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ON THE BLOCKING FORCE OF STEADY-STATE FLOW OF HERSCHEL-BULKLEY FLUID

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ABSTRACT. The paper is devoted to the study of the blocking force of Herschel-Bulkley fluid in the case of steady-state flow. To this aim, we consider a mathematical model which describes the steady-state flow of a Herschel-Bulkley fluid in a bounded domain. We give the mathematical formulation of the blockage phenomenon and we establish the existence of blocking force. We also focus on behaviour of the flow with respect to the blocking force.

1. INTRODUCTION

The Herschel-Bulkley fluid is the most generalized model describing the behavior of Non-Newtonian viscoplastic fluids, this incompressible fluid has been studied and used by mathematicians, physicists and engineers. While this model describes adequately a large class of flows. It has been used to model the flow of metals, plastic solids and a variety of polymers. Due to existence of yield limit, the model can capture phenomena connected with the development of discontinuous stresses. Physical experiments and numerical studies of the flow of Herschel-Bulkley fluids prove that when the yield stress increases, the rigid zones become larger and may completely block the flow. This property is called the blocking phenomenon. The literature concerning this topic is extensive; see e.g. [5–7, 9–12].

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Our paper deals with the steady-state flow of Herschel Bulkley. The main object is the study the behaviour of the flow. We provide a generalization of a result obtained by Mihai, Patrick et al. [14] for Bingham fluid to the steady-state flow of Herschel-Bulkley model, An important property of the Herschel-Bulkley model concerns the existence of rigid zones which are located in the interior of the flow. As the external loads decrease the rigid zones become larger and may completely block the flow if the forces become lower than a certain value which stands for a maximal blocking force.

The paper is organized as follows. In Section 2 we present the mechanical problem of the steady-state flow of Herschel-Bulkley fluid in a bounded domain $\Omega \subset \mathbb{R}^n$. We introduce some notations and preliminaries. In addition, we derive the variational formulation of the problem. In Section 3, we show the mathematical formulation of blockage phenomenon and we prove the existence of blocking force. Section 4 is devoted to study of the behaviour of the flow with respect to the blocking force.

2. Problem Statement

We consider a mathematical problem modelling the steady-state flow of the rigid viscoplastic and incompressible Herschel-Bulkley fluid in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2,\ 3)$, with the boundary Γ of class C^1. The fluid is acted upon by given volume forces of density $\mathbf{f}$. On Γ we suppose that the velocity is equal to zero.

We denote by S_n the space of symmetric tensors on $\mathbb{R}^n$. We define the inner product and the Euclidean norm on $\mathbb{R}^n$ and S_n, respectively, by

$$\mathbf{u}\cdot\mathbf{v} = u_i v_i \quad \forall\, \mathbf{u},\ \mathbf{v} \in \mathbb{R}^n \ \text{ and } \ \sigma\cdot\tau = \sigma_{ij}\tau_{ij} \quad \forall\, \sigma,\ \tau \in S_n.$$

$$|\mathbf{u}| = (\mathbf{u}\cdot\mathbf{u})^{\frac{1}{2}} \quad \forall \mathbf{u} \in \mathbb{R}^n \ \text{ and } \ |\sigma| = (\sigma\cdot\sigma)^{\frac{1}{2}} \quad \forall \sigma \in S_n.$$

Here and below, the indices i and j run from 1 to n and the summation convention over repeated indices is used. We denote by σ^D the deviator of $\sigma = (\sigma_{ij})$ given by

$$\sigma^D = \left(\sigma_{ij}^D\right), \quad \sigma_{ij}^D = \sigma_{ij} - \frac{\sigma_{kk}}{n}\delta_{ij},$$

where $\delta = (\delta_{ij})$ denotes the identity tensor.

Let $1 < p \leq 2$. We consider the rate of deformation operator defined for every $\mathbf{u} \in W^{1,p}(\Omega)^n$ by

$$D(\mathbf{u}) = (D_{ij}(\mathbf{u})), \quad D_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

The steady-state flow of Herschel-Bulkley fluid can be described the following mechanical problem.

