ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **11** (2022), no.12, 1313–1343
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.12.11

THE RELAXED MAXIMUM PRINCIPLE FOR G-STOCHASTIC CONTROL SYSTEMS WITH CONTROLLED JUMPS

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ABSTRACT. This paper is concerned with optimal control of systems driven by G-stochastic differential equations (G-SDEs), with controlled jump term. We study the relaxed problem, in which admissible controls are measure-valued processes and the state variable is governed by an G-SDE driven by a counting measure valued process called relaxed Poisson measure such that the compensator is a product measure. Under some conditions on the coefficients, using the G-chattering lemma, we show that the strict and the relaxed control problems have the same value function. Additionally, we derive a maximum principle for this control problem.

1. INTRODUCTION

We consider a stochastic control problem where the state variable is a solution of a SDE driven by a G-Brownian motion with jumps, the control enters both the drift and the jump term. More precisely the system evolves according to the SDE

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2020 *Mathematics Subject Classification.* 93E20, 60H07, 60H10, 60H30.
Key words and phrases. Relaxed optimal control, G-Brownian motion, sublinear expectation, stochastic control, relaxed maximum principle, jump process .
Submitted: 07.12.2022; *Accepted:* 22.12.2022; *Published:* 24.12.2022.

$$\text{(1.1)} \qquad \begin{cases} dx_t = b(t,x_t,u_t)dt + \sigma(t,x_t)dB_t + \gamma(t,x_t,u_t)d\langle B\rangle_t \\ \qquad\quad + \int_\Gamma f(t,x_{t^-},\theta,u_t)\widetilde{N}(dt,d\theta) \\ x_0 = x \end{cases},$$

on some space of sublinear expectation $(\Omega, H, \widehat{\mathbb{E}}, F^{\mathcal{P}})$, where $F^{\mathcal{P}}$ is the universal filtration, with $\mathcal{P}$ is a tight family of possibly mutually singular probability measures, and b, σ, γ, f are given deterministic functions, u is the control process. We consider here an independent Poisson random measure N; whose compensator is given by $v(d\theta)dt$.

The expected cost to be minimized over the class of admissible controls is defined by:

$$\text{(1.2)} \quad J(x;u) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[g(x_T) + \int_0^T h(t,x_t,u_t)dt\right] = \widehat{\mathbb{E}}\left[g(x_T) + \int_0^T h(t,x_t,u_t)dt\right],$$

where x is the initial condition of the process $(x_t)_{t\in[0,T]}$.

We defined then the value function V by:

$$\text{(1.3)} \qquad V(x) := \inf_{u\in\mathcal{U}} J(x;u),$$

where $\mathcal{U}$ is the set of admissible controls, a control process that verify (1.3) is called optimal.

In the recent years the framework of G-expectation has found increasing application in the domain of finance and economics, e.g., Epstein and Ji [16, 17] study the asset pricing with ambiguity preferences, Beissner [5] who studies the equilibrium theory with ambiguous volatility, and many others see e.g [6, 48, 49], also see [25–27]. for numerical methods. The motivation is that many systems are subject to model uncertainty or ambiguity due to incomplete information, or vague concepts and principles. Aspects of model ambiguity such as volatility uncertainty have been studied by Peng (2007, 2008, 2010, [39–41]) who introduced a sublinear expectation with a process called G-Brownian motion, also by Denis and Martini [13] who suggested a structure based on quasi-sure analysis from abstract potential theory to construct a similar structure using a tight family $\mathcal{P}$ of possibly mutually singular probability measures.

The strict control problem may fail to have an optimal solution, if we don't impose some kind of convexity assumption. In this case, we must embed the space of strict controls into a larger space that has nice properties of compactness and convexity. This space is that of probability measures on A, where A is the set of values taken by the strict control. These measure valued processes are called relaxed controls. In the classical framework, the first existence result of an optimal relaxed control is proved by Fleming [18], for the SDEs with uncontrolled diffusion coefficient and no jump term. For such systems of SDEs a maximum principle has been established in [2, 3, 34]. The case where the control variable appears in the diffusion coefficient has been solved in [14]. The existence of an optimal relaxed control of SDEs, where the control variable enters in the jump term was derived by Kushner [31], also recently the work given by H. Ben Gherbal and B. Mezerdi [7] of relaxed stochastic maximum principle in optimal control of diffusions with controlled jumps, for which the state variable is governed by a SDE driven by a counting measure valued process called relaxed Poisson measure, where the existence of an optimal relaxed control and a Pontryagin maximum principle were proved.

In the G-framework, the existence of an optimal relaxed control is established In 2018, by Redjil and Choutri [42], where a stochastic differential equation is considered without jump term and an uncontrolled diffusion coefficient, the celebrate Chattering lemma was generalized in the G-framework and the existence of relaxed optimal control was proved. The same result in the case with jump term is recently proved by A. Redjil, H. Ben Gherbal and O. Kebiri [43].

In this paper, we establish a Pontryagin maximum principle for the relaxed control problem given by (2.1) and (2.2). More precisely we derive necessary conditions for optimality satisfied by an optimal control. The proof is based on the results obtained in [43], Pontryagin's maximum principle for nearly optimal strict controls and some stability results of trajectories and adjoint processes with respect to the control variable. In our case the diffusion coefficient is uncontrolled because our control set is not convex, by this the controlled case require more work and two-adjoint equation. We let this case as a future work.

The motivation of our work came from applications in finance when a jump process models the stock price where we can't estimate exactly the coefficients

of the noise. The uncertainty in the noise coefficient will produce a G-SDE with jump, if then we want to control this dynamic, this lead a controlled G-SDE with jump.

The rest of the paper is organized as follows: in the next section, we formulate the control problem, and introduce the assumptions of the model. Section 3 is devoted to the proof of the approximation and stability results. In the last section, we state and prove a maximum principle for our relaxed control problem, which is the main result of this paper.

2. Formulation of the problem

2.1. **G-Strict control problem.** We consider a control problem of systems governed by stochastic differential equations on some sublinear expectation space $(\Omega, H, \widehat{\mathbb{E}}, F^{\mathcal{P}})$, such that $F^{\mathcal{P}}$ the universal filtration defined by $F^{\mathcal{P}} = \left\{\widehat{\mathcal{F}}_t^{\mathcal{P}}\right\}_{t\geq 0}$, where $\widehat{\mathcal{F}}_t^{\mathcal{P}} := \bigcap_{\mathbb{P}\in\mathcal{P}} (\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}})$ for $t \geq 0$, such that $\mathcal{F}_t^{\mathbb{P}} := \mathcal{F}_t^{+} \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_t^{+})$ is the right continous $\mathbb{P}-$completed filtrations generated by a $G-$Brownian motion B and an independent Poisson measure N, with compensator $\nu(d\theta)dt$, and $\mathcal{N}_{\mathcal{P}} := \bigcap_{\mathbb{P}\in\mathcal{P}} \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$ the family of polar sets, where $\mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$ is the $\mathbb{P}-$negligible sets. The jumps are confined to a compact set Γ, and set

$$\widetilde{N}(dt, d\theta) = N(dt, d\theta) - \nu(d\theta)dt.$$

Consider a compact set A in $\mathbb{R}^k$ and let $\mathcal{U}$ the class of measurable, adapted processes $u : [0;T] \times \Omega \longrightarrow A$, such that $u \in M_G^2(0,T)$. For any $u \in \mathcal{U}$, we consider the following stochastic differential equation (SDE)

$$\begin{cases} dx_t = b(t, x_t, u_t)dt + \sigma(t, x_t)dB_t + \gamma(t, x_t, u_t)d\langle B\rangle_t \\ \qquad\quad + \displaystyle\int_\Gamma f(t, x_{t^-}, \theta, u_t)\widetilde{N}(dt, d\theta) \\ x_0 = x, \end{cases} \tag{2.1}$$

where

$$b : [0;T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n$$
$$\sigma : [0;T] \times \mathbb{R}^n \longrightarrow \mathcal{M}_{n\times d}(\mathbb{R})$$

$$\gamma : [0;T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^{n\times n}$$
$$f : [0;T] \times \mathbb{R}^n \times \Gamma \times A \longrightarrow \mathbb{R}^n$$

are bounded, measurable and continuous functions.

The expected cost is given by

$$J(u) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[g(x_T) + \int_0^T h(t,x_t,u_t)dt\right] = \widehat{\mathbb{E}}\left[g(x_T) + \int_0^T h(t,x_t,u_t)dt\right], \tag{2.2}$$

where,

$$g : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$h : [0;T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R},$$

be bounded and continuous functions.

The problem is to minimize the functional $J(.)$ over $\mathcal{U}$. A control that solves this problem is called optimal.

We note that from the result of [35], equation (2.1) has a unique solution, under these assumptions (A):

(A1) Let be b, σ, γ and f bounded and lipschitz continuous with respect to the state variable x uniformly in (t,u), also we suppose that $\gamma(t,x,.)$ is a symmetric $d\times d$ matrix with each element.

(A2) For all $(t,x,\theta) \in [0,T]\times\mathbb{R}^n\times\Gamma$ the functions $b(t,x,.)$, $f(t,x,\theta,.)$ and $\gamma(t,x,.)$ are continuous in $u\in\mathcal{U}$.

(A3) $b(.,x,.)$, $\gamma(.,x,.)$ and $\sigma(.,x)$ taking value in $M_G^2(0,T)$ and $f(.,x,.,.)$ takes value in $\hat{H}_G^2(0,T)$.

