ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **11** (2022), no.12, 1281–1290
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.12.9

SYMBOLIC APPROACH TO THE QUADRATIC DECOMPOSITION OF APPELL SEQUENCES

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ABSTRACT. In this paper, we characterize the four derived sequences obtained by the symbolic approach to the quadratic decomposition of Appll sequences. Moreover, we prove that the two monic polynomial sequences associated to such quadratic decomposition are also Appell sequences.

1. INTRODUCTION

Let $\mathcal{P}$ be the linear space of polynomials in one variable with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. $\langle u, f\rangle$ denotes the action of u in $\mathcal{P}'$ on f in $\mathcal{P}$ and by $(u)_n := \langle u, x^n\rangle\,,\ n \geq 0,$ the moments of u with respect to the monomial sequence $\{x^n\}_{n\geq 0}$. When $(u)_0 = 1,$ the linear functional u is said to be normalized. In this work we need to recall some operations in $\mathcal{P}'$, (see [4, 6]). For any u in $\mathcal{P}'$, any q in $\mathcal{P}$, let $Du = u'$and qu be respectively the derivative, the left multiplication of the linear functionals defined by duality:

$$\left\langle u^{'}, f\right\rangle := -\left\langle u, f'\right\rangle,$$
$$\left\langle qu, f\right\rangle := \left\langle u, qf\right\rangle.$$

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2020 *Mathematics Subject Classification.* 33C45, 42C05.
Key words and phrases. Orthogonal polynomials, Appell sequences, quadratic decomposition, lowering operator.
Submitted: 22.11.2022; *Accepted:* 08.12.2022; *Published:* 20.12.2022.

Recall that a linear operator $L : \mathcal{P} \to \mathcal{P}$ has a transpose ${}^tL : \mathcal{P}' \to \mathcal{P}'$defined by

$$\left\langle {}^tL\left(u\right),\ f\right\rangle = \left\langle u,\ L\left(f\right)\right\rangle,\ \ u \in P',\ f \in \mathcal{P}, \tag{1.1}$$

then D is the differential operator. Thus, the differentiation operator D on forms is minus the transpose of the differentiation operator D on polynomials.

The linear functional u is called regular (quasi-definite) if we can associate with it a polynomials sequence $\{P_n\}_{n\geq 0}$ such that

$$\langle u, P_nP_m\rangle = k_n\delta_{n,m} \tag{1.2}$$

with $k_n \neq 0$, for every $n, m \geq 0$, $\{P_n\}_{n\geq 0}$ is said to be orthogonal with respect to u [4, 6, 7].

Definition 1.1. *[4] A sequence of monic polynomials $\{P_n\}_{n\geq 0}$ is called orthogonal with respect to the linear functional u if the following orthogonality conditions hold*

$$\begin{array}{ll} \langle u, P_n\left(x\right)P_m\left(x\right)\rangle = 0, & n \neq m, \\ \langle u, P_n^2\left(x\right)\rangle \neq 0, & n \geq 0, \end{array}$$

where $\deg P_n = n$*, for every* $n \geq 0$.

In this case, $\{P_n\}_{n\geq 0}$ satisfies the following two order recurrence relation:

$$\begin{cases} P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_nP_{n-1}(x), & n \geq 1, \\ P_0(x) = 1, \quad P_1(x) = x - \beta_0 \end{cases}$$

where $\beta_n = \dfrac{\langle u, xP_n^2(x)\rangle}{\langle u, P_n^2(x)\rangle}$ and $\gamma_{n+1} = \dfrac{\left\langle u, P_{n+1}^2(x)\right\rangle}{\langle u, P_n^2(x)\rangle} \neq 0$, $n \geq 0$.

We will denote as $\left\{P_n^{[1]}\right\}_{n\geq 0}$ the MPS obtained from given MPS by a single differentiation $P_n^{[1]}\left(x\right) := \dfrac{1}{n+1}P'_{n+1}\left(x\right)$, $n \geq 0$.