Problem P1. Find the velocity field $\mathbf{u} = (u_i) : \Omega \longrightarrow \mathbb{R}^n$ and the stress field $\sigma = (\sigma_{ij}) : \Omega \longrightarrow S_n$ such that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \operatorname{div}\sigma + \mathbf{f} \text{ in } \Omega. \tag{2.1}$$

$$\left.\begin{array}{ll} \sigma^D = \mu |D(\mathbf{u})|^{p-2} D(\mathbf{u}) + g\dfrac{D(\mathbf{u})}{|D(\mathbf{u})|} & \text{if } |D(\mathbf{u})| \neq 0 \\ |\sigma^D| \leq g & \text{if } |D(\mathbf{u})| = 0 \end{array}\right\} \text{ in } \Omega. \tag{2.2}$$

$$\operatorname{div}\mathbf{u} = 0 \text{ in } \Omega. \tag{2.3}$$

$$\mathbf{u} = 0 \text{ on } \Gamma. \tag{2.4}$$

Here $\operatorname{div}\sigma = (\sigma_{ij,j})$ and $\operatorname{div}\mathbf{u} = u_{i,i}$. The flow is given by the equation (2.1) where the density is assumed equal to one. Equation (2.2) represents the constitutive law of Herschel-Bulkley fluid where $\mu > 0$ and $g \geq 0$ represent respectively the consistency and yield limit of the fluid, $1 < p \leq 2$ is the power law index. (2.3) represents the incompressibility condition. (2.4) gives the adherence condition on the boundary Γ.

Remark 2.1.

1. The Bingham fluid represents a particular case of Herschel-Bulkley fluid corresponding to $p = 2$.

2. In the constitutive law of Herschel-Bulkley fluid (2.2)*, the viscosity and hydrostatic pressure are given, respectively, by*

$$\eta = \mu |D(\mathbf{u})|^{p-2} \textit{ and } \pi = -\frac{1}{n}\sigma_{kk}. \tag{2.5}$$

Let us introduce the function spaces

$$W_{p,\text{div}} = \left\{\mathbf{v} \in W_0^{1,p}(\Omega)^n : \operatorname{div}(\mathbf{v}) = 0 \text{ in } \Omega\right\}, \tag{2.6}$$

$W_{p,\text{div}}$ is a Banach space equipped with the norm

$$\|\mathbf{v}\|_{W_{p,\text{div}}} = \|\mathbf{v}\|_{W^{1,p}(\Omega)^n}\,, \tag{2.7}$$

Moreover, Korn's inequality holds in the space $W_{p,\text{div}}$, see [12], which means that there exists a positive constant C_0 depending only on Ω and Γ such that

$$C_0 \|D(\mathbf{v})\|_{L^p(\Omega)^{n\times n}} \geq \|\mathbf{v}\|_{W_{p,\text{div}}} \quad \forall \mathbf{v} \in W_{p,\text{div}}. \tag{2.8}$$

Denoting by p' the conjugate of p. We introduce the convective operator

$$B : W_{p,\text{div}} \times W_{p,\text{div}} \times W_{p,\text{div}} \longrightarrow \mathbb{R}, \quad B(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}dx. \tag{2.9}$$

We begin by recalling the following lemma see [12], which gives some properties of the convective operator B.

Lemma 2.1. *Suppose that*

$$\frac{3n}{n+2} \leq p \leq 2. \tag{2.10}$$

Then, B *is trilinear, continuous on* $W_{p,\text{div}} \times W_{p,\text{div}} \times W_{p,\text{div}}$. *Moreover,* $\forall (\mathbf{u},\mathbf{v},\mathbf{w}) \in W_{p,\text{div}} \times W_{p,\text{div}} \times W_{p,\text{div}}$ *we have* $B(\mathbf{u},\mathbf{v},\mathbf{w}) = -B(\mathbf{u},\mathbf{w},\mathbf{v})$.

For the rest of this paper, we choose $\frac{3n}{n+2} \leq p \leq 2$. The use of Green's formula permits us to derive the following variational formulation of the mechanical problem (P1), see [12].

