frac{1-u^{\frac{1}{n}}}{u^{\frac{1}{n}}}\frac{1-v^{\frac{1}{n}}}{v^{\frac{1}{n}}}}\right]^n \end{aligned}$$

$$= \left[\frac{(uv)^{\frac{1}{n}}}{(uv)^{\frac{1}{n}} + (1-u^{\frac{1}{n}})v^{\frac{1}{n}} + (1-v^{\frac{1}{n}})u^{\frac{1}{n}} + (1-\theta)(1-u^{\frac{1}{n}})(1-v^{\frac{1}{n}})} \right]^n .$$

Then finally: $C_n(u,v) = \frac{uv}{[1+\theta(u^{\frac{1}{n}}+v^{\frac{1}{n}}-(uv)^{\frac{1}{n}}-1)]^n}$.

This expression generalizes the Ali-Michael and Haq.'s law because if $n = 1$*, we find it. However, if we do simultaneously* $n = 1$ *and* $\theta = 1$*, we find Gumbel's law. It is moreover part of the archimedean family with the generating function* $\psi_\theta(t) = \log \frac{1-\theta(1-t)}{t}$.

Proposition 3.6. *Construction of copulas from maximal values. If X and Y are two random vectors which made up of the random variables and for which we set ourselves the objective of finding the maximum or optimal component.*

For the choice of the maximum of the first vector, let's consider $(X_1^1, \ldots, X_n^1), \ldots, (X_1^d, \ldots, X_n^d)$ *samples of random vectors consisting of n independent and identically distributed random variables, with distribution marginal F. Let* $M_j = \max(X_1^j, \ldots, X_n^j)$ *be the maximum of j-th vector or block.*

Let's consider that the maximum of all the maxima is taken on the i-th component, we will have $M_i = \max(X_1^j, \ldots, X_n^j) = (X_1^i, \ldots, X_n^i)$ *with* $1 \le j \le d$.

Moreover, $M_i = (X_1^i, \ldots, X_n^i)$ *being a maximum value, the copula associated with it will be given by* $(u_1, \ldots, u_n) \to C^i(u_1^{\frac{1}{i}}, \ldots, u_n^{\frac{1}{i}})$*. Moreover, according the theorem (2.2) or (2.3), if it admits a limit when* $i \uparrow +\infty$ *then this limit will be an extreme value copula. Moreover, we know that the class of extreme value copulas coincides with that of max-stable copulas i.e. those which obey the next property:*

$$C^n(u_1^{\frac{1}{n}}, \ldots, u_d^{\frac{1}{n}}) = C(u_1, \ldots, u_d); \forall n \ge 1 \text{ and } (u_1, \ldots, u_d) \in [0,1]^d.$$

In this case, the normalization of M_i *leads to the vector* $(\frac{M_1^i - a_1^i}{b_1^i}, \ldots, \frac{M_n^i - a_n^i}{b_n^i})$ *where*a_j^i *and* b_j^i *are suitably chosen normalization sequences with* $1 \le j \le n; 1 \le i \le d$*. Roughly, if* $(X_1, \ldots, X_n)$ *is a sequence of n independent and identically distributed and distribution function* $F(x) = P(X_j \le x); 1 \le j \le n$.

The study of the behavior of extreme events is done by considering the maximum of each of the d random vectors considered.

Remark 3.1. *In practice, these random variables can be chosen as a block of values taken by a random variable over the course of a year, a quarter,etc. The set of values in a block constitutes a random vector.*

Proposition 3.7. *Moreover, the distribution function of $M_i = (X_1^i, \dots X_n^i)$ will therefore be $F_i(x) = P(M_i \leq x) = P(X_1^i \leq x, \dots, X_n^i \leq x) = P(X_1^i \leq x).P(X_2^i \leq x)\dots P(X_n^i \leq x) = [F(x)]^n$ due to the fact that the random variables are independent and identically distributed and have the same law F(x).*

In reality, $F_i(x)$ represents a kind of joint law linked to the different marginal laws constituted by the random variables $X_j^i; 1 \leq j \leq n; 1 \leq i \leq d$.