Denote by $(a)_n$ the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & \text{if } n = 0, \\ a(a-1)(a-2)...(a-n+1), & \text{if } n \geq 1. \end{cases} \tag{1.3}$$

Definition 1.2. *A linear mapping $\mathcal{M}$ of $\mathcal{P}$ into itself is called lowering operator when $\mathcal{M}(1) = 0$ and $\deg(\mathcal{M}(x^n)) = n-1$, $n \geq 1$.*

Definition 1.3. *An Appell sequence is a MPS $\{P_n\}_{n\geq 0}$ such that $P_n^{[1]}(x) = \frac{1}{n+1}P'_{n+1}\left(x\right)$, $n \geq 0$, [1, 4, 9].*

Definition 1.4. *A MPS $\{P_n\}_{n\geq 0}$ is called an $\mathcal{M}-$Appell sequence with respect to a lowering operator $\mathcal{M}$ if $P_n(x) = P_n^{[1]}(x;\mathcal{M})$ for all integers $n \geq 0$, [2, 3]*

2. Symbolic approach of Appell sequences

Let $\{P_n\}_{n\geq 0}$ a MPS, it is always possible to associate with it two MPS $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ and two sequences $\{S_n\}_{n\geq 0}$ and $\{T_n\}_{n\geq 0}$ such that

$$P_{2n}(x) = Q_n(x^2 + a) + xS_{n-1}(x^2 + a), \tag{2.1}$$

$$P_{2n+1}(x) = T_n(x^2 + a) + xR_{n-1}(x^2 + a). \tag{2.2}$$

where $deg\ S_n \leq n$, $deg\ T_n \leq n$, $n \geq 0$, $S_{-1}(x) = 0$, [4, 5].

As a consequence, a MPS $\{P_n\}_{n\geq 0}$ is symmetric, that is $P_n(-x) = (-1)^n P_n(x)$, $n \geq 0$, if and only if $S_n(x) = T_n(x) = 0$, $n \geq 0$, [5, 8].

Theorem 2.1. *Let $\{P_n\}_{n\geq 0}$ an appell polynomials sequence satisfied (2.1) and (2.2), then the sequences $\{Q_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$, $\{S_n\}_{n\geq 0}$ and $\{T_n\}_{n\geq 0}$ are given by*

$$\begin{aligned} Q_n(x) &= \frac{1}{(2n+1)(n+1)}\left(L_{-1}Q_{n+1}\right), && n \geq 0, \\ R_n(x) &= \frac{1}{(2n+3)(n+1)}\left(L_{1}R_{n+1}\right), && n \geq 0, \\ S_n(x) &= \frac{1}{(2n+3)(n+2)}\left(L_{1}S_{n+1}\right), && n \geq 0, \\ T_n(x) &= \frac{1}{(2n+3)(n+1)}\left(L_{-1}T_{n+1}\right), && n \geq 0, \end{aligned} \tag{2.3}$$

where the operateur L_α with $\alpha = 1$ or $\alpha = -1$ defined by

$$L_\alpha = 2L + \alpha D,$$

where $L = D(x-a)D$.

Proof. By differenting (2.1), we have

$$2nP_{2n-1}^{[1]}(x) = 2xnQ_{n-1}^{[1]}(x^2 + a) + S_{n-1}(x^2 + a) + 2x^2 S'_{n-1}(x^2 + a),$$

replacing n by $n + 1$, we obtain

$$2(n+1)P_{2n+1}^{[1]}(x) = 2x(n+1)Q_n^{[1]}(x^2 + a) + S_n(x^2 + a) + 2x^2 S_n^{'}(x^2 + a),$$

using (2.2) , we get

$$\begin{aligned}&2(n+1)\left[T_n(x^2+a)+xR_{n-1}(x^2+a)\right]\\&=2x(n+1)Q_n^{[1]}(x^2+a)+S_n(x^2+a)+2x^2S_n^{'}(x^2+a),\end{aligned}$$