(A4) The functions g and $h(.,x,.)$ are taking value in $M_G^2(0,T)$ and bounded. Moreover we suppose that g is lipschitz continuous, and h is lipschitz continuous with respect to the state variable x uniformly in time and control (t,u).

2.2. **The G-relaxed control problem.** Let (A,d) be a separable metric space and $\mathcal{P}(A)$ be the space of probability measures on the set A endowed with its Borel σ-algebra $\mathcal{B}(A)$. The class $M([0,T]\times A)$ of relaxed controls we consider in this paper is a subset of the set $M([0,T]\times A)$ of Radon measures $\nu(dt,da)$ on $[0,T]\times A$ equipped with the topology of stable convergence of measures, whose projections

on $[0,T]$ coincide with the Lebesgue measure dt, and whose projection on A coincide with some probability measure $\mu_t(da) \in \mathcal{P}(A)$ i.e. $\nu(da,dt) := \mu_t(da)dt$. The topology of stable convergence of measures is the coarsest topology which makes the mapping

$$q \mapsto \int_0^T \int_A \varphi(t,a)q(dt,da)$$

continuous, for all bounded measurable functions $\varphi(t,a)$ such that for fixed t, $\varphi(t,\cdot)$ is continuous. Equipped with this topology, $M := M([0,T]\times A)$ is a separable metrizable space. Moreover, it is compact whenever A is compact. The topology of stable convergence of measures implies the topology of its weak convergence. For further details see [14, 15].

Now we present the following definitions:

Definition 2.1. *Lip(Ω) is the set of random variables of the form*

$$\xi := \varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$$

for some bounded Lipschitz continuous function ϕ on $\mathbb{R}^{d\times n}$ and $0 \le t_1 \le t_2 \le \cdots \le t_n \le T$. The coordinate process $(B_t,\ t \ge 0)$ is called G-Brownian motion whenever B_1 is G-normally distributed under $\widehat{\mathbb{E}}\,[\cdot]$ and for each $s,t \ge 0$ and $t_1,t_2,\ldots,t_n \in [0,t]$ we have

$$\widehat{\mathbb{E}}[\varphi(B_{t_1},\ldots,B_{t_n},B_{t+s}-B_t)] = \widehat{\mathbb{E}}[\psi(B_{t_1},\ldots,B_{t_n})],$$

where $\psi(x_1,\ldots,x_n) = \widehat{\mathbb{E}}[\varphi(x_1,\ldots,x_n,\sqrt{s}B_1)]$. This property implies that the increments of the G-Brownian motion are independent and that $B_{t+s}-B_t$ and B_s are $N(0,s\Sigma)$-distributed.

Next, we introduce the class of relaxed stochastic controls on $(\Omega_T,\mathcal{H},\widehat{\mathbb{E}})$, where $\mathcal{H}$ is a vector lattice of real functions on Ω such that $\mathrm{Lip}(\Omega_T) \subset \mathcal{H}$.

Definition 2.2. *A relaxed stochastic control on $(\Omega_T, Lip(\Omega_T),\widehat{\mathbb{E}})$ is a random measure $q(\omega,dt,da) = \mu_t(\omega,da)dt$ such that for each subset $C \in \mathcal{B}(A)$, the process $(\mu_t(C))_{t\in[0,T]}$ is $F^{\mathcal{P}}$-progressively measurable i.e. for every $t \in [0,T]$, the mapping $[0,t]\times\Omega \to [0,1]$ defined by $(s,\omega) \mapsto \mu_s(\omega,A)$ is $B([0,t])\otimes\widehat{\mathcal{F}}_t^{\mathcal{P}}$-measurable. In particular, the process $(\mu_t(C))_{t\in[0,T]}$ is adapted to the universal filtration $F^{\mathcal{P}}$. We denote by $\mathcal{R}$ the class of relaxed stochastic controls.*

The set $\mathcal{U}([0,T])$ of strict controls constituted of $F^{\mathcal{P}}$-adapted processes u taking values in the set A, embeds into the set $\mathcal{R}$ of relaxed controls through the

mapping:

$$\Phi : \quad \mathcal{U}([0,T]) \ni u \mapsto \Phi(u)(dt,da) = \delta_{u_t}(da)dt \in \mathcal{R}.$$

Definition 2.3. *Let μ a relaxed representation of an admissible control u , for each $\Gamma_0 \subset \Gamma$, Γ_0 is a Borel set $(\Gamma_0 \in B(\Gamma))$ and $A_0 \subset A$, $(A_0 \in \mathcal{B}(A))$, we define:*

$$N^{\mu}([0,t], A_0, \Gamma_0) \underset{\text{we denote by}}{:=} N^{\mu}(t, A_0, \Gamma_0) = \int_0^t \int_{\Gamma_0} 1_{A_0}(u_s) N(ds, d\theta).$$

N^{μ} is the number of jumps of $\int_0^t \int_{\Gamma_0} \theta N(ds, d\theta)$ on $[0,t]$ with values in Γ_0 and where $u_s \in A_0$ at the jump times s.

Since $1_{A_0}(u_s) = \mu_s(A_0)$; then the compensator of the counting measure valued process N^{μ} is

$$v(d\theta)\mu_t(da)dt = \mu_t \otimes v(da, d\theta).$$

Definition 2.4. *A relaxed Poisson measure N^{μ} is a counting measure valued process such that its compensator is the product measure of the relaxed control μ with the compensator v of N such that for each $\Gamma_0 \subset \Gamma$, Γ_0 is a Borel set $(\Gamma_0 \in \mathcal{B}(\Gamma))$ and $A_0 \subset A$, $(A_0 \in \mathcal{B}(A))$, the processes*

$$Z^{\mu} = \widetilde{N}^{\mu}(t, A_0, \Gamma_0) = N^{\mu}(t, A_0, \Gamma_0) - \mu(t, A_0)\nu(\Gamma_0)$$

are $\widehat{\mathcal{F}}_t^{\mathcal{P}}-$martingales and orthogonal for disjoint $\Gamma_0 \times A_0$, because according to [7], the processes Z^{μ} are $\mathcal{F}_t^{\mathbb{P}}$- martingales for each $\mathbb{P} \in \mathcal{P}$, and so is an $\widehat{\mathcal{F}}_t^{\mathcal{P}}$-martingale, also are orthogonal for disjoint $\Gamma_0 \times A_0$.

Proposition 2.1. *For any bounded measurable function φ with real values, the process Y given by:*

$$Y_t = \int_0^t \int_{\Gamma} \int_A \varphi(s, x_{s^-}, \theta, a) N^{\mu}(dt, d\theta, da) - \int_0^t \int_{\Gamma} \int_A \varphi(s, x_{s^-}, \theta, a) v(d\theta)\mu_s(da)ds$$

is an $\widehat{\mathcal{F}}_t^{\mathcal{P}}-$martingale.

Proof. By the definition of $G-$martingale [41] Y is an $\widehat{\mathcal{F}}_t^{\mathcal{P}}-$martingale means that Y is an $\mathcal{F}_t^{\mathbb{P}}$-supermartingale for each probability sufficiently. And this is verified in our case because from H. Ben Gherbal and B. Mezerdi [7] the process Y is an $\mathcal{F}_t^{\mathbb{P}}$-martingale for each $\mathbb{P} \in \mathcal{P}$. □

Proposition 2.2. *Consider a sequence of $(\mu_s^n \otimes \nu)_n$ converging weakly to $\mu_s \otimes \nu$ on $\Omega \times [0,T] \times A \times \Gamma$, there exists a sequence of orthogonal martingale measures $\widetilde{N}^n$*

defined on $\Omega \times [0,T] \times A \times \Gamma$, *such that for each bounded function* φ*:*

$$\int_0^t \int_A \int_\Gamma \varphi(s, X^\mu_{s^-}, \theta, a)\widetilde{N}^n(ds, d\theta, da)$$
$$\underset{n\to\infty}{\longrightarrow} \int_0^t \int_A \int_\Gamma \varphi(s, X^\mu_{s^-}, \theta, a)\widetilde{N}^\mu(ds, d\theta, da) \textit{ quasi-surely}.$$

Proof. Given a fixed probability measure, we have from [7]

$$\mathbb{P} \in \mathcal{P} \Longrightarrow \int_0^t \int_A \int_\Gamma \varphi(s, x^\mu_{s^-}, \theta, a)\widetilde{N}^n(ds, d\theta, da)$$
$$\to \int_0^t \int_A \int_\Gamma \varphi(s, x^\mu_{s^-}, \theta, a)\widetilde{N}^\mu(ds, d\theta, da)\ \mathbb{P}\text{-surely},$$

this means that we have the convergence outside a polar set, which means that we have quasi surely convergence. □

Now we present our relaxed controlled system, the G-SDE with controlled jumps in terms of relaxed Poisson measure is given by:

$$\text{(2.3)} \quad \begin{cases} dx^\mu_t = \int_A b(t, x^\mu_t, a)\mu_t(da)dt + \sigma(t, x^\mu_t)dB_t + \int_A \gamma(t, x^\mu_t, a)\mu_t(da)d\langle B\rangle_t \\ \qquad + \int_A \int_\Gamma f(t, x^\mu_{t^-}, \theta, a)\widetilde{N}^\mu(dt, d\theta, da) \\ x^\mu_0 = 0. \end{cases}$$

The cost functional is given by:

$$J(\mu) = \widehat{\mathbb{E}}\left[\int_0^T \int_A h(t, x^\mu_t, a)\mu_t(da)\, dt + g(x^\mu_T)\right].$$

3. Approximation of trajectories and stability results

The next lemma, which called G-chattering lemma gives the approximation of a relaxed control by a sequence of strict controls order for the relaxed control problem. This result is considered essential in showing that the relaxed control problem is a truly an extension of the strict one. We refer to [43] to more detail of this subsection.