Usually F(x) is unknown. Let's look for it. As it is done for the central-limit theorem, we will observe the limit behavior of the random vector M_i. From the outset, we know that the fact of being a maximum value, the copula which will be associated will be a function such as: $(u_1, \dots, u_n) \to C^i(u_1^{\frac{1}{i}}, \dots, u_n^{\frac{1}{i}})$.

Let's prove that this copula of extreme values really exists.

Let H: (x,y)$\longrightarrow \frac{1}{1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}}$ be the function joined to the margins F and G. It follows that $F(x) = \lim_{y\to+\infty} H(x,y) = \frac{1}{1+e^{-x}}$ and $G(y) = \lim_{x\to+\infty} H(x,y) = \frac{1}{1+e^{-y}}$. So $F_i(x) = [\frac{1}{1+e^{-x}}]^n$. We also know that the limited development of $\frac{1}{1+u}$, when $u \longmapsto 0$, is: $1-u+\frac{u^2}{2}+\dots$. We can therefore write that $\frac{1}{1+e^{-x}} = 1-e^{-x}$, car $e^{-x} \longmapsto 0$ lorsque $x \longmapsto +\infty$. Thus, we can say that $\frac{1}{1+e^{-x-\log n}} = 1 - e^{-x-\log n} = 1 - \frac{e^{-x}}{e^{\log n}} = 1 - \frac{e^{-x}}{n}$. On the other hand, we know that $(1-\frac{a}{n})^n \longmapsto \exp(-a)$ when $n \longmapsto +\infty(7.1)$. So, $\lim(1-\frac{e^{-x}}{n})^n = \exp(-e^{-x})$ when $n \to +\infty$. Afterwards, according to (7.1), $(1-\frac{e^{-\theta x}}{n})^n$ tends to $\exp(-e^{-\theta x})$ when $n \to +\infty$. Since $F_i(x) = P(M_i \leq x)$ normalizing it using sequences suitably chosen, we will have:

$$
\begin{aligned}
P(\frac{M_i - a_i}{b_i} \leq x) &= P(M_i - a_i \leq b_i x) = P(M_i \leq b_i x + a_i) \\
&= P(X_1^i \leq b_i x + a_i, \dots, X_n^i \leq b_i x + a_i) \\
&= [F(b_i x + a_i)]^n = F_i(b_i x + a_i).
\end{aligned}
$$

We can also say that $F_\theta(x)$ is the distribution function of the limit of the distribution function of M_i suitably normalized by a_i and b_i such that $limF_i(a_i x + b_i) = limF_n(a_n x + b_n) = F_\theta(x); \forall x \in \mathbb{R}$ for any point of continuity $x \in [0,1]$.

In this specific case, $\theta = 1$, according the Theorem 2.2 or 2.3, there exist this extreme values copula. It is precisely from Gumbel because, knowing that it is defined in the following way:

$$C(u,v) = \exp\{-((-\ln u))^\theta + (-\ln(v))^\theta)^{\frac{1}{\theta}}\}.$$

If we set $\psi^{-1}(t) = \exp(-e^{-\theta t)})$ *we will have*

$$y = \exp(-e^{-\theta x}) \Longleftrightarrow \log y = -e^{-\theta x} \Longrightarrow \ln(-\log y) = -\theta x$$

$$\Longrightarrow x = -\frac{1}{\theta}\ln(-\log y) \Longrightarrow \psi(t) = -[\ln(-\log t)]^{\frac{1}{\theta}}.$$

This expression is from a Gumbel copula generator.

Our objective being to propose a copula of managing statistical data from two or more different sources, we will consider another block of data in order to proceed with the construction of said copula which can make it possible to study the dependencies between these blocks.