then

$$\begin{cases}2(n+1)T_n(x^2+a)=S_n(x^2+a)+2x^2S_n^{'}(x^2+a)\\ R_{n-1}(x^2+a)=Q_n^{[1]}(x^2+a)\end{cases}$$

x^2+a to replace by x, we have

(2.4) $$2(n+1)T_n(x)=S_n(x)+2(x-a)S_n^{'}(x),\quad n\geq 0,$$

(2.5) $$R_{n-1}(x)=Q_n^{[1]}(x),\qquad n\geq 0.$$

By differenting (2.2) , we have

$$(2n+1)\,P_{2n}^{[1]}(x)=2xT_n^{'}(x^2+a)+R_n(x^2+a)+2nx^2R_{n-1}^{[1]}(x^2+a),$$

using (2.1) , we get

$$\begin{aligned}&(2n+1)\left[Q_n(x^2+a)+xS_{n-1}(x^2+a)\right]\\&=2xT_n^{'}(x^2+a)+R_n(x^2+a)+2nx^2R_{n-1}^{[1]}(x^2+a),\end{aligned}$$

then

$$\begin{cases}(2n+1)\,Q_n(x^2+a)=R_n(x^2+a)+2nx^2R_{n-1}^{[1]}(x^2+a),\\ (2n+1)\,S_{n-1}(x^2+a)=2T_n^{'}(x^2+a),\end{cases}$$

x^2+a to replace by x, we have

(2.6) $$(2n+1)\,Q_n(x)=R_n(x)+2n(x-a)R_{n-1}^{[1]}(x),\quad n\geq 0,$$

(2.7) $$(2n+1)\,S_{n-1}(x)=2T_n^{'}(x),\quad n\geq 0.$$

n to rplace by $n+1$ in (2.6) , we obtain

$$(2n+3)\,Q_{n+1}(x)=R_{n+1}(x)+2(n+1)(x-a)R_n^{[1]}(x),\quad n\geq 0,$$

by differentiating, we get

$$(2n+3)\,(n+1)Q_n^{[1]}(x)=R_{n+1}'(x)+2\left((x-a)R_{n+1}^{'}(x)\right)'.$$

Using (2.5) , then

$$(2n+3)\,(n+1)R_n(x)=R'_{n+1}(x)+2\left((x-a)R'_{n+1}(x)\right)'. \tag{2.8}$$

From (2.6)

$$\begin{aligned}(2n+1)\,Q_n(x)&=Q_n^{[1]}(x)+2n(x-a)R_{n-1}^{[1]}(x)\\&=\frac{Q'_{n+1}(x)}{n+1}+2(x-a)R'_n(x)\\&=\frac{Q'_{n+1}(x)}{n+1}+2(x-a)\left(Q_n^{[1]}(x)\right)'\\&=\frac{Q'_{n+1}(x)}{n+1}+2(x-a)\left(\frac{Q'_{n+1}(x)}{n+1}\right)'\\\Longrightarrow (2n+1)\,(n+1)Q_n(x)&=Q'_{n+1}(x)+2(x-a)Q''_{n+1}(x),\end{aligned}$$

so,

$$(2n+1)\,(n+1)Q_n(x)=\left(2(x-a)Q'_{n+1}(x)\right)'-Q'_{n+1}(x). \tag{2.9}$$

From (2.8) and (2.9) can be write

$$R_n(x)=\frac{1}{(2n+3)\,(n+1)}\left[2(x-a)D^2+3D\right]R_{n+1}(x),$$

and

$$Q_n(x)=\frac{1}{(2n+1)\,(n+1)}\left[2(x-a)D^2+D\right]Q_{n+1}(x).$$

Further, n to replace by $n+1$ in(2.7) , we find

$$(2n+3)\,S_n(x)=2T'_{n+1}(x),$$

replacing the above expression in (2.4) , we obtain

$$\begin{aligned}(2n+3)\,(n+1)T_n(x)&=T'_{n+1}(x)+2(x-a)\left(T'_{n+1}(x)\right)'\\&=T'_{n+1}(x)+2\left[\left((x-a)T'_{n+1}(x)\right)'-T'_{n+1}(x)\right]\\&=2\left((x-a)T'_{n+1}(x)\right)'-T'_{n+1}(x).\end{aligned}$$