Problem P_f. For prescribed data $\mathbf{f} \in W'_{p,\text{div}}$. Find $\mathbf{u} \in W_{p,\text{div}}$ satisfying the variational inequality

$$\begin{aligned} &B(\mathbf{u},\mathbf{u},\mathbf{v}-\mathbf{u}) + \mu \int_\Omega |D(\mathbf{u})|^{p-2} D(\mathbf{u}) \cdot D(\mathbf{v}-\mathbf{u})\, dx \\ &+ g\int_\Omega |D(\mathbf{v})|\, dx - g\int_\Omega |D(\mathbf{u})|\, dx \geq \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})\, dx \quad \forall \mathbf{v} \in W_{p,\text{div}}. \end{aligned} \tag{2.11}$$

By taking $\mathbf{v} = 0$, respectively $\mathbf{v} = 2\mathbf{u}$ in (2.11), the following equation holds

$$\mu \int_\Omega |D(\mathbf{u})|^p\, dx + g\int_\Omega |D(\mathbf{u})|\, dx = \int_\Omega \mathbf{f}\cdot\mathbf{u}dx. \tag{2.12}$$

This implies using again (2.11)

$$B(\mathbf{u},\mathbf{u},\mathbf{v})+\mu\int_\Omega |D(\mathbf{u})|^{p-2}D(\mathbf{u})\cdot D(\mathbf{v})\,dx + g\int_\Omega |D(\mathbf{v})|\,dx \geq \int_\Omega \mathbf{f}\cdot\mathbf{v}dx \quad \forall\mathbf{v}\in W_{p,\text{div}}. \tag{2.13}$$

Consequently, the steady-state flow of Herschel-Bulkley fluid can be also described by the system (2.12), (2.13)

3. Blockage property

This section is consecrated to the study of blockage property of Herschel-Bulkley fluid. To do this, let us recall the following standard definition, see [14].

Definition 3.1. *We will say that the fluid is blocked in the domain Ω if $\mathbf{u}=0$ a.e. in Ω is solution to the variational problem (P_f).*

We prove the following proposition, which gives the variational interpretation of blockage property.

Proposition 3.1. *The fluid is blocked in the domain Ω if and only if*

$$g\int_\Omega |D(\mathbf{v})|\,dx \geq \int_\Omega \mathbf{f}\cdot\mathbf{v}dx \quad \forall\mathbf{v}\in W_{p,\text{div}}. \tag{3.1}$$

Proof. The first implication is an immediate consequence of the definition of blockage property. For the second one, we proceed as follows. Suppose that (3.1) holds. In particular, we have

$$g\int_\Omega |D(\mathbf{u})|\,dx \geq \int_\Omega \mathbf{f}\cdot\mathbf{u}dx. \tag{3.2}$$

Subtracting the inequalities (2.12) and (3.1) , we find

$$\mu\int_\Omega |D(\mathbf{u})|^{p}\,dx \leq B(\mathbf{u},\mathbf{u},\mathbf{v})+\mu\int_\Omega |D(\mathbf{u})|^{p-2}D(\mathbf{u})\cdot D(\mathbf{v})\,dx + g\int_\Omega |D(\mathbf{v})|\,dx - \int_\Omega \mathbf{f}\cdot\mathbf{v}dx \quad \forall\mathbf{v}\in W_{p,\text{div}}. \tag{3.3}$$

Thus, the result can be obtained by setting $\mathbf{v}=0$ as test function in (3.3) and using Korn's inequality. □

Hence, the mathematical study of blockage property consists in finding the relationship between the yield limit g and the density of volume forces $\mathbf{f}$ such that the inequality (3.1) holds.

We say that $\mathbf{f}$ is a blocking force if the inequality (3.1) is satified.

We suppose from now on that

$$\mathbf{f} \in L^{\infty}(\Omega)^{n}. \tag{3.4}$$

Proposition 3.2. *Denote by j the functionel given by $j : W_{p,\text{div}} \longrightarrow [0, +\infty[$,*

$$j(\mathbf{v}) = g \int_{\Omega} |D(\mathbf{v})| \, dx, \forall \mathbf{v} \in W_{p,\text{div}}.$$

Then the set $j(0)$ of all blocking forces is nonempty closed and convex.