Let $(Y_1^1, \ldots, Y_n^1), \ldots, (Y_1^n, \ldots, Y_n^d)$ *d samples of vectors from another block of n independent and identically distributed random variables. By doing so, in leads to a distribution function* $G(y) = \frac{1}{1+e^{-\alpha . y}}; \alpha > 0$. *With the two blocks of data, we can then propose to find the copula which thinks their data namely* $(X_1^1, \ldots, X_n^1), \ldots, (X_1^d, \ldots, X_n^d)$ *and* $(Y_1^1, \ldots, Y_n^1), \ldots, (Y_1^d, \ldots, Y_n^d)$. *Let* C *be the copula associated with the marginal functions F and G, we know that if H is the function joined to these marginals, according Sklar,*$H(x, y) = C(F(x), G(y)) = P(X \leq x, Y \leq y)$. *If moreover, the margins are continuous then C is unique. The reciprocal will be:* $C(u, v) = H(F^{-1}(u), G^{-1}(v))$.

Let's determine the inverse functions $F^{-1}(u)$ *and* $G^{-1}(v)$. *Since*

$$y = F(x) = \frac{1}{1+e^{-\theta . x}} \Longleftrightarrow y(1+e^{-\theta . x}) = 1 \Longleftrightarrow e^{-\theta . x} = \frac{1-y}{y}$$

$$\Longrightarrow x = (\theta)^{-1}\log\frac{y}{1-y},$$

$F^{-1}(u) = \frac{1}{\theta}\log\frac{u}{1-u}$ *and* $G^{-1}(v) = \frac{1}{\alpha}\log\frac{v}{1-v}$. *Since we have considered that H(x,y), the joint distribution function of the marginal F(x) and G(y), is given by*

$$\frac{1}{1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}},$$

$\theta \geq 0$, *let us show that it is indeed a distribution function.*

From the outset, H is positive because it is made up of positive terms. Moreover, the quantity $e^{-x}+e^{-y}+(1-\theta)e^{-x-y}$ *tends to 0 as x and y tend to infinity: which means that* $H \in [0,1] \forall x, y \in \mathbb{R}$.

Let us show that it is 2-increasing. We can clearly state that H is absolutely continuous. It remains to show that it admits a density of probability and that this one is *differentiable in its turn*

$$\frac{\partial H_\theta(x,y)}{\partial x} = -\frac{-e^{-x}-(1-\theta)e^{-x-y}}{[1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}]^2} = \frac{e^{-x}+(1-\theta)e^{-x-y}}{[1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}]^2}.$$

In the same way, the partial derivative in y will be:

$$\frac{\partial H_\theta(x,y)}{\partial y} = \frac{e^{-y}+(1-\theta)e^{-x-y}}{[1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}]^2}.$$

Then the derivative with the two variables x and y will be:

$$\begin{aligned}
&\frac{\partial^2 H_\theta(x,y)}{\partial y\partial x}\\
&= \Big\{ -(1-\theta)e^{-x-y}[1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}]^2 + 2[e^{-y}+(1-\theta)e^{-x-y}]\\
&\quad [e^{-y}+(1-\theta)e^{-x-y}][1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}]\Big\}\\
&\quad \times \Big\{[1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}]^4\Big\}^{-1}\\
&= \Big\{[1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}][-(1-\theta)e^{-x-y}[1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}]\\
&\quad + 2[e^{-y}+(1-\theta)e^{-x-y}][e^{-y}+(1-\theta)e^{-x-y}]\Big\}\\
&\quad \times \Big\{[1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}]^4\Big\}^{-1}.
\end{aligned}$$

The sign of this derivate does not depend on the numerator, but all the terms are positive so: $\frac{\partial^2 H(x,y)}{\partial y \partial x} \geq 0; \forall x, y \in \mathbb{R}$. *Therefore, H(x,y) is indeed a distribution function joined to the margins F(x) and G(y), because it responds to all the previous approach allowing them to be constructed.*

Consequently, it is also a distribution function of the family of extreme values and therefore of the archimedean family.