Lemma 3.1 (see [42])**.** *Let* (A, d) *be a separable metric space and assume that* A *is a compact set. Let* $(\mu_t)_t$ *be an* $F^{\mathcal{P}}$*-progressively measurable process with values in* $P(A)$*. Then there exists a sequence* $(u^n_t)_{n\geq 0}$ *of* $F^{\mathcal{P}}$*-progressively measurable processes*

with values in A such that the sequence of random measures $\delta_{u_t^n}(da)dt$ converges in the sense of stable convergence (thus, weakly) to $\mu_t(da)dt$ quasi-surely:

$$\mu_t^n(da)dt = \delta_{u_t^n}(da)dt \longrightarrow \mu_t(da)dt \quad \textit{quasi-surely.}$$

Lemma 3.2 (See [43]). *Under our assumption (A), for every $\mathbb{P} \in \mathcal{P}$, it holds that*

(1)

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le T} |x_t^n - x_t^\mu|^2\right] = 0, \tag{3.1}$$

and

$$\lim_{n\to\infty} J^{\mathbb{P}}(u^n) = J^{\mathbb{P}}(\mu). \tag{3.2}$$

(2) *Moreover,*

$$\inf_{u\in\mathcal{U}} J^{\mathbb{P}}(u) = \inf_{\mu\in\mathcal{R}} J^{\mathbb{P}}(\mu), \tag{3.3}$$

and there exists a relaxed control $\hat{\mu}_{\mathbb{P}} \in \mathcal{R}$ such that $J^{\mathbb{P}}(\hat{\mu}_{\mathbb{P}}) = \inf_{\mu\in\mathcal{R}} J^{\mathbb{P}}(\mu)$.

The next theorem gives the stability of the stochastic differential equations with respect to the control variable, and that the two problems has the same infimum of the expected costs.

Theorem 3.1 (See [43].). *Under our assumption (A) we have:*

(1) *Let μ be a relaxed control and let x^μ the corresponding trajectory. Then there exists a sequence (u^n) of strict controls such that:*

$$\lim_{n\longrightarrow\infty} \widehat{\mathbb{E}}\left[\sup_{0\le t\le T} |x_t^n - x_t^\mu|^2\right] = 0,$$

where x_t^n denotes the trajectory associated to u^n.

(2) *Let $J(u^n)$ and $J(\mu)$ be the cost functional corresponding respectively to u^n and μ (where $dt\delta_{u^n}(t)(da)$ converges weakly to $dt\mu_t(da)$ quasi-surely). Then, there exists a subsequence (u^{n_k}) of (u^n) such that*

$$\lim_{k\longrightarrow\infty} J(u^{n_k}) = J(\mu).$$

4. Maximum principle for relaxed control problems

Our main goal in this section is to establish optimality necessary conditions for relaxed control problem, where the system is described by a G-SDE driven by a relaxed Poisson measure. The proof is based on the G-chattering lemma, we derive necessary conditions of near optimality satisfied by a sequence of strict controls. By using stability properties of the state equations and adjoint processes, we obtain the maximum principle for our relaxed problem.

4.1. **The maximum principle for strict control.**

Under the above hypothesis, (2.1) has a unique strong solution and the cost functional (2.2) is well defined from $\mathcal{U}$ into $\mathbb{R}$, for more details see [43]. The purpose of this subsection is to derive optimality necessary conditions satisfied by an optimal strict control. The proof is based on the strong perturbation of the optimal control u^*, which defined by:

$$u_t^h = \begin{cases} \nu_t & \text{if } t \in [t_0; t_0 + h] \\ u_t^* & \text{otherwise,} \end{cases}$$

where $0 \leq t_0 < T$ is fixed, h is sufficiently small, and ν is an arbitrary $A-$valued $\mathcal{F}_{t_0}^{\mathbb{P}}-$measurable random such that under every $\mathbb{P} \in \mathcal{P}$, $\mathbb{E}^{\mathbb{P}} |\nu|^2 < \infty$. Let x_t^h denotes the trajectory associated with u_t^h, then

$$\begin{cases} x_t^h = x_t^* \quad ; \ t \leq t_0 \\ dx_t^h = b(t, x_t^h, \nu_t)dt + \sigma(t, x_t^h)dB_t + \gamma(t, x_t^h, \nu_t)d\langle B\rangle_t \\ \qquad + \displaystyle\int_\Gamma f(t, x_{t^-}^h, \theta, \nu_t)\widetilde{N}(dt, d\theta) \quad ; t_0 < t < t_0 + h \\ dx_t^h = b(t, x_t^h, u_t^*)dt + \sigma(t, x_t^h)dB_t + \gamma(t, x_t^h, u_t^*)d\langle B\rangle_t \\ \qquad + \displaystyle\int_\Gamma f(t, x_{t^-}^h, \theta, u_t^*)\widetilde{N}(dt, d\theta) \quad ; t_0 + h < t < T. \end{cases}$$

We first have

Lemma 4.1. *Under assumptions (A_1)-(A_2), we have for every $\mathbb{P} \in \mathcal{P}$, it holds that*

$$\lim_{h \to 0} \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} \left| x_t^h - x_t^* \right|^2 \right] = 0, \tag{4.1}$$

and,

$$\lim_{h\to 0}\widehat{\mathbb{E}}\left[\sup_{t\in[t_0;T]}\left|x_t^h-x_t^*\right|^2\right]=0. \tag{4.2}$$

Proof. Under every $\mathbb{P}\in\mathcal{P}$, the G-SDEs (2.1) and (2.3) becomes standard SDEs driven by a standard Brownian motion B and a Poisson measure $\widetilde{N}$, the proof of (4.1) follows from H. Ben Gherbal and B. Mezerdi [7], we sketch it here. Using the fact that under $\mathbb{P}\in\mathcal{P}$, $\widetilde{N}$ is a martingale and B is a continuous martingale whose quadratic variation process $\langle B\rangle$ is such that $\pi_t=\frac{d\langle B\rangle_t}{dt}$ is bounded by a deterministic $d\times d$ symmetric positive definite matrix $\overline{\sigma}$, and x^h satisfy

$$dx_t^h=b(t,x_t^h,\nu_t)dt+\sigma(t,x_t^h)dB_t+\pi_t\gamma(t,x_t^h,\nu_t)dt+\int_\Gamma f(t,x_{t^-}^h,\theta,\nu_t)\widetilde{N}(dt,d\theta),$$

then the result gives by a standard arguments from stochastic calculus, for more detail see [7].

For the second limit, set

$$\varsigma_h=\sup_{t\in[t_0;T]}\left|x_t^h-x_t^*\right|^2,$$

if there is a $\theta>0$ such that $\widehat{\mathbb{E}}\left[\varsigma_h\right]\geq\theta$, we can find a probability $\mathbb{P}\in\mathcal{P}$ such that $\widehat{\mathbb{E}}\left[\varsigma_h\right]\geq\theta-\varepsilon;\,\varepsilon\to 0$. Since $\mathcal{P}$ is weakly compact, there exists a subsequence $\left(\mathbb{P}_{n_k}\right)_{k\geq 1}$ that converges weakly to some $\mathbb{P}\in\mathcal{P}$, hence

$$\lim_{h\to 0}\mathbb{E}^{\mathbb{P}}\left[\varsigma_h\right]=\lim_{h\to 0}\lim_{k\to\infty}\mathbb{E}^{\mathbb{P}_{n_k}}\left[\varsigma_h\right]\geq\lim\inf_{k\to\infty}\mathbb{E}^{\mathbb{P}_{n_k}}\left[\varsigma_h\right]\geq\theta.$$

This contradicts (4.1). This complete the proof. □

Since u^* is optimal, then

$$J(u^*)\leq J(u^h)=J(u^*)+h\frac{dJ(u^h)}{dh}\bigg|_{h=0}+\circ(h).$$

Thus, a necessary condition for optimality is that

$$\frac{dJ(u^h)}{dh}\bigg|_{h=0}\geq 0.$$

Note that under every $\mathbb{P} \in \mathcal{P}$, the following properties hold, because $b(t,x,u)$, $h(t,x,u)$, $\gamma(t,x,u)$ and $f(t,x_{t^-},\theta,u)$ are sufficiently integrable.

$$\frac{1}{h}\int_t^{t+h} \mathbb{E}^{\mathbb{P}}\left[|k(s,x_s,u_s)-k(t,x_t,u_t)|^2\right] \xrightarrow{h\to 0} 0 \; dt - a.e \tag{4.3}$$

$$\frac{1}{h}\int_t^{t+h} \mathbb{E}^{\mathbb{P}}\left[|\gamma(s,x_s,u_s)-\gamma(t,x_t,u_t)|^2\right] \xrightarrow{h\to 0} 0 \; d\langle B\rangle_t - a.e \tag{4.4}$$

$$\frac{1}{h}\int_\Gamma\int_t^{t+h} \mathbb{E}^{\mathbb{P}}\left[|f(s,x_{s^-},\theta,u_s)-f(t,x_{t^-},\theta,u_t)|^2\right] \upsilon(d\theta) \xrightarrow{h\to 0} 0 \; dt - a.e, \tag{4.5}$$

where k stands for b or h.