Proof. Since $j(\mathbf{v}) \geq 0$ for any $\mathbf{v} \in W_{p,\text{div}}$, we deduce that $\partial j(0)$ contains $\mathbf{f} = 0$. According to [6] the set $\partial j(0)$ is closed and convex. □

Proposition 3.3. *Let $\mathbf{f} \in \partial j(0)$, $\mathbf{f} \neq 0$ and set $M = sup\{\lambda > 0 | \lambda \mathbf{f} \in \partial j(0)\}$. Then $M < +\infty$ and $M\mathbf{f} \in \partial j(0)$.*

Proof. The set $\{\lambda > 0 \mid \lambda \mathbf{f} \in \partial j(0)\}$ is nonempty since it contains $\lambda = 1$ (in fact it contains $]0, 1]$). Since $\mathbf{f} \neq 0$, there is $v_0 \in W_{p,div}$ satisfying $(\mathbf{f}, v_0) > 0$. If λ_0 is large enough we have $(\lambda_0 \mathbf{f}, v_0) > j(v_0)$so that $\lambda_0 \mathbf{f} \notin \partial j(0)$. Consequently $\{\lambda > 0 \mid \lambda \mathbf{f} \in \partial j(0)\} \subset]0, \lambda_0[$ and $M \leq \lambda_0 < +\infty$. Let $(\lambda_n)_n$ be a sequence converging towards M and verifying $\lambda_n \mathbf{f} \in \partial j(0)$. Hence $\lambda_n \mathbf{f} \to M\mathbf{f}$ and since $\partial j(0)$ is closed we deduce that $M\mathbf{f} \in \partial j(0)$. □

Definition 3.2. *Let $\mathbf{f}$ be a blocking force and let M be defined as in the Proposition 4. We call $\widetilde{\mathbf{f}} = M\mathbf{f}$ the maximal blocking force associated with $\mathbf{f}$.*

Proposition 3.4. *Let $\mathbf{f} \in \partial j(0), \mathbf{f} \neq 0$. Then the maximal blocking force is given by $\widetilde{\mathbf{f}} = M_1 \mathbf{f}$ where*

$$M_1 = inf_{(\mathbf{f},v)\neq 0} \frac{j(v)}{|(\mathbf{f}, v)|}.$$

Proof. Remark that $\mathbf{f} \in \partial j(0)$ iff $|(\mathbf{f}, v)| \leq j(v)$ $\forall v \in W_{p,\text{div}}$ and observe that $M = M_1$, where $M = sup\{\lambda > 0 \mid \lambda \mathbf{f} \in \partial j(0)\}$. □

Let $\mathbf{f}$ be a blocking force. We denote by C the set

$$C = \left\{ \mathbf{v} \in W_{p,\text{div}} \mid g \int_{\Omega} |D(\mathbf{v})| \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \right\}. \tag{3.5}$$

It is straightforward to verify that the set C is a convex cone in $W_{p,\text{div}}$.

4. Behaviour of the Flow

Let us introduce for $\varepsilon > 0$ the perturbed blocking force

$$\mathbf{f}_\varepsilon = \left(1 + \varepsilon^{p-1}\right) \mathbf{f}, \tag{4.1}$$

and denote by $\mathbf{u}_\varepsilon$ the solution of the corresponding problem, i.e.

$$\begin{aligned} &B\left(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v} - \mathbf{u}_\varepsilon\right) + \mu \int_\Omega \left|D\left(\mathbf{u}_\varepsilon\right)\right|^{p-2} D\left(\mathbf{u}_\varepsilon\right) \cdot D\left(\mathbf{v} - \mathbf{u}_\varepsilon\right) dx \\ &+g \int_\Omega \left|D\left(\mathbf{v}\right)\right| dx - g \int_\Omega \left|D\left(\mathbf{u}_\varepsilon\right)\right| dx \geq \int_\Omega \mathbf{f}_\varepsilon \cdot \left(\mathbf{v} - \mathbf{u}_\varepsilon\right) dx \quad \forall \mathbf{v} \in W_{p,\text{div}}. \end{aligned} \tag{4.2}$$

The above inequality can be written in equivalent form

$$\mu \int_\Omega \left|D\left(\mathbf{u}_\varepsilon\right)\right|^p dx + g \int_\Omega \left|D\left(\mathbf{u}_\varepsilon\right)\right| dx = \int_\Omega \mathbf{f}_\varepsilon \cdot \mathbf{u}_\varepsilon dx, \tag{4.3}$$

$$\begin{aligned} B\left(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}\right) + \mu \int_\Omega \left|D\left(\mathbf{u}_\varepsilon\right)\right|^{p-2} D\left(\mathbf{u}_\varepsilon\right) \cdot D\left(\mathbf{v}\right) dx + g \int_\Omega \left|D\left(\mathbf{v}\right)\right| dx \\ \geq \int_\Omega \mathbf{f}_\varepsilon \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in W_{p,\text{div}}. \end{aligned} \tag{4.4}$$

Setting now

$$\mathbf{w}_\varepsilon = \frac{\mathbf{u}_\varepsilon}{\varepsilon}, \quad \forall \varepsilon > 0. \tag{4.5}$$

In the following we establish a convergence result for $\left(\mathbf{w}_\varepsilon\right)_{\varepsilon>0}$ when ε tends to 0.