Hence

$$T_n(x)=\frac{2}{(2n+3)\,(n+1)}\left[(x-a)D^2+D\right]T_{n+1}(x).$$

From (2.7) and on account of (2.4) , can be write

$$\begin{aligned}(2n+3)\,S_n(x) &= 2T_{n+1}^{'}(x)\\ &= \frac{1}{n+2}\Big(S_{n+1}(x)+2(x-a)S_{n+1}^{'}(x)\Big)^{\prime}\\ &= \frac{1}{(2n+3)\,(n+2)}\left[\Big(2(x-a)S_{n+1}^{'}(x)\Big)^{\prime}+S_{n+1}^{'}(x\right].\end{aligned}$$

□

Proposition 2.1. *Let* $\{P_n\}_{n\geq 0}$ *an appell polynomials sequence satisfied* (2.1) *and* (2.2) , *then either* $\{P_n\}_{n\succeq 0}$ *is symmetric or there exists an integer* $p \geq 0$ *such that* $S_p(x) \neq 0$ *(respectively,* $T_p(x) \neq 0$*). In this case, we have*

(2.10)
$$\begin{aligned}S_n(x) &= 0, \quad 0\leq n\leq p-1,\ p\geq 1,\\ T_n(x) &= 0, \quad 0\leq n\leq p-1,\ p\geq 1,\end{aligned}$$

(2.11)
$$S_{p+n}(x) = \binom{n+p+1}{n}\frac{(p+\frac{3}{2})_n}{(\frac{3}{2})_n}S_p\widetilde{S}_n(x), \qquad n\geq 0,$$

(2.12)
$$T_{p+n}(x) = \binom{n+p}{n}\frac{(p+\frac{3}{2})_n}{(\frac{1}{2})_n}T_p\widetilde{T}_n(x), \qquad n\geq 0,$$

where $\widetilde{S}_n$ *and* $\widetilde{T}_n$ *are two monic polynomials fulfilling* $\deg \widetilde{S}_n(x) = n$ *and* $\deg \widetilde{T}_n(x) = n$*,* $n \geq 0$, $(a)_n$ *is given by* (1.3).

Proof. Firstly, If $\{P_n\}_{n\succeq 0}$ is a symmetric sequence, then

$$S_n(x) = T_n(x) = 0, \qquad n\geq 0.$$

Reciprocally, if $S_n(x) = 0$, $n \geq 0$, then from (2.4)

$$T_n(x) = 0, \qquad n\geq 0.$$

Moreover, if $T_n(x) = 0$, $n \geq 0$, then from (2.7)

$$S_n(x) = 0, \qquad n\geq 0.$$

Now, if $\{P_n\}_{n\geq 0}$ is not a symmetric sequence, then it exists the smallest integer $p \geq 0$ such that

$$\begin{aligned} S_p(x) &\neq 0 \\ &\text{and} \\ S_n(x) &= 0, \qquad 0 \leq n \leq p-1,\ p \geq 1. \end{aligned}$$

From (2.7), we have

$$T_n(x) = \text{constant}, \qquad 0 \leq n \leq p.$$

On the other hand, using (2.4) , we obtain

$$\begin{aligned} T_n(x) &= 0, \qquad 0 \leq n \leq p-1, \\ &\text{and} \\ 2(p+1)T_p &= S_p(x) + 2(x-a)S'_p(x), \end{aligned}$$

this leads to

$$S_p(x) = \text{constant} = S_p \neq 0,$$

with

$$S_p = (p+1)T_p.$$

Taking into account (2.4) and (2.7) , we have

$$\begin{aligned} \deg(S_{n+p}) &= n, \ \ n \geq 0 \\ &\text{and} \\ \deg(T_{n+p}) &= n, \ \ n \geq 0. \end{aligned}$$