Lemma 4.2. *Under assumptions (A_1)-(A_3), it holds that*

$$\lim_{h\to 0} \widehat{\mathbb{E}}\left[\left|\frac{x_t^h - x_t^*}{h} - z_t\right|^2\right] = 0.$$

Proof. We proceed as in H. Ben Gherbal and B. Mezerdi [7], let

$$y_t^h = \frac{x_t^h - x_t^*}{h} - z_t.$$

Then, we have for $t \in [t_0; t_0+h]$

$$\begin{cases} dy_t^h = \frac{1}{h}\left[b(t,x_t^*+h(y_t^h+z_t),\nu_t) - b(t,x_t^*,u_t^*) - hb_x(t,x_t^*,u_t^*)z_t\right]dt \\ \quad +\frac{1}{h}\left[\sigma(t,x_t^*+h(y_t^h+z_t)) - \sigma(t,x_t^*) - h\sigma_x(t,x_t^*)z_t\right]dB_t \\ \quad +\frac{1}{h}\left[\gamma(t,x_t^*+h(y_t^h+z_t),\nu_t) - \gamma(t,x_t^*,u_t^*) - h\gamma_x(t,x_t^*,u_t^*)z_t\right]d\langle B\rangle_t \\ \quad +\frac{1}{h}\int_\Gamma\left[f(t,x_{t^-}^*+h(y_{t^-}^h+z_{t^-}),\nu_t) - f(t,x_{t^-}^*,u_t^*) - hf_x(t,x_{t^-}^*,u_t^*)z_{t^-}\right]\widetilde{N}(dt,d\theta) \\ y_{t_0}^h = -\left[b(t_0,x_{t_0}^*,\nu_{t_0}) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right]. \end{cases}$$

Hence,

$$y_{t_0+h}^h = \frac{1}{h}\int_{t_0}^{t_0+h}\left[b(t,x_t^*+h(y_t^h+z_t),\nu_t) - b(t,x_t^*,\nu_t)\right]dt$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\left[b(t,x_t^*,\nu_t)-b(t,x_{t_0}^*,\nu_t)\right]dt$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\left[b(t,x_{t_0}^*,\nu_t)-b(t_0,x_{t_0}^*,\nu_t)\right]dt+\frac{1}{h}\int_{t_0}^{t_0+h}\left[b(t_0,x_{t_0}^*,u_{t_0}^*)-b(t,x_t^*,u_t^*)\right]dt$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\left[\sigma(t,x_t^*+h(y_t^h+z_t))-\sigma(t,x_t^*)\right]dB_t$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\left[\gamma(t,x_t^*+h(y_t^h+z_t),\nu_t)-\gamma(t,x_t^*,\nu_t)\right]d\langle B\rangle_t$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\left[\gamma(t,x_t^*,\nu_t)-\gamma(t,x_{t_0}^*,\nu_t)\right]d\langle B\rangle_t$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\left[\gamma(t,x_{t_0}^*,\nu_t)-\gamma(t_0,x_{t_0}^*,\nu_t)\right]d\langle B\rangle_t$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\left[\gamma(t_0,x_{t_0}^*,u_{t_0}^*)-\gamma(t,x_t^*,u_t^*)\right]d\langle B\rangle_t$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\int_{\Gamma}\left[f(t,x_{t^-}^*+h(y_{t^-}^h+z_{t^-}),\theta,\nu_t)-f(t,x_{t^-}^*,\theta,\nu_t)\right]\widetilde{N}(dt,d\theta)$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\int_{\Gamma}\left[f(t,x_{t^-}^*,\theta,\nu_t)-f(t,x_{t_0^-}^*,\theta,\nu_t)\right]\widetilde{N}(dt,d\theta)$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\int_{\Gamma}\left[f(t,x_{t_0^-}^*,\theta,\nu_t)-f(t_0,x_{t_0^-}^*,\theta,\nu_t)\right]\widetilde{N}(dt,d\theta)$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\int_{\Gamma}\left[f(t_0,x_{t_0^-}^*,\theta,\nu_t)-f(t_0,x_{t_0^-}^*,\theta,u_{t_0}^*)\right]\widetilde{N}(dt,d\theta)$$

$$+\frac{1}{h}\int_{t_0}^{t_0+h}\int_{\Gamma}\left[f(t_0,x^*_{t_0^-},\theta,u^*_{t_0})-f(t,x^*_{t^-},\theta,u^*_t)\right]\widetilde{N}(dt,d\theta)$$
$$-\int_{t_0}^{t_0+h} b_x(t,x^*_t,u^*_t)z_t dt-\int_{t_0}^{t_0+h}\gamma_x(t,x^*_t,u^*_t)z_t d\langle B\rangle_t$$
$$-\int_{t_0}^{t_0+h}\sigma_x(t,x^*_t)z_t dB_t-\int_{t_0}^{t_0+h}\int_{\Gamma} f_x(t,x^*_{t^-},\theta,u^*_t)z_t\widetilde{N}(dt,d\theta).$$

Then, under every $\mathbb{P}\in\mathcal{P}$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left|y^h_{t_0+h}\right|^2 \le C\Bigg[&\mathbb{E}^{\mathbb{P}}\sup_{t_0\le t\le t_0+h}\left|x^h_t-x^*_t\right|^2\\
&+\sup_{t_0\le t\le t_0+h}\mathbb{E}^{\mathbb{P}}\left|b(t,x^*_{t_0},\nu_{t_0})-b(t_0,x^*_{t_0},\nu_{t_0})\right|^2 dt\\
&+\tfrac{1}{h}\mathbb{E}^{\mathbb{P}}\int_{t_0}^{t_0+h}\left|b(t_0,x^*_{t_0},u^*_{t_0})-b(t,x^*_t,u^*_t)\right|^2 dt+\mathbb{E}^{\mathbb{P}}\sup_{t_0\le t\le t_0+h}\left|x^*_t-x^*_{t_0}\right|^2\\
&+\sup_{t_0\le t\le t_0+h}\mathbb{E}^{\mathbb{P}}\left|\gamma(t,x^*_{t_0},\nu_{t_0})-\gamma(t_0,x^*_{t_0},\nu_{t_0})\right|^2 d\langle B\rangle_t\\
&+\tfrac{1}{h}\mathbb{E}^{\mathbb{P}}\int_{t_0}^{t_0+h}\left|\gamma(t_0,x^*_{t_0},u^*_{t_0})-\gamma(t,x^*_t,u^*_t)\right|^2 d\langle B\rangle_t\\
&+\mathbb{E}^{\mathbb{P}}\int_{t_0}^{t_0+h}\int_{\Gamma}\left|\nu-u^*_{t_0}\right|^2\upsilon(d\theta)dt+\mathbb{E}^{\mathbb{P}}\int_{t_0}^{t_0+h}|z_t|^2 dt\\
&+\sup_{t_0\le t\le t_0+h}\mathbb{E}^{\mathbb{P}}\int_{\Gamma}\left|f(t,x^*_{t_0^-},\theta,\nu_{t_0})-f(t_0,x^*_{t_0^-},\theta,\nu_{t_0})\right|^2\upsilon(d\theta)\\
&+\tfrac{1}{h}\mathbb{E}^{\mathbb{P}}\int_{t_0}^{t_0+h}\int_{\Gamma}\left|f(t_0,x^*_{t_0^-},\theta,u^*_{t_0})-f(t,x^*_{t^-},\theta,u^*_t)\right|^2\upsilon(d\theta)dt\Bigg].
\end{aligned}\tag{4.6}$$

By lemma (4.1), and the properties (4.3), (4.4) and (4.5), it is easy to see that for each $\mathbb{P}\in\mathcal{P}$, $\mathbb{E}^{\mathbb{P}}\left|y^h_{t_0+h}\right|^2$ tends to 0 as $h\to 0$.

Finally, we deduce that $\widehat{\mathbb{E}}\left|y^h_{t_0+h}\right|^2$ tends to 0 as $h\to 0$ by the same way as in the proof of lemma (4.1). For $t\in[t_0+h;T]$, we denote $x^{h,\lambda}_t=x^*_t+\lambda h(y^h_t+z_t)$, then

y_t^h satisfies the following SDE:

$$\begin{aligned}
dy_t^h &= \frac{1}{h}\left[b(t,x_t^*+h(y_t^h+z_t),u_t^*)-b(t,x_t^*,u_t^*)\right]dt\\
&+\frac{1}{h}\left[\sigma(t,x_t^*+h(y_t^h+z_t))-\sigma(t,x_t^*)\right]dB_t\\
&+\frac{1}{h}\left[\gamma(t,x_t^*+h(y_t^h+z_t),u_t^*)-\gamma(t,x_t^*,u_t^*)\right]d\langle B\rangle_t\\
&+\frac{1}{h}\int_\Gamma\left[f(t,x_{t^-}^*+h(y_{t^-}^h+z_{t^-}),\theta,u_t^*)-f(t,x_{t^-}^*,\theta,u_t^*)\right]\widetilde{N}(dt,d\theta)\\
&-b_x(t,x_t^*,u_t^*)z_tdt-\sigma_x(t,x_t^*)z_tdB_t-\int_\Gamma f_x(t,x_{t^-}^*,\theta,u_t^*)z_t\widetilde{N}(dt,d\theta),
\end{aligned}$$