Theorem 4.1. *Suppose that* $\mathbf{f}$ *is a blocking force. Then,* $\left(\mathbf{w}_\varepsilon\right)_{\varepsilon>0}$ *convergences strongly when,* ε *tends to* 0, *in* $W_{p,\text{div}}$ *to* $\mathbf{w}$ *solution of the following variational inequality*

$$w \in C \mid \mu \int_\Omega \left|D\left(\mathbf{w}\right)\right|^{p-2} D\left(\mathbf{w}\right) \cdot D\left(\mathbf{v} - \mathbf{w}\right) dx \geq \int_\Omega \mathbf{f} \cdot \left(\mathbf{v} - \mathbf{w}\right) dx \quad \forall \mathbf{v} \in C. \tag{4.6}$$

Proof. The system becomes, taking into account (4.5)

$$\mu \varepsilon^{p-1} \int_\Omega \left|D\left(\mathbf{w}_\varepsilon\right)\right|^p dx + g \int_\Omega \left|D\left(\mathbf{w}_\varepsilon\right)\right| dx = \left(1 + \varepsilon^{p-1}\right) \int_\Omega \mathbf{f} \cdot \mathbf{w}_\varepsilon dx, \tag{4.7}$$

$$\varepsilon^2 B(\mathbf{w}_\varepsilon, \mathbf{w}_\varepsilon, \mathbf{v}) + \mu\varepsilon^{p-1} \int_\Omega |D(\mathbf{w}_\varepsilon)|^{p-2} D(\mathbf{w}_\varepsilon) \cdot D(\mathbf{v})\, dx +$$
$$g \int_\Omega |D(\mathbf{v})|\, dx \geq \left(1 + \varepsilon^{p-1}\right) \int_\Omega \mathbf{f} \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in W_{p,\text{div}}.$$

Equation (4.7) gives

$$\mu\varepsilon^{p-1} \int_\Omega |D(\mathbf{w}_\varepsilon)|^p\, dx + g \int_\Omega |D(\mathbf{w}_\varepsilon)|\, dx - \int_\Omega \mathbf{f} \cdot \mathbf{w}_\varepsilon dx$$
$$= \varepsilon^{p-1} \int_\Omega \mathbf{f} \cdot \mathbf{w}_\varepsilon dx \tag{4.8}$$

Suppose that f is a a blocking force, then (4.8) gives

$$\mu \int_\Omega |D(\mathbf{w}_\varepsilon)|^p\, dx \leq g \int_\Omega |D(\mathbf{w}_\varepsilon)|\, dx. \tag{4.9}$$

We deduce makig use Korn's inequality and some algebraic manupilations that

$$\|\mathbf{w}_\varepsilon\|_{W_{p,\text{div}}} \leq c. \tag{4.10}$$

Hence, we can extract a subsequence still denoted by $(\mathbf{w}_\varepsilon)_{\varepsilon>0}$ such that

$$\mathbf{w}_\varepsilon \longrightarrow \mathbf{w} \text{ in } W_{p,\text{div}} \text{ weakly.} \tag{4.11}$$

Rellich-Kondrachof's compactness theorem allows us to get after a new extraction

$$\mathbf{w}_\varepsilon \longrightarrow \mathbf{w} \text{ in } L^p(\Omega)^n \text{ strongly and a.e. in } \Omega. \tag{4.12}$$

Therefore, equation (4.8) gives again

$$g \int_\Omega |D(\mathbf{w}_\varepsilon)|\, dx \leq \left(1 + \varepsilon^{p-1}\right) \int_\Omega \mathbf{f} \cdot \mathbf{w}_\varepsilon dx.$$

There by allowing to find

$$g \lim \int_\Omega |D(\mathbf{w}_\varepsilon)|\, dx \leq \liminf \int_\Omega \mathbf{f} \cdot \mathbf{w}_\varepsilon dx. \tag{4.13}$$