We can determine this copula:

$$C_{ik}(u,v) = H_{ik}(F^{-1}(u), G^{-1}(v)),$$

$$C_{ik}(u,v) = \frac{1}{1+e^{-F^{-1}(u)}+e^{-G^{-1}(v)}+(1-\theta)e^{-F^{-1}(u)-G^{-1}(v)}}.$$

We have already established that F(x)=lim $H(x,y)$ *when* $y \to +\infty$ *and G(x)=lim* $H(x,y)$ *when* $x \to +\infty$. *From which* $F(x) = \frac{1}{1+e^{-x}}$ *and* $G(y) = \frac{1}{1+e^{-y}}$. *The reciprocal functions of these margins are defined by:*

$$F^{-1}(u)=\log\tfrac{u}{1-u} \text{ and } G^{-1}(v)=\log\tfrac{v}{1-v},$$

$$\begin{aligned}
C_{ik}(u,v) &= \left[\frac{1}{1+e^{-F^{-1}(u)}+e^{-G^{-1}(v)}+(1-\theta)e^{-F^{-1}(u)-G^{-1}(v)}}\right]^n\\
&= \left[\frac{1}{1+e^{-\log\frac{u}{1-u}}+e^{-\log\frac{v}{1-v}}+(1-\theta)e^{-\log\frac{u}{1-u}-\log\frac{v}{1-v}}}\right]^n\\
&= \left|\frac{1}{1+\frac{1-u}{u}+\frac{1-v}{v}+(1-\theta)\frac{1-u}{u}\frac{1-v}{v}}\right]^n\\
&= \left[\frac{uv}{uv+v(1-u)+u(1-v)+(1-\theta)(1-u)(1-v)}\right]^n\\
&= \left[\frac{uv}{1-\theta(1-u-v+uv)}\right]^n.
\end{aligned}$$

This expression is also that of generalized Ali-Michael and Haq.

If $n=1$ we find his usual expression. If $\theta=1$ we find generalized Gumbel. If $n=1$ and $\theta=1$ we have a usual expression of Gumbel.

It clearly appears that this last expression has a wide spectrum, because we find the different laws defined above with the particularity of managing the statistical data by taking into account the parameters linked to the position of the random variables.

Remark 3.2. *We can obtain a case of independence by considering that H(x,y) is given by $\frac{1}{1+e^{-x}+e^{-y}}$. With the same approach, we will have:*

$$C(u,v)=H(F^{-1}(u),G^{-1}(v)),$$

and further

$$\begin{aligned}
C(u,v) &= \frac{1}{1+e^{-F^{-1}(u)}+e^{-G^{-1}(v)}} = \frac{1}{1+e^{-\log\frac{u}{1-u}}+e^{-\log\frac{v}{1-v}}}\\
&= \frac{uv}{uv+v(1-u)+u(1-v)+(1-u)(1-v)}\\
&= \frac{uv}{uv+v-uv+u-uv+1-v-u+uv}=uv.
\end{aligned}$$

Proposition 3.8. *Proposition of an Ali-Michael and Haq. copula which takes into account the sample size. This one obviously takes into account the size of the chosen sample. Indeed, we have established that for p samples of size n and if F_i is the* *distribution function of* M_i*, the maximum vector over all samples, we have:*

$$\begin{aligned} F_i(x) &= P(M_i \leq x) = P(X_1^i \leq x, \ldots, X_n^i \leq x) \\ &= P(X_1^i \leq x).P(X_2^i \leq x) \ldots P(X_n^i \leq x) = [F(x)]^n. \end{aligned}$$

In an analogous way, we can establish that G_k *is the distribution function of* N_k *where k is the supposed index that makes* $N_j = \max(Y_1^j, \ldots, Y_n^j)$ *maximal. Thus* $G_k(y) = [G(y)]^n$*. Consider that the joint distribution function H(x,y) is a logistic function of Ali-Michael and Haq., i.e.:*

$$H(x, y) = \frac{1}{1 + e^{-x} + e^{-y} + (1 - \theta)e^{-x-y}}.$$

Thus F(x) and G (y), associated marginal functions will be logistic functions of Gumbel form. Let H_{ik} *be the joint function of* F_i *and* G_k *we will have:*

$$\begin{aligned} &H_{ik} = P(M_i \leq x, N_k \leq y \\ &= P(X_1^i \leq x)\, P(X_2^i \leq x)\ \ldots\ P(X_n^i \leq x)\, P(Y_1^k \leq x)\, P(X_2^k \leq x)\ \ldots\ P(X_n^k \leq x) \\ &= [F(x)]^n [G(y)]^n = [F(x)G(y)]^n = [H(x, y]^n, \end{aligned}$$

because X and Y are independent. From here:

$$H_{ik}(x, y) = \left(\frac{1}{1 + e^{-x} + e^{-y} + (1 - \theta)e^{-x-y}} \right)^n.$$