then,

$$\begin{aligned}
y_t^h &= y_{t_0+h}^h+\int_{t_0+h}^t\int_0^1 b_x(s,x_s^{h,\lambda},u_s^*)y_s^hd\lambda ds+\int_{t_0+h}^t\int_0^1\sigma_x(s,x_s^{h,\lambda})y_s^hd\lambda dB_s\\
&+\int_{t_0+h}^t\int_0^1\gamma_x(s,x_s^{h,\lambda},u_s^*)y_s^hd\lambda d\langle B\rangle_s+\int_0^1\int_{t_0+h}^t\int_\Gamma f_x(s,x_s^{h,\lambda},\theta,u_s^*)y_s^hd\lambda\widetilde{N}(ds,d\theta)+\rho_t^h,
\end{aligned}$$

where,

$$\begin{aligned}
\rho_t^h &= \int_{t_0+h}^t\int_0^1 b_x(s,x_s^{h,\lambda},u_s^*)z_sd\lambda ds+\int_{t_0+h}^t\int_0^1\sigma_x(s,x_s^{h,\lambda})z_sd\lambda dB_s\\
&+\int_{t_0+h}^t\int_0^1\gamma_x(s,x_s^{h,\lambda},u_s^*)z_sd\lambda d\langle B\rangle_s+\int_{t_0+h}^t\int_0^1\int_\Gamma f_x(s,x_s^{h,\lambda},\theta,u_s^*)z_sd\lambda\widetilde{N}(ds,d\theta)\\
&-\int_{t_0+h}^t b_x(s,x_s^*,u_s^*)z_sds-\int_{t_0+h}^t\gamma_x(s,x_s^*,u_s^*)z_sd\langle B\rangle_s-\int_{t_0+h}^t\sigma_x(s,x_s^*)z_sdB_s\\
&-\int_{t_0+h}^t\int_\Gamma f_x(s,x_{s^-}^*,\theta,u_s^*)z_s\widetilde{N}(ds,d\theta),\ .
\end{aligned}$$

Hence, under every $\mathbb{P} \in \mathcal{P}$, we have

$$\mathbb{E}^{\mathbb{P}} \left|y_t^h\right|^2 \leq \mathbb{E}^{\mathbb{P}} \left|y_{t_0+h}^h\right|^2 + K\mathbb{E}^{\mathbb{P}} \int_{t_0+h}^{t} \left| \int_0^1 b_x(s, x_s^{h,\lambda}, u_s^h) y_s^h d\lambda \right|^2 ds$$
$$+K\mathbb{E}^{\mathbb{P}} \int_{t_0+h}^{t} \left| \int_0^1 \sigma_x(s, x_s^{h,\lambda}) y_s^h d\lambda \right|^2 ds + K\mathbb{E}^{\mathbb{P}} \int_{t_0+h}^{t} \left| \int_0^1 \gamma_x(s, x_s^{h,\lambda}, u_s^h) y_s^h d\lambda \right|^2 d\langle B\rangle_s$$
$$+K\mathbb{E}^{\mathbb{P}} \int_{t_0+h}^{t} \int_{\Gamma} \left| \int_0^1 f_x(s, x_s^{h,\lambda}, \theta, u_s^h) y_s^h d\lambda \right|^2 \upsilon(d\theta) ds + K\mathbb{E}^{\mathbb{P}} \left|\rho_t^h\right|^2 .$$

Since b_x, σ_x, γ_x and f_x are bounded, then

$$\mathbb{E}^{\mathbb{P}} \left|y_t^h\right|^2 \leq \mathbb{E}^{\mathbb{P}} \left|y_{t_0+h}^h\right|^2 + C\mathbb{E}^{\mathbb{P}} \int_0^t \left|y_s^h\right|^2 ds + K\mathbb{E}^{\mathbb{P}} \left|\rho_t^h\right|^2 .$$

We conclude by the continuity of b_x, σ_x, γ_x and f_x, and the dominated convergence that $\lim_{h\to 0} \rho_t^h = 0$. Hence by the Gronwall lemma, and (4.6) we get

$$\lim_{h\to 0} \sup_{t_0+h\leq t\leq T} \mathbb{E}^{\mathbb{P}} \left|y_t^h\right|^2 = 0.$$

Finally, we deduce that $\widehat{\mathbb{E}} \left|y_t^h\right|^2$ tends to 0 as $h \to 0$ by the same way as in the proof of lemma (4.1).

The second estimate is proved in a similar way. □

Choose t_0 such that (4.3), (4.4) and (4.5) holds, then we have

Corollary 4.1. *Under assumptions (A_1)-(A_3), one has*

$$0 \leq \left.\frac{dJ(u^h)}{dh}\right|_{h=0} \leq \widehat{\mathbb{E}} \left[g_x(x_T^*) z_T + \int_0^T h_x(t, x_t^*, u_t^*) z_t dt \right], \tag{4.7}$$

where the process z is the solution of the linear SDE

$$\begin{cases} dz_t = b_x(t, x_t^*, u_t^*) z_t dt + \sigma_x(t, x_t^*) z_t dB_t + \gamma_x(t, x_t^*, u_t^*) z_t d\langle B\rangle_t \\ \quad + \int_{\Gamma} f_x(t, x_{t^-}^*, \theta, u_t^*) z_{t^-} \widetilde{N}(dt, d\theta);\ t_0 \leq t \leq T \\ z_{t_0} = \left[b(t_0, x_{t_0}^*, \nu_{t_0}) - b(t_0, x_{t_0}^*, u_{t_0}^*) \right]. \end{cases} \tag{4.8}$$

We use the same notations as in the proof of lemma (4.2), to prove this corollary.

proof . We have by the definition of J that

$$\frac{1}{h}\left[J(u^h)-J(u^*)\right] \leq \frac{1}{h}\widehat{\mathbb{E}}\left[\int_{t_0}^{T} h(t,x_t^h,u_t^h)dt+g(x_T^h)\right]-\widehat{\mathbb{E}}\left[\int_{t_0}^{T} h(t,x_t^*,u_t^*)dt+g(x_T^*)\right],$$

then,

$$\begin{aligned}0 \leq \tfrac{1}{h}\left[J(u^h)-J(u^*)\right] \leq \widehat{\mathbb{E}}\Bigg[&\int_0^1 g_x(x_T^{h,\lambda})z_T d\lambda+\tfrac{1}{h}\int_0^T\int_0^1 h_x(t,x_t^{h,\lambda},u_t^h)z_t^h d\lambda dt\\ &+\int_0^T\int_0^1 h_u(t,x_t^*,u_t^{h,\lambda})u_t d\lambda dt\Bigg].\end{aligned}$$

From lemma (4.2), we obtain (4.7) by letting h tend to 0. □

Let us introduce the adjoint process, which is a G-backward stochastic differential equation (G-BSDE in short). We proceed as in [8], [47] and [7].

By the integration by parts formula, we can see that the solution of dz_t is given by $z_t=\varphi_t\eta_t$ where

$$\begin{cases} d\varphi(t,\tau)=b_x(t,x_t^*,u_t^*)\varphi(t,\tau)dt+\sigma_x(t,x_t^*)\varphi(t,\tau)dB_t\\ \quad+\int_\Gamma f_x(t,x_{t^-}^*,\theta,u_t^*)\varphi(t^-,\tau)\widetilde{N}(dt,d\theta)+\gamma_x(t,x_t^*,u_t^*)d\langle B\rangle_t \quad 0\leq\tau\leq t\leq T,\\ \varphi(\tau,\tau)=I_d,\end{cases}$$

and

$$\begin{cases} d\eta_t=\psi_t\left\{b_u(t,x_t^*,u_t^*)u_t-\int_\Gamma f_u(t,x_{t^-}^*,\theta,u_t^*)u_t\upsilon(d\theta)\right\}dt\\ \quad-\psi_{t^-}\int_\Gamma\left(f_x(t,x_{t^-}^*,\theta,u_t^*)+I_d\right)^{-1}f_u(t,x_{t^-}^*,\theta,u_t^*)u_tN(dt,d\theta)\\ \quad+\psi_t\gamma_u(t,x_t^*,u_t^*)u_td\langle B\rangle_t\\ \eta_0=0,\end{cases}$$

with ψ_t is the inverse of φ satisfying suitable integrability conditions, and it is the solution of the following equation

$$\begin{cases} d\psi(t,\tau) = \{\sigma_x(t,x_t^*)\psi(t,\tau)\sigma_x(t,x_t^*) - b_x(t,x_t^*,u_t^*)\psi(t,\tau) \\ \quad -\int_\Gamma f_x(t,x_{t^-}^*,\theta,u_t^*)\psi(t^-,\tau)\upsilon(d\theta)\Big\}dt \\ \quad -\sigma_x(t,x_t^*)\psi(t,\tau)dB_t - \gamma_x(t,x_t^*,u_t^*)d\langle B\rangle_t \\ \quad -\psi(t^-,\tau)\int_\Gamma \left(f_x(t,x_{t^-}^*,\theta,u_t^*)+I_d\right)^{-1} f_x(t,x_{t^-}^*,\theta,u_t^*)N(dt,d\theta) \\ \qquad 0\le\tau\le t\le T, \\ \psi(\tau,\tau) = I_d. \end{cases}$$

Remark 4.1.