This yields

$$g \int_\Omega |D(\mathbf{w})|\, dx \leq \int_\Omega \mathbf{f} \cdot \mathbf{w} dx. \tag{4.14}$$

Consequently, since f is a blocking force, the following equation holds

$$g\int_{\Omega}|D(\mathbf{w})|\,dx=\int_{\Omega}\mathbf{f}\cdot\mathbf{w}dx. \tag{4.15}$$

Taking $\mathbf{w}$ as test function in inequality (4.8), it implies that

$$\varepsilon^{2}B(\mathbf{w}_{\varepsilon},\mathbf{w}_{\varepsilon},\mathbf{w})+\mu\varepsilon^{p-1}\int_{\Omega}|D(\mathbf{w}_{\varepsilon})|^{p-2}D(\mathbf{w}_{\varepsilon})\cdot D(\mathbf{w})\,dx$$
$$+g\int_{\Omega}|D(\mathbf{w})|\,dx\geq\left(1+\varepsilon^{p-1}\right)\int_{\Omega}\mathbf{f}\cdot\mathbf{w}dx$$

This gives, making use (4.15)

$$\varepsilon^{3-p}B(\mathbf{w}_{\varepsilon},\mathbf{w}_{\varepsilon},\mathbf{w})+\mu\int_{\Omega}|D(\mathbf{w}_{\varepsilon})|^{p-2}D(\mathbf{w}_{\varepsilon})\cdot D(\mathbf{w})\,dx\geq g\int_{\Omega}|D(\mathbf{w})|\,dx \tag{4.16}$$

Moreover, Lemma 1 permits us to obtain the estimate

$$|B(\mathbf{w}_{\varepsilon},\mathbf{w}_{\varepsilon},\mathbf{w})|\leq\|\mathbf{w}_{\varepsilon}\|^{2}_{W_{p,\mathrm{div}}}\|\mathbf{w}\|_{W_{p,\mathrm{div}}}. \tag{4.17}$$

From another hand, it is well known that the non linear terme $\mu\int_{\Omega}|D(\mathbf{w}_{\varepsilon})|^{p-2}D(\mathbf{w}_{\varepsilon})\cdot D(\mathbf{w})\,dx$ converges to $\mu\int_{\Omega}|D(\mathbf{w})|^{p}\,dx,$ see the reference [4]. Consequently, by passing to the limit, one can find, keeping in mind account (4.17),

$$\mu\int_{\Omega}|D(\mathbf{w})|^{p}\,dx\geq g\int_{\Omega}|D(\mathbf{w})|\,dx. \tag{4.18}$$

We get thanks to (4.9)

$$\liminf\mu\int_{\Omega}|D(\mathbf{w}_{\varepsilon})|^{p}\,dx\leq g\liminf\int_{\Omega}|D(\mathbf{w}_{\varepsilon})|\,dx.$$

So, using (4.13) we can infer that

$$\liminf\mu\int_{\Omega}|D(\mathbf{w}_{\varepsilon})|^{p}\,dx\leq\liminf\int_{\Omega}\mathbf{f}\cdot\mathbf{w}_{\varepsilon}dx$$

Which implies that

$$\mu\int_{\Omega}|D(\mathbf{w})|^{p}\,dx\leq\int_{\Omega}\mathbf{f}\cdot\mathbf{w}dx \tag{4.19}$$

Putting together (4.15), (4.18) and (4.19) we obtain

$$\mu\int_{\Omega}|D(\mathbf{w})|^{p}\,dx=g\int_{\Omega}|D(\mathbf{w})|\,dx=\int_{\Omega}\mathbf{f}\cdot\mathbf{w}dx \tag{4.20}$$

Which implies in particular that $\mathbf{w} \in C$. Furthermore, by (4.8) we get

$$\varepsilon^{3-p} B\left(\mathbf{w}_{\varepsilon}, \mathbf{w}_{\varepsilon}, \mathbf{v}\right) + \mu \int_{\Omega} \left|D\left(\mathbf{w}_{\varepsilon}\right)\right|^{p-2} D\left(\mathbf{w}_{\varepsilon}\right) \cdot D\left(\mathbf{v}\right) dx +$$
$$\frac{1}{\varepsilon^{p-1}} \left(g \int_{\Omega} \left|D\left(\mathbf{v}\right)\right| dx - \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \right) \geq \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \ \ \forall \mathbf{v} \in W_{p,\mathrm{div}}.$$

By choosing $\mathbf{v} \in C$ in the above inquality, the passage to the limit leads to

$$\mu \int_{\Omega} \left|D\left(\mathbf{w}\right)\right|^{p-2} D\left(\mathbf{w}\right) \cdot D\left(\mathbf{v}\right) dx \geq \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \ \ \forall \mathbf{v} \in C. \tag{4.21}$$

Combining (4.20) and (4.21) yields the inequality (4.6).