On the other hand, according Sklar $H_{ik}(x, y) = C_{ik}(F(x), G(y))$ *and vice versa:*

$$\begin{aligned} C_{ik}(u, v) &= H_{ik}(F^{-1}(u), G^{-1}(v)) \\ &= \left(\frac{1}{1 + e^{-F^{-1}(u)} + e^{-G^{-1}(v)} + (1 - \theta)e^{-F^{-1}(u) - G^{-1}(v)}} \right)^n \\ &= \left[\frac{1}{1 + e^{\log \frac{1-u}{u}} + e^{\log \frac{1-v}{v}} + (1 - \theta)e^{\log \frac{1-u}{u}} + \log \frac{1-v}{v}} \right]^n \\ &= \left[\frac{1}{1 + \frac{1-u}{u} + \frac{1-v}{v} + (1 - \theta)\frac{1-u}{u}\frac{1-v}{v}} \right]^n \\ &= \left[\frac{uv}{uv + 1 - uv + 1 - vu + (1 - \theta)(1 - u)(1 - v)} \right]^n \\ &= \left[\frac{uv}{uv + v - uv + u - uv + (1 - \theta)(1 - v - u + uv)} \right]^n. \end{aligned}$$

Then finally:

$$C_{ik}(u,v) = \left[\frac{uv}{1-\theta+\theta(u+v)+(1-\theta)uv}\right]^n.$$

This expression takes into account the size of the samples which turns out to be same. Subsequently, we will examine the case where the sizes are different.

Proposition 3.9. *Cases where the samples are not of the same size. Let i be the size of the first sample and k the size of the second sample, knowing that,* $F(x) = \frac{1}{1+e^{-x}}$ *and* $G(y) = \frac{1}{1+e^{-y}}$*, we can write that:* $F_i(x) = (\frac{1}{1+e^{-x}})^i$ *and* $G_k(y) = (\frac{1}{1+e^{-y}})^k$*. We can then determine their reciprocal functions. They will be:*

$$y^{\frac{1}{i}} = \frac{1}{1+e^{-x}} \Longleftrightarrow y^{\frac{1}{i}} + y^{\frac{1}{i}}e^{-x} = 1 \Longleftrightarrow y^{\frac{1}{i}}e^{-x} = 1 - y^{\frac{1}{i}} \Longrightarrow x = \log\frac{u^{\frac{1}{i}}}{1-u^{\frac{1}{i}}}.$$

From where: $F_i^{-1}(u) = \log\frac{u^{\frac{1}{i}}}{1-u^{\frac{1}{i}}}$ *and* $G_k^{-1}(v) = \log\frac{v^{\frac{1}{k}}}{1-v^{\frac{1}{k}}}$*. Then, according to Sklar,* $H_{ik}(x,y) = C_{ik}(F_i(x), G_k(y))$*, from which,* $C_{ik}(u,v) = H_{ik}(F_i^{-1}(u), G_k^{-1}(v))$*, which gives*