(1) *From Ito's formula, we can easily check that*

$$d\left(\varphi(t,\tau)\psi(t,\tau)\right) = 0, \text{ and } \varphi(\tau,\tau)\psi(\tau,\tau) = I_d.$$

(2) *If* $\tau = 0$, *we simply write* $\varphi(t,0) = \varphi_t$ *and* $\psi(t,0) = \psi_t$.

Then the equality (4.7) will become

$$\begin{aligned} \frac{dJ(u^h)}{dh}\Big|_{h=0} = \widehat{\mathbb{E}}\Bigg[\int_0^T \{h_x(t,x_t^*,u_t^*)\varphi_t\eta_t + h_u(t,x_t^*,u_t^*)u_t\}\,dt \\ +g_x(x_T^*)\varphi_T\eta_T\Bigg]. \end{aligned} \tag{4.9}$$

Set

$$X = \int_0^T h_x(t,x_t^*,u_t^*)\varphi_t^*dt + g_x(x_T^*)\varphi_T^*$$

$$y_t = \widehat{\mathbb{E}}\left[X\,\middle|\,\mathcal{F}_t^{\mathbb{P}}\right] - \int_0^t h_x(s,x_s^*,u_s^*)\varphi_s^*ds + \int_0^t dk_s,$$

then, we have

$$\begin{aligned} y_T &= \widehat{\mathbb{E}}\left[X\,\middle|\,\mathcal{F}_t^{\mathbb{P}}\right] - \int_0^T h_x(s,x_s^*,u_s^*)\varphi_s^*ds + \int_0^T dk_t \\ &= X - \int_0^T h_x(s,x_s^*,u_s^*)\varphi_s^*ds = g_x(x_T^*)\varphi_T^* + \int_0^T dk_t. \end{aligned} \tag{4.10}$$

Replacing (4.10) in (4.9), we obtain

$$\frac{dJ(u^h)}{dh}\bigg|_{h=0} = \widehat{\mathbb{E}}\left[\int_0^T \{h_x(t,x_t^*,u_t^*)\varphi_t^*\eta_t + h_u(t,x_t^*,u_t^*)u_t\}\,dt + y_T\eta_T\right]. \tag{4.11}$$

By the Ito representation theorem of a G-martingale (see [41]), there exist processes $Q \in M_G^2(0,T)$, $S \in S(d)$ and $R \in \mathcal{L}_G^2(0,T)$ satisfying

$$\widehat{\mathbb{E}}\left[X\left|\mathcal{F}_t^{\mathbb{P}}\right.\right] = \widehat{\mathbb{E}}[X] + \int_0^t Q_s dB_s + \int_0^t \varphi_s^* S_s d\langle B\rangle_s - 2\int_0^t \varphi_s^* G(S_s)ds + \int_0^t\int_\Gamma R_s(\theta)\widetilde{N}(ds,d\theta),$$

where G the generator $G : S(d) \to \mathbb{R}$ satisfying the uniformly elliptic condition, i.e., there exists a $\beta > 0$ such that, for each $A, \overline{A} \in S(d)$ with $A \geq \overline{A}$,

$$G(A) - G(\overline{A}) \geq \beta tr[A - \overline{A}].$$

Hence,

$$\begin{aligned} y_t = \widehat{\mathbb{E}}[X] &- \int_0^t \left(h_x(s,x_s^*,u_s^*)\varphi_s + 2\varphi_s^* G(S_s)\right)ds + \int_0^t Q_s dB_s \\ &+ \int_0^t\int_\Gamma R_s(\theta)\widetilde{N}(ds,d\theta) + \int_0^t dk_s + \int_0^t \varphi_s^* S_s d\langle B\rangle_s. \end{aligned}$$

Now, let us calculate $\widehat{\mathbb{E}}[y_T\eta_T]$, we have

$$dy_t = -\left(h_x(s,x_s^*,u_s^*)\varphi_s + 2\varphi_s^* G(S_s)\right)dt + Q_t dB_t + \int_\Gamma R_t(\theta)\widetilde{N}(dt,d\theta) + dk_t + \varphi_s^* S_t d\langle B\rangle_t,$$

by the integration by parts formula we get

$$\begin{aligned} d(y_t\eta_t) = {} & y_t\psi_t\left[b_u(t,x_t^*,u_t^*)u_t - \int_\Gamma f_u(t,x_s^*,\theta,u_s^*)u_t\upsilon(d\theta)\right]dt \\ & - y_t\psi_{t^-}\int_\Gamma \left(f_x + Id\right)^{-1} f_u u_t N(dt,d\theta) - \left(\eta_t\varphi_t^* h_x + 2\eta_t\varphi_s^* G(S_t)\right)dt \\ & + \eta_t Q_t dB_t + \int_\Gamma \eta_t R_t(\theta)\widetilde{N}(dt,d\theta) \\ & + \left\{y_t\psi_t\gamma_u(t,x_t^*,u_t^*)u_t + q_t\sigma_x\eta_t\varphi_t^* + \eta_t\varphi_s^* S_t\right\}d\langle B\rangle_t \end{aligned}$$

$$+\int_{\Gamma} R_t(\theta)\psi_t \left(f_x + Id\right)^{-1} f_u u_t \upsilon(d\theta)dt + \eta_t \varphi_t^* dk_t.$$

If we define the adjoint process by: $p_t = y_t\psi_t$, then

$$\begin{aligned} d(y_t\eta_t) &= p_t b_u u_t dt - p_t \int_{\Gamma} f_u u_t \upsilon(d\theta)dt - p_t \int_{\Gamma} \left(f_x + Id\right)^{-1} f_u u_t \widetilde{N}(dt, d\theta) \\ &\quad -p_t \int_{\Gamma} \left(f_x + Id\right)^{-1} f_u u_t \upsilon(d\theta)dt - \left(\eta_t \varphi_t^* h_x + 2\eta_t \varphi_s^* G(S_t)\right) dt + \eta_t Q_t dB_t \\ &\quad + \left\{p_t \gamma_u(t, x_t^*, u_t^*)u_t + q_t \sigma_x \eta_t \varphi_t^* + \eta_t \varphi_s^* S_t\right\} d\langle B\rangle_t + \eta_t \varphi_t^* dk_t \\ &\quad + \int_{\Gamma} \eta_t R_t(\theta)\widetilde{N}(dt, d\theta) + \int_{\Gamma} R_t(\theta)\psi_t \left(f_x + Id\right)^{-1} f_u u_t \upsilon(d\theta)dt, \end{aligned}$$

hence,

$$\begin{aligned} y_T\eta_T &= \int_0^T p_t b_u u_t dt - \int_0^T \int_{\Gamma} p_t f_u u_t \upsilon(d\theta)dt - \int_0^T \int_{\Gamma} p_t \left(f_x + Id\right)^{-1} f_u u_t \widetilde{N}(dt, d\theta) \\ &\quad - \int_0^T \int_{\Gamma} p_t \left(f_x + Id\right)^{-1} f_u u_t \upsilon(d\theta)dt - \int_0^T \left(\eta_t \varphi_t^* h_x + 2\eta_t \varphi_s^* G(S_t)\right) dt + \int_0^T \eta_t Q_t dB_t \\ &\quad + \int_0^T \left\{p_t \gamma_u(t, x_t^*, u_t^*)u_t + q_t \sigma_x \eta_t \varphi_t^* + \eta_t S_t\right\} d\langle B\rangle_t + \int_0^T \eta_t \varphi_t^* dk_t \\ &\quad + \int_0^T \int_{\Gamma} \eta_t R_t(\theta)\widetilde{N}(dt, d\theta) + \int_0^T \int_{\Gamma} R_t(\theta)\psi_t \left(f_x + Id\right)^{-1} f_u u_t \upsilon(d\theta)dt, \end{aligned}$$

take the G-expectation, we obtain

$$\begin{aligned} \widehat{\mathbb{E}}\left[y_T\eta_T\right] &= \widehat{\mathbb{E}}\left[\int_0^T p_t b_u u_t dt + \int_0^T \left[\int_{\Gamma} R_t(\theta)\psi_t \left(f_x + Id\right)^{-1}\right.\right. \\ &\quad \left. -p_t \left(\left(f_x + Id\right)^{-1} + Id\right)\right] f_u u_t \upsilon(d\theta)dt + \int_0^T \left\{p_t \gamma_u(t, x_t^*, u_t^*)u_t + q_t \sigma_x \eta_t \varphi_t^*\right. \\ &\quad \left.\left. + \eta_t \varphi_s^* S_t\right\} d\langle B\rangle_t + \int_0^T \eta_t \varphi_t^* dk_t - \int_0^T \left(\eta_t \varphi_t^* h_x + 2\eta_t \varphi_s^* G(S_t)\right) dt\right]. \end{aligned}$$

We define the adjoint process r by

$$r_t(\theta) = R_t(\theta)\psi_t \left(f_x + Id\right)^{-1} - p_t \left(\left(f_x + Id\right)^{-1} + Id\right),$$

hence

$$\begin{aligned}\widehat{\mathbb{E}}\left[y_T\eta_T\right] = \widehat{\mathbb{E}}\Bigg[&\int_0^T \left\{p_t b_u u_t + \int_\Gamma r_t(\theta) f_u u_t \upsilon(d\theta)\right\} dt \\ &- \int_0^T \left(\eta_t\varphi_t^* h_x + 2\eta_t\varphi_s^* G(S_t)\right) dt \\ &+ \int_0^T \left\{\gamma_u(t, x_t^*, u_t^*)u_t + q_t\sigma_x\eta_t\varphi_t^* + \eta_t\varphi_s^* S_t\right\} d\langle B\rangle_t + \int_0^T \eta_t\varphi_t^* dk_t\Bigg].\end{aligned}$$