Our object now is to prove the strong convergence. For this aim, we procede as follows. The use of (4.7) and (4.8) permits us to affirm that for every $\mathbf{v} \in W_{p,\mathrm{div}}$ we have

$$\varepsilon^{2} B\left(\mathbf{w}_{\varepsilon}, \mathbf{w}_{\varepsilon}, \mathbf{v}\right) + \mu \varepsilon^{p-1} \int_{\Omega} \left|D\left(\mathbf{w}_{\varepsilon}\right)\right|^{p-2} D\left(\mathbf{w}_{\varepsilon}\right) \cdot D\left(\mathbf{v} - \mathbf{w}_{\varepsilon}\right) dx$$
$$\geq \left(1 + \varepsilon^{p-1}\right) \int_{\Omega} \mathbf{f} \cdot \left(\mathbf{v} - \mathbf{w}_{\varepsilon}\right) dx - g \left(\int_{\Omega} \left|D\left(\mathbf{v}\right)\right| dx - \int_{\Omega} \left|D\left(\mathbf{w}_{\varepsilon}\right)\right| dx \right)$$

It follows, by setting $\mathbf{v} = \mathbf{w}$

$$-\varepsilon^{2} B\left(\mathbf{w}_{\varepsilon}, \mathbf{w}_{\varepsilon}, \mathbf{w}\right) + \mu \varepsilon^{p-1} \int_{\Omega} \left|D\left(\mathbf{w}_{\varepsilon}\right)\right|^{p-2} D\left(\mathbf{w}_{\varepsilon}\right) \cdot D\left(\mathbf{w}_{\varepsilon} - \mathbf{w}\right) dx$$
$$\leq \left(1 + \varepsilon^{p-1}\right) \int_{\Omega} \mathbf{f} \cdot \left(\mathbf{w}_{\varepsilon} - \mathbf{w}\right) dx + g \int_{\Omega} \left(\left|D\left(\mathbf{w}\right)\right| - \left|D\left(\mathbf{w}_{\varepsilon}\right)\right|\right) dx. \tag{4.22}$$

Further, since f is a blocking force and $\mathbf{w} \in W_{p,\mathrm{div}}$, one can verify that

$$g \int_{\Omega} \left(\left|D\left(\mathbf{w}\right)\right| - \left|D\left(\mathbf{w}_{\varepsilon}\right)\right|\right) dx \leq \int_{\Omega} \mathbf{f} \cdot \left(\mathbf{w} - \mathbf{w}_{\varepsilon}\right) dx$$

Consequently, inequality (4.22) becomes

$$\mu \int_{\Omega} \left|D\left(\mathbf{w}_{\varepsilon}\right)\right|^{p-2} D\left(\mathbf{w}_{\varepsilon}\right) \cdot D\left(\mathbf{w}_{\varepsilon} - \mathbf{w}\right) dx \leq \int_{\Omega} \mathbf{f} \cdot \left(\mathbf{w}_{\varepsilon} - \mathbf{w}\right) dx + \varepsilon^{3-p} B\left(\mathbf{w}_{\varepsilon}, \mathbf{w}_{\varepsilon}, \mathbf{w}\right). \tag{4.23}$$

This becomes

$$\begin{aligned} &\mu \int_\Omega \left(|D(\mathbf{w}_\varepsilon)|^{p-2} D(\mathbf{w}_\varepsilon) - |D(\mathbf{w})|^{p-2} D(\mathbf{w})\right) \cdot D(\mathbf{w}_\varepsilon - \mathbf{w})\, dx \\ &\leq \int_\Omega \mathbf{f} \cdot (\mathbf{w}_\varepsilon - \mathbf{w})\, dx - \mu \int_\Omega |D(\mathbf{w})|^{p-2} D(\mathbf{w}) D(\mathbf{w}_\varepsilon - \mathbf{w})\, dx \\ &+ c\varepsilon^{3-p} \|\mathbf{w}_\varepsilon\|^2_{W_{p,\text{div}}} \|\mathbf{w}\|_{W_{p,\text{div}}}. \end{aligned} \tag{4.24}$$