$$\begin{aligned}
C_{ik}(u,v) &= \frac{1}{1+e^{\log\frac{1-u^{\frac{1}{i}}}{u^{\frac{1}{i}}}} + e^{\log\frac{1-v^{\frac{1}{k}}}{v^{\frac{1}{k}}}} + (1-\theta)e^{\log\frac{1-u^{\frac{1}{i}}}{u^{\frac{1}{i}}}+\log\frac{1-v^{\frac{1}{k}}}{v^{\frac{1}{k}}}}} \\
&= \frac{1}{1+e^{\log\frac{1-u^{\frac{1}{i}}}{u^{\frac{1}{i}}}} + e^{\log\frac{1-v^{\frac{1}{k}}}{v^{\frac{1}{k}}}} + (1-\theta)e^{\log\frac{1-u^{\frac{1}{i}}}{u^{\frac{1}{i}}}\frac{1-v^{\frac{1}{k}}}{v^{\frac{1}{k}}}}} \\
&= \frac{1}{1+\frac{1-u^{\frac{1}{i}}}{u^{\frac{1}{i}}} + \frac{1-v^{\frac{1}{k}}}{v^{\frac{1}{k}}} + (1-\theta)\frac{1-u^{\frac{1}{i}}}{u^{\frac{1}{i}}}\frac{1-v^{\frac{1}{k}}}{v^{\frac{1}{k}}}} \\
&= \frac{u^{\frac{1}{i}}v^{\frac{1}{k}}}{1+v^{\frac{1}{k}}(1-u^{\frac{1}{i}}) + u^{\frac{1}{i}}(1-v^{\frac{1}{k}}) + (1-\theta)(1-v^{\frac{1}{k}})(1-u^{\frac{1}{i}})} \\
&= \frac{u^{\frac{1}{i}}v^{\frac{1}{k}}}{1+v^{\frac{1}{k}} + u^{\frac{1}{i}} - 2u^{\frac{1}{i}}v^{\frac{1}{k}} + (1-\theta)(1-v^{\frac{1}{k}})(1-u^{\frac{1}{i}})} \\
&= \frac{u^{\frac{1}{i}}v^{\frac{1}{k}}}{2-\theta+v^{\frac{1}{k}} + u^{\frac{1}{i}} - 2u^{\frac{1}{i}}v^{\frac{1}{k}} + (\theta-1)v^{\frac{1}{k}}) + (\theta-1)u^{\frac{1}{i}}) + (1-\theta)u^{\frac{1}{i}}v^{\frac{1}{k}}} \\
&= \frac{u^{\frac{1}{i}}v^{\frac{1}{k}}}{2-\theta+\theta(v^{\frac{1}{k}} + u^{\frac{1}{i}}) - (\theta+1)u^{\frac{1}{i}}v^{\frac{1}{k}}}.
\end{aligned}$$

More clearly:

$$C_{ik}(u,v) = \frac{u^{\frac{1}{i}}v^{\frac{1}{k}}}{2-\theta+\theta(v^{\frac{1}{k}}+u^{\frac{1}{i}})-(\theta+1)u^{\frac{1}{i}}v^{\frac{1}{k}}}.$$

Thus is defined this other copula which takes into account the size of the samples and even the dependence parameter present in the expression of Ali-Michael and Haq. From our point of view, this is a significant advantage,

4. Conclusion

We have thus proposed approaches that allow us to manage the random vectors as well as the different random variables that constitute them: this facilitates the management of statistical data. These can be macroeconomic, climatic, hydrological. . . .

Indeed, the measurement of two or more random variables is a widespread practice in statistics, but it is often discouraged by its complexity of data management. These different approaches that have led to the construction of some copulas that we believe innovative can bring a plus to the world of statistics.

References

[1] C. Fontaine: *Utilisation de copules paramétriques en présence des données observatoires: cadre théorique et modélisations,* Université Montpellier, 2016.

[2] G. Mazo: *Construction et estimation de copules en grande dimension* , Université de Grenoble, 2014.

[3] S. Loisel: *Copule de Gumbel,* 2007-2008.

[4] M.H. Toupin: *Nouveau test d'adéquation pour les copules basé sur le processus de spearman* , Janvier 2008.

[5] V.Y.B. Loyara: *Modélisation des risques de portefeuille par l'approche des copules multivariées* , Université de Ouaga 1 Pr Kizerbo, 2019.

[6] N. Kadi: *Estimation non-paramétrique de la distribution et densité de copules*, Université de Sherbooke, Quebec, Canada, avril 2014.

[7] Silimani: *Estimation des paramètres d'une copule*, Université Alger, 2011-2012.

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