By the replacing in (4.11), we get

$$\begin{aligned}\left.\frac{dJ(u^h)}{dh}\right|_{h=0} = \widehat{\mathbb{E}}\Bigg[&\int_0^T \Big\{h_u(s, x_s^*, u_s^*) + p_s b_u(s, x_s^*, u_s^*) \\ &+ \int_\Gamma r_s(\theta) f_u(s, x_s^*, \theta, u_s^*)\upsilon(d\theta)\Big\} u_s ds \\ &+ \int_0^T \left\{p_t\gamma_u(t, x_t^*, u_t^*)u_t + q_t\sigma_x\eta_t\varphi_t^* + \eta_t\varphi_s^* S_t\right\} d\langle B\rangle_t \\ &- \int_0^T 2\eta_t\varphi_s^* G(S_t)dt + \int_0^T \eta_t\varphi_t^* dk_t\Bigg] \geq 0.\end{aligned} \tag{4.12}$$

Finally, based on the remark 5.2 in [47] if we assume that in equation (4.12) $k = 0\ q.s$, and we define the Hamiltonian H from $[0;T]\times\mathbb{R}^n\times A\times\mathbb{R}^n\times\mathbb{R}^{n\times m}\times L_m^2$ into $\mathbb{R}$ by

$$\begin{aligned}H(t, x, u, p, q, r(.)) = &\ h(t, x_t, u_t) + pb(t, x_t, u_t) + q\sigma(t, x_t) \\ &+ \int_\Gamma r_t(\theta) f(s, x_t, \theta, u_t)\upsilon(d\theta),\end{aligned} \tag{4.13}$$

and

$$F(t,x,u,p,q,r(.)) = \int_0^T \{p_t\gamma_u(t,x_t^*,u_t^*)u_t + q_t\sigma_x\eta_t\varphi_t^* + \eta_t\varphi_s^*S_t\}\, d\langle B\rangle_t$$
$$- \int_0^T 2\eta_t\varphi_s^* G(S_t)dt.$$

We get from (4.12) the next theorem.

Theorem 4.1 (maximum principle for strict control)**.** *Let u^* be the optimal strict control minimizing the cost $J(.)$ over $\mathcal{U}$, and denote by x^* the corresponding optimal trajectory. Then there exists a unique triple of square integrable adapted processes (p,q,r) which is the unique solution of the backward G-SDE*

$$\text{(4.14)} \quad \begin{cases} dp_t = -\Big\{ h_x(t,x_t^*,u_t^*) + p_t b_x(t,x_t^*,u_t^*) \\ \qquad + \int_\Gamma r_t(\theta) f(t,x_{t^-}^*,\theta,u_t^*)\upsilon(d\theta)\Big\}dt \\ \qquad - \{\gamma_x(t,x_t^*,u_t^*)p_t + q_t\sigma_x(t,x_t^*)\}\, d\langle B\rangle_t \\ \qquad + q_t dB_t + \int_\Gamma r_t(\theta)\widetilde{N}(dt,d\theta) + dk_t \\ p_T = g_x(x_T^*),\ k_0 = 0, \end{cases}$$

such that, if we assume that $b = 0$ and $h = 0$, then for all $\nu \in \mathcal{U}$ the following inequality holds

$$\widehat{\mathbb{E}}\left[H(t,x_t^*,\nu_t,p_t) - H(t,x_t^*,u_t^*,p_t) + G(t,x_t^*,u_t^*,p,q,r(.))\right] \geq 0.dt - a.e.$$

Where the Hamiltonian H is defined by (4.13).

4.2. **The maximum principle for near optimal controls.** In this subsection, we establish necessary conditions of near optimality satisfied by a sequence of nearly optimal strict controls. This result is based on Ekeland's variational principle, which is given by the following lemma.

Lemma 4.3. *[Ekeland's variational principle] Let (E,d) be a complete metric space and $f : E \to \overline{\mathbb{R}}$ be lower semicontinuous and bounded from below. Given $\varepsilon > 0$,*

suppose $u^\varepsilon \in E$ *satisfies* $f(u^\varepsilon) \leq \inf(f) + \varepsilon$. *Then for any* $\lambda > 0$, *there exists* $\nu \in E$ *such that*

- $f(\nu) \leq f(u^\varepsilon)$
- $d(u^\varepsilon, \nu) \leq \lambda$
- $f(\nu) \leq f(\omega) + \frac{\varepsilon}{\lambda} d(\omega, \nu)$ *for all* $\omega \neq \nu$.

To apply Ekeland's variational principle, we have to endow the set $\mathcal{U}$ of strict controls with an appropriate metric. For any u and $\nu \in \mathcal{U}$, we set

$$d(u, \nu) = \mathbb{P} \otimes dt \left\{ (\omega, t) \in \Omega \times [0; T] \, ; u(t, \omega) \neq \nu(t, \omega) \right\},$$

where $\mathbb{P} \otimes dt$ is the product measure of $\mathbb{P}$ with the Lebesgue measure dt.

Remark 4.2. *It is easy to see that* $(\mathcal{U}, d)$ *is a complete metric space, and it well known that the cost functional* J *is continuous from* $\mathcal{U}$ *into* $\mathbb{R}$. *For more detail see [32].*

Now, let $\mu^* \in \mathcal{R}$ be an optimal relaxed control and denote by x^{μ^*} the trajectory of the system controlled by μ^*. From lemma (3.1), there exists a sequence (u^n) of strict controls such that

$$\mu_t^n(da)dt = \delta_{u_t^n}(da)dt \longrightarrow \mu_t^*(da)dt \quad \text{quasi-surely,}$$

and for every $\mathbb{P} \in \mathcal{P}$

$$\lim_{n \to \infty} \mathbb{E}^{\mathbb{P}} \left[\left| x_t^n - x_t^{\mu^*} \right|^2 \right] = 0,$$

where x^n is the solution of (2.3) corresponding to μ^n.

According to the optimality of μ^* and lemma (4.3), there exists a sequence (ε_n) of positive numbers with $\lim_{n \to \infty} \varepsilon_n = 0$ such that

$$J(u^n) = J(\mu^n) \leq J(\mu^*) + \varepsilon_n = \inf_{u \in \mathcal{U}} J(u) + \varepsilon_n.$$

A suitable version of lemma (4.3) implies that, given any $\varepsilon_n > 0$, there exists $u^n \in \mathcal{U}$ such that

$$J(u^n) \leq J(u) + \varepsilon_n d(u^n, u), \, \forall u \in \mathcal{U}. \tag{4.15}$$

Let us define the perturbation

$$u_t^{n,h} = \begin{cases} \nu_t & \text{if } \; t \in [t_0; t_0 + h] \\ u_t^n & \text{otherwise.} \end{cases}$$

From (4.15) we have

$$0 \leq J(u^{n,h}) - J(u^n) + \varepsilon_n d(u^{n,h}, u^n).$$

Using the definition of d it holds that

$$0 \leq J(u^{n,h}) - J(u^n) + \varepsilon_n Ch, \tag{4.16}$$

where C is a positive constant.

Now, we can introduce the next theorem which is the main result of this section.

Theorem 4.2. *For each $\varepsilon_n > 0$, there exists $(u^n) \in \mathcal{U}$ such that there exists a unique triple of square integrable adapted processes (p^n, q^n, r^n) which is the solution of the backward SDE*

$$\begin{cases} dp_t^n = -\left\{h_x(t, x_t^n, u_t^n) + p_t^n b_x(t, x_t^n, u_t^n) \right. \\ \qquad \left. + \displaystyle\int_\Gamma r_t^n(\theta) f(t, x_{t^-}^n, \theta, u_t^n) \upsilon(d\theta) \right\} dt \\ \qquad - \left\{\gamma_x(t, x_t^n, u_t^n) p_t + q_t^n \sigma_x(t, x_t^n)\right\} d\langle B\rangle_t \\ \qquad + q_t^n dB_t + \displaystyle\int_\Gamma r_t^n(\theta) \widetilde{N}(dt, d\theta) + dk_t^n \\ p_T^n = g_x(x_T^n), \ k_0^n = 0, \end{cases} \tag{4.17}$$

such that, if we assume that in equation (4.17)

$$\forall n, h_x(t, x_t^n, u_t^n) = 0, b_x(t, x_t^n, u_t^n) = 0,$$

then for all $\nu \in \mathcal{U}$

$$\begin{aligned} &\widehat{\mathbb{E}}\left[H(t, x_t^n, \nu_t, p_t^n) - H(t, x_t^n, u_t^n, p_t^n)\right. \\ &\left. + G^n(t, x_t^*, u_t^*, p, q, r(.))\right] + C\varepsilon_n \geq 0 \quad dt - a.e.. \end{aligned} \tag{4.18}$$

Here C is a positive constant.

Proof. From the inequality (4.16), we use the same method as in the previous subsection, we obtain (4.18). □

4.3. **The relaxed stochastic maximum principle.** Now, we can introduce the next theorem, which is the main result of this section.