Let us observe now that for every $x, y \in \mathbb{R}^n$,

$$\left(|x|^{p-2} x - |y|^{p-2} y\right) \cdot (x - y) \geq c \frac{|x-y|^2}{(|x| + |y|)^{2-p}}, \quad 1 < p \leq 2.$$

So, applying the above inequality, (4.24) can be rewritten as

$$\mu \int_\Omega \frac{|D(\mathbf{w}_\varepsilon - \mathbf{w})|^2}{(|D(\mathbf{w}_\varepsilon)| + |D(\mathbf{w})|)^{2-p}} dx \leq c \left| \int_\Omega \mathbf{f} \cdot (\mathbf{w}_\varepsilon - \mathbf{w})\, dx \right|$$
$$+ c\varepsilon^{3-p} \|\mathbf{w}_\varepsilon\|^2_{W_{p,\text{div}}} \|\mathbf{w}\|_{W_{p,\text{div}}} + c\mu \left| \int_\Omega |D(\mathbf{w})|^{p-2} D(\mathbf{w}) D(\mathbf{w}_\varepsilon - \mathbf{w})\, dx \right|.$$

Which gives exploiting Korn's and Hölder's inequalities

$$\|\mathbf{w}_\varepsilon - \mathbf{w}\|^p_{W_p} \leq c \left(\int_\Omega (|D(\mathbf{w}_m)| + |D(\mathbf{w})|)^p\, dx \right)^{\frac{2-p}{2}} \left(\left| \int_\Omega \mathbf{f} \cdot (\mathbf{w}_\varepsilon - \mathbf{w})\, dx \right| \right.$$
$$\left. + \varepsilon^{3-p} \|\mathbf{w}_\varepsilon\|^2_{W_{p,\text{div}}} \|\mathbf{w}\|_{W_{p,\text{div}}} + \mu \int_\Omega |D(\mathbf{w})|^{p-2} D(\mathbf{w}) D(\mathbf{w}_\varepsilon - \mathbf{w})\, dx \right)^{\frac{p}{2}}.$$

Passing to the limit, we conclude, using (4.11) and taking into account the fact that $|D(\mathbf{w})|^{p-2} D(\mathbf{w})$ belongs is bounded of $L^{p'}(\Omega)^n$ that

$$\mathbf{w}_\varepsilon \longrightarrow \mathbf{w} \text{ in } W_{p,\text{div}} \text{ strongly.}$$

Which permits us to achieve the proof. □

Corollary 4.1. *Denoting by* w_0 *the unique solution of the variational equation given by*

$$\mu \int_\Omega |D(\mathbf{w}_0)|^{p-2} D(\mathbf{w}_0) \cdot D(\mathbf{v})\, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in W_{p,\text{div}}. \tag{4.25}$$

Then, the following estimates hold

$$(4.26) \qquad \|D(\mathbf{w})\|_{L^p(\Omega)^{n\times n}} \leq \|D(\mathbf{w}_0)\|_{L^p(\Omega)^{n\times n}} \quad \textit{and} \quad \int_\Omega \mathbf{f}\cdot\mathbf{w}dx \leq \int_\Omega \mathbf{f}\cdot\mathbf{w}_0 dx.$$

Proof. We can infer by setting $\mathbf{w}$ as test function in (4.25) that

$$\mu\int_\Omega |D(\mathbf{w}_0)|^{p-2} D(\mathbf{w}_0)\cdot D(\mathbf{w})\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{w}dx.$$

This yields, using Hölder's inequality

$$\mu\|D(\mathbf{w}_0)\|^{p-1}_{L^p(\Omega)^{n\times n}}\|D(\mathbf{w})\|_{L^p(\Omega)^{n\times n}} \geq \int_\Omega \mathbf{f}\cdot\mathbf{w}dx = \mu\|D(\mathbf{w})\|^p_{L^p(\Omega)^{n\times n}}.$$

Which allows us to get the first estimate. The second estimate becomes an immediate consequence of the first one. □

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