Therefore, it exists two nonzero sequences $\{\xi_n\}_{n\geq 0}$ and $\{\nu_n\}_{n\geq 0}$ such that

$$\begin{aligned} S_{n+p}(x) &= \xi_n \widetilde{S}_n(x), \qquad n \geq 0, \\ T_{n+p}(x) &= \nu_n \widetilde{T}_n(x), \qquad n \geq 0, \end{aligned}$$

where $\widetilde{S}_n$ and $\widetilde{T}_n$ are two monic polynomials of degree n, $n \geq 0$, $\xi_0 = T_p$ and $\nu_0 = 2(p+1)T_p$.

According to (2.4) and (2.7) , we deduce that

$$\begin{aligned}\xi_n &= \binom{n+1+p}{n}\frac{\left(p+\frac{3}{2}\right)_n}{\left(\frac{3}{2}\right)_n}\xi_0,\\ \nu_n &= \frac{n+\frac{1}{2}}{n+p+1}\xi_n, \quad n\geq 0.\end{aligned}$$

$\square$

Now, given a MPS $\{P_n\}_{n\succeq 0}$, we construct the sequence $\{P_n^{[1]}(.;\mathcal{M})\}_{n\geq 0}$ defined by

(2.13) $$P_n^{[1]}(x;\mathcal{M}) = \chi_n(\mathcal{M}P_{n+1})(x), \qquad n\geq 0,$$

where $\chi_n \in \mathbb{C}-\{0\}$, $n\geq 0$, is chosen for making $P_n^{[1]}(.;\mathcal{M})$ monic. Thus, we have

(2.14) $$P_n^{[1]}(x;L_\alpha) = \frac{1}{(n+1)\left[2(n+1)+\alpha\right]}(L_\alpha P_{n+1})(x), \qquad n\geq 0.$$

where $\alpha \neq -2(n+1)$, $n\geq 0$. Consequently, relations (2.13) and (2.14) become

$$\begin{aligned}Q_n(x) &= Q_n^{[1]}(x;L_{-1}), \qquad n\geq 0,\\ R_n(x) &= R_n^{[1]}(x;L_1), \qquad n\geq 0.\end{aligned}$$

According to the definition 1.4, the Theorem 2.1 allows us to conclude that $\{Q_n\}_{n\succeq 0}$ is $L_{-1}-$Appell and $\{R_n\}_{n\succeq 0}$ is L_1-Appell.

Moreover, from (2.3) and (2.11), (2.12) given in Proposition 2.1, we may say that the sequences $\{\widetilde{S}_n\}_{n\geq 0}$ and $\{\widetilde{T}_n\}_{n\geq 0}$ are L_1 and $L_{-1}-$Appell, respectively.

3. $L_\alpha-$APPELL SEQUENCES

Let $\{P_n\}_{n\geq 0}$ be a MPS with dual sequence $\{u_n\}_{n\geq 0}$ and let $\{u_n^{[1]}L_\alpha\}_{n\geq 0}$ be the dual sequence of $\{P_n^{[1]}(.;L_\alpha)\}_{n\geq 0}$.

From (1.1) , the transpose ${}^tL_\alpha$ defined by

$$\begin{aligned}\left\langle {}^tL_\alpha u, p\right\rangle &= \langle u, L_\alpha p\rangle = \langle u, (2L+\alpha D)p\rangle\\ &= \left\langle (2\,{}^tL-\alpha D)u, p\right\rangle, \qquad p\in\mathcal{P},\end{aligned}$$

then

$${}^tL_\alpha = 2\,{}^tL-\alpha D.$$

And as ${}^{t}D = -D$ this leads to ${}^{t}L = L$. Thus, ${}^{t}L_\alpha := L_{-\alpha}$ where L_α is defined on $\mathcal{P}$ and $\mathcal{P}'$.