Theorem 4.3. *[The relaxed stochastic maximum principle] Let μ^* be an optimal relaxed control minimizing the functional J over $\mathcal{R}$, and let $x_t^{\mu^*}$ be the corresponding optimal trajectory. Then there exists a unique triple of square integrable and adapted processes $(p^{\mu^*}, q^{\mu^*}, r^{\mu^*})$ which is the solution of the backward SDE*

$$
\begin{cases}
dp_t^{\mu^*} = -\Big\{ \int_A h_x(t, x_t^{\mu^*}, a)\mu_t^*(da) + \int_A p_t^n b_x(t, x_t^{\mu^*}, a)\mu_t^*(da) \\
\qquad + \int_A \int_\Gamma r_t^{\mu^*}(\theta) f(t, x_{t^-}^{\mu^*}, \theta, a)\mu_t^* \otimes \upsilon(da, d\theta) \Big\} dt \\
\qquad - \Big\{ \gamma_x(t, x_t^{\mu^*}, a) p_t^{\mu^*} \mu_t^*(da) + q_t^{\mu^*} \sigma_x(t, x_t^{\mu^*}) \Big\} d\langle B \rangle_t + q_t^{\mu^*} dB_t \\
\qquad + \int_A \int_\Gamma r_t^{\mu^*}(\theta) \widetilde{N}^{\mu^*}(dt, d\theta, da) + dk_t^{\mu^*} \\
p_T^{\mu^*} = g_x(x_T^{\mu^*}),\ k_0^{\mu^*} = 0,
\end{cases}
\tag{4.19}
$$

such that if we assume that $b = 0$ and $h = 0$, then for all $\nu \in \mathcal{U}$

$$
\begin{aligned}
0 \leq \widehat{\mathbb{E}} \Big[& H(t, x_t^{\mu^*}, \nu_t, p_t^{\mu^*}, q^{\mu^*}, r_t^{\mu^*}(.)) - \int_\Gamma H(t, x_t^{\mu^*}, a, p_t^{\mu^*}, q^{\mu^*}, r_t^{\mu^*}(.))\mu_t^*(da) \\
& + G^{\mu^*}(t, x_t^*, u_t^*, p, q, r(.)) \Big] \qquad dt - a.e.
\end{aligned}
\tag{4.20}
$$

The proof of this theorem is based on the following stability result of G-BSDEs with jumps. Note that this theorem is proved in the classical problems by Hu and Peng [23], and by H. Ben Gherbal and B. Mezerdi [7] in the case with jump.

4.3.1. *Stability theorem for G-BSDEs with jump.* Let us denote by $M_G^2(0,T)$ the subset of $\mathcal{L}_G^2(0,T)$ consisting of $\mathcal{F}_t-$progressively measurable processes. consider the following G-BSDE with jump depending on a parameter n. Using the fact that under $\mathbb{P} \in \mathcal{P}$, $\widetilde{N}$ is a martingale and B is a continuous martingale whose quadratic variation process $\langle B \rangle$ is such that $\pi_t = \frac{d\langle B \rangle_t}{dt}$ is bounded by a deterministic $d \times d$ symmetric positive definite matrix $\overline{\sigma}$, and p_t^n satisfy

$$
\left\{
\begin{aligned}
dp_t^n = &-\Big\{ h_x(t,x_t^n,u_t^n) + p_t^n\left(b_x(t,x_t^n,u_t^n) - \pi_t\gamma_x(t,x_t^*,u_t^*)\right) \\
&\quad - \pi_t q_t^n \sigma_x(t,x_t^n) + \int_\Gamma r_t^n(\theta) f(t,x_{t^-}^n,\theta,u_t^n)\upsilon(d\theta)\Big\}dt \\
&\quad + q_t^n dB_t + \int_\Gamma r_t^n(\theta)\widetilde{N}(dt,d\theta) + dk_t^n \\
p_T^n = g_x(x_T^n);\ & k_0^n = 0.
\end{aligned}
\right. \tag{4.21}
$$

Then we have

$$
p_t^n = p_T^n + \int_t^T F^n(s,p_s^n,q_s^n,r_s^n)ds - \int_t^T q_s^n dB_s - \int_t^T\int_\Gamma r_s^n(\theta)N^n(ds,d\theta) - K_T^n + K_t^n \quad t\in[0;T],
$$

with

$$
\begin{aligned}
F^n(s,p_s^n,q_s^n,r_s^n) = &-h_x(t,x_t^n,u_t^n) + p_t^n\left(b_x(t,x_t^n,u_t^n) - \pi_t\gamma_x(t,x_t^*,u_t^*)\right) \\
&- \pi_t q_t^n\sigma_x(t,x_t^n) + \int_\Gamma r_t^n(\theta) f(t,x_{t^-}^n,\theta,u_t^n)\upsilon(d\theta).
\end{aligned}
$$

Using the linearity of the adjoint equation, it is not difficult to check that the following assumptions are verified:

(1) For any n, $(p,q,r)\in\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}$, $F^n(.,p,q,r)\in M_G^2(0,T)$ and $p_T^n\in\mathcal{L}_G^2(0,T)$.

(2) There exists a constant $C_0>0$ such that

$$
\begin{aligned}
&|F^n(s,p_1,q_1,r_1) - F^n(s,p_2,q_2,r_2)| \\
&\le C_0\left(|p_1-p_2| + |q_2-q_2| + \int_\Gamma |r_1-r_2|\,\upsilon(d\theta)\right) \quad P.a.s\ \ a.e\ \ t\in[0;T],
\end{aligned}
$$

(3) $E\left(|p_T^n - p_T^*|^2\right)\overrightarrow{n\to\infty}\ 0$,

(4) $\forall t\in[0;T]$,

$$
\lim_{n\to\infty}\mathbb{E}^{\mathbb{P}}\left[\left|\int_t^T\left(F^n(s,p_s^*,q_s^*,r_s^*) - F^*(s,p_s^*,q_s^*,r_s^*)\right)ds\right|^2\right] = 0.
$$

Theorem 4.4 (Stability theorem for G-BSDEs with jumps)**.** *Let* (p^{n}, q^{n}, r^{n}) *and* (p^{*}, q^{*}, r^{*})*, be the solutions of (4.17) and (4.19), respectively. We have*

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\left[|p^n - p^*|^2 + \int_t^T |q^n - q^*|^2 \, ds + \int_t^T \int_\Gamma |r^n - r^*|^2 \, \upsilon(d\theta)ds + |k^n - k^*|^2\right] = 0.$$

Proof. Under every $\mathbb{P} \in \mathcal{P}$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}} |p_t^n - p_t^*|^2 + \int_t^T |q_s^n - q_s^*|^2 \, ds + \int_t^T \int_\Gamma |r_s^n - r_s^*|^2 \, \upsilon(d\theta)ds \\
&\leq 2\mathbb{E}^{\mathbb{P}} |\alpha_t^n|^2 + 2\mathbb{E}^{\mathbb{P}} \left(\int_t^T [F^n(s, p_s^n, q_s^n, r_s^n) - F^n(s, p_s^*, q_s^*, r_s^*)] \, ds\right)^2 \\
&\leq 2\mathbb{E}^{\mathbb{P}} |\alpha_t^n|^2 + 2(T-t)E \int_t^T |F^n(s, p_s^n, q_s^n, r_s^n) - F^n(s, p_s^*, q_s^*, r_s^*)|^2 \, ds,
\end{aligned}$$

with

$$\begin{aligned}
\alpha_t^n = p_T^n - p_T^* + \int_t^T [F^n(s, p_s^*, q_s^*, r_s^*) - F^*(s, p_s^*, q_s^*, r_s^*)] \, ds + (k_T^* - k_T^n) \\
+ (k_t^* - k_t^n) \, .
\end{aligned}$$

Because of the assumption 2, we get

$$\mathbb{E}^{\mathbb{P}} |p_t^n - p_t^*|^2 \leq \frac{2}{3}\mathbb{E}^{\mathbb{P}} |\alpha_t^n|^2 + \frac{1}{6}\int_t^T \mathbb{E}^{\mathbb{P}} |p_s^n - p_s^*|^2 \, ds \tag{4.22}$$

$$\mathbb{E}^{\mathbb{P}} \int_t^T |q_s^n - q_s^*|^2 \, ds \leq \frac{4}{3}\mathbb{E}^{\mathbb{P}} |\alpha_t^n|^2 + \frac{2}{3}\int_t^T \mathbb{E}^{\mathbb{P}} |p_s^n - p_s^*|^2 \, ds \tag{4.23}$$

$$\mathbb{E}^{\mathbb{P}} \int_t^T \int_\Gamma |r_s^n - r_s^*|^2 \, \upsilon(d\theta)ds \leq \frac{4}{3}\mathbb{E}^{\mathbb{P}} |\alpha_t^n|^2 + \frac{2}{3}\int_t^T \mathbb{E}^{\mathbb{P}} |p_s^n - p_s^*|^2 \, ds \, . \tag{4.24}$$

By the assumptions 3, 4 and the stability theorem of G-BSDE without jump, see [21], we deduce that $\lim\limits_{n\to\infty} \mathbb{E}^{\mathbb{P}} |\alpha_t^n|^2 = 0$, then $\lim\limits_{n\to\infty} \mathbb{E}^{\mathbb{P}} |p_t^n - p_t^*|^2 = 0$ and

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}} \int_t^T |q_s^n - q_s^*|^2 \, ds = 0.$$

Hence, by (4.24) we get

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}} \int_t^T \int_\Gamma |r_s^n - r_s^*|^2 \, \upsilon(d\theta) ds = 0.$$

Finally, by the aggregation property we conclude the desired result. □

Proof of Theorem (4.3) . By passing to the limit in inequality (4.18), and using lemma (4.3), we get easily the inequality (4.20). □

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