Now it is easy to prove that

$$L_\alpha(pq) = p(L_\alpha q) + q(L_\alpha p) + 4xp'q', \qquad p,\ q \in \mathcal{P}, \tag{3.1}$$

$$L_\alpha(pu) = u(L_\alpha p) + p(L_\alpha u) + 4xp'u', \qquad p \in \mathcal{P},\ u \in \mathcal{P}'. \tag{3.2}$$

Lemma 3.1. *The sequence $\{u_n^{[1]}(L_\alpha)\}_{n\geq 0}$ satisfies*

$$L_{-\alpha}(u_n^{[1]}(L_\alpha)) = (n+1)\left[2(n+1)+\alpha\right] u_{n+1}, \qquad n \geq 0. \tag{3.3}$$

Proof. According to (1.2), we have

$$\begin{aligned}
\left\langle u_n^{[1]}(L_\alpha),\ P_m^{[1]}(x;L_\alpha)\right\rangle &= \delta_{n,m} \qquad n,\ m\geq 0,\\
\left\langle u_n^{[1]}(L_\alpha),\ L_\alpha(P_{m+1})\right\rangle &= (n+1)\left[2(n+1)+\alpha\right]\delta_{n,m} \qquad n,\ m\geq 0,\\
\left\langle L_{-\alpha}(u_n^{[1]}(L_\alpha), P_{m+1}\right\rangle &= (n+1)\left[2(n+1)+\alpha\right]\delta_{n,m} \qquad n,\ m\geq 0. \qquad (3.4)
\end{aligned}$$

In particular,

$$\left\langle L_{-\alpha}(u_n^{[1]}(L_\alpha)), L_{\alpha m+1}\right\rangle = 0, \qquad m\geq n+1,\ n\geq 0.$$

This implies [9, 10]

$$L_{-\alpha}(u_n^{[1]}(L_\alpha)) = \sum_{k=0}^{n+1}\mu_{n,k}\, u_k, \qquad n\geq 0$$

with $\mu_{n,k} = \left\langle L_{-\alpha}(u_n^{[1]}(L_\alpha)), P_k\right\rangle$, $0\leq k\leq n+1$. Consequently, from (3.3),we get (3.4). $\square$

Proposition 3.1. *Let $\{P_n\}_{n\geq 0}$ be a MPS, $\{P_n\}_{n\geq 0}$ is a L_α –Appell sequence if and only if its dual sequence $\{u_n\}_{n\geq 0}$ satisfies*

$$u_n = \frac{1}{n!2^n\left(1+\frac{\alpha}{2}\right)_n} L_{-\alpha}^n(u_0), \qquad n\geq 0. \tag{3.5}$$

Proof. Necessary condition, from (3.3), the sequence $\{u_n\}_{n\geq 0}$ satisfies

$$L_{-\alpha}(u_n) = (n+1)\left[2(n+1)+\alpha)\right] u_{n+1}, \qquad n\geq 0. \tag{3.6}$$

For $n=0$, we have

$$u_1 = \frac{1}{2+\alpha} L_{-\alpha}u_0.$$

By recurrence, we obtain (3.5).

Sufficient condition, from (3.5), we have (3.6) is satisfied. Comparing (3.6) with (3.3), we obtain

$$L_{-\alpha}(u_n^{[1]}(L_\alpha)) = L_{-\alpha}u_n, \qquad n \geq 0.$$

The operator L_α satisfies $L_\alpha(\mathcal{P}) = \mathcal{P}$ and $L_{-\alpha}$ is on $\mathcal{P}'$. Then $u_n^{[1]}(L_\alpha) = u_n$, $n \geq 0$. □

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