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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR FRACTIONAL BSDES WITH WEAK MONOTONICITY COEFFICIENTS

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ABSTRACT. In this paper, we deal with the fractional backward stochastic differential equations (F-BSDEs in short) with Hurst parameter $H \in (\frac{1}{2}, 1)$ when the driver g is weak monotone. Via an approximation theory, we derive the existence and uniqueness of solutions to F-BSDEs. The comparison theorem is also established.

1. INTRODUCTION

In the 20th century, a new concept was born in the field of fractional calculus it is called fractional Brownian motion (fBm in short), which presented by Kolmogorov [10] as a method to generate a spirals Gaussians in Hilbert spaces. After some years, Mandelbrot and Van Ness [11] studied their properties and incorporated a concept fBm into financial models, we also inform you that there are many fields of application of fBm, for example: physics , hydrology, economy, telecommunications.

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The fBm with Hurst parameter $H \in (0,1)$ is a continuous normal process $B^H = \{B_t^H, t \geq 0\}$ whose covariance is given by:

$$\mathbb{E}\left(B_s^H B_t^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

For $H = \frac{1}{2}$, the process B^H is a standard Wiener process. However, since B^H with $H > \frac{1}{2}$ is not a Markov process, nor a semi-martingale, we cannot use the classic technique of stochastic calculus to find a concept for the stochastic integral associated to fBm. Basically, two different kinds of integrals were identified and developed in relation to fBm:

- The first kind is Stieltjes-Riemann integral (see Young, [15]).
- The second class of these integrals presented by Decreusefond [5] in (1998) is the Skorokhod integral, it has been defined as the adjoint of the derivative in the cadre of the Malliavin calculus.

Linear backward stochastic differential equations (LBSDEs in short) were studied by Bismut in (1973) [4] and for the non-linear case with non-stochastic terminal time were first presented by Pardoux-Peng in (1990) [13] where they get the existence and uniqueness results. Since then, these pioneering works have been widely used in many areas such as: optimal control [7], financial mathematics [6] and in the probability representation to the solutions of PDEs.

Backward stochastic differential equations (BSDEs for short) driven by fBm were studied by several authors. However, compared to the extensive search for backward SDEs driven by the standard Brownian motion (Bm in short), only a few were accomplished and there are many questions remain open, because we cannot directly apply the usual methods here and the main drawback to this is that fBm is neither a semimartingale, nor a Markov process. The BSDEs driven by fractional Brownian motion (F-BSDEs for short) were first studied by Biagini et al. [3] with Hurst parameter H greater than $\frac{1}{2}$. In 2005 Bender [2] studied a linear F-BSDEs with Hurst index $0 < H < 1$. After several years, Hu & Peng [9] studied a linear and a non-linear F-BSDEs associated to the stochastic integral is the Skorokhod integral, for that, they use the technique of quasi-conditional expectation and Malliavin calculus. In this paper we study the BSDEs driven by fBm with Hurst parameter H greater than $\frac{1}{2}$. We establish existence and uniqueness of solutions of this kind of equation under weak monotonicity condition and establish a comparison theorem, which indicates that our assumptions are much weaker than those in the paper of Hu & Peng [9].

The organization of our paper is as follows: In section 2, we give some definitions and results about fractional stochastic integral, which will be needed throughout this paper. The existence and uniqueness result for the solution of backward SDE driven by fBm under weak monotonicity condition is given in section 3. Finally, we prove the comparison theorem in section 4.

2. Background on fBm and fractional stochastic calculus

In this section we will present some basic concepts related to fractional stochastic calculus and fractional Browian motion.

Let $(\Omega, \mathcal{G}, \mathbb{P}, \mathcal{G}_t, t \geq 0)$ be a stochastic space such that $\mathcal{G}_t$ is an increasing family contains all $\mathbb{P}$-null elements of $\mathcal{G}_t$, also we assume that the filtration $\mathcal{G}_t$ is governed by a fBm $B^H = \{B_t^H, t \geq 0\}$.

We suppose $H \in \left(\frac{1}{2}, 1\right)$ during all this paper. For y is real, we put $\psi(y) = H(2H-1)|y|^{2H-2}$.

Let ν and κ be two measurables functions on $[0, T]$, we define

$$\langle \nu, \kappa \rangle_t = \int_0^t \int_0^t \psi(r-u) \nu_r \kappa_u dr du,$$

and $\langle \nu, \nu \rangle_t = ||\nu||_t^2$. Note that, $\forall t \in [0, T]$, $\langle \nu, \kappa \rangle_t$ is a product space of Hilbert.

Let $\mathcal{Q}$ be the completion of the measurable functions on $[0, T]$ under $||\cdot||_T$. The components of $\mathcal{Q}$ can be distributions.

We note by ∇_T the set of all polynomials which can be written in the following form

$$K(\omega) = k\left(\int_0^T \nu_1(t) dB_t^H, \cdots, \int_0^T \nu_p(t) dB_t^H\right),$$

where k is a polynomial of p variables and $p \geq 0$.

Then, for a component F belong in ∇_T their Malliavin derivative $\mathcal{D}_s^H$ is defined as follows:

$$\mathcal{D}_s^H K = \sum_{i=1}^p \frac{\partial f}{\partial x_i}\left(\int_0^T \nu_1(t) dB_t^H, \cdots, \int_0^T \nu_p(t) dB_t^H\right) \nu_1(s), \, s \in [0, T].$$

Because the non-convergence operator $\mathcal{D}^H : \mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}) \to (\Omega, \mathcal{G}, \mathcal{Q})$ is not openable. So, we can consider the completion of ∇_T by the space $\mathbb{D}^{1,2}$ endowed with the following norm

$$||K||_{1,2}^2 = \mathbb{E}\,|K|^2 + \mathbb{E}\left|\left|\mathcal{D}_s^H K\right|\right|_T^2.$$

We can now give another derivative

$$\mathcal{D}_t^H K = \int_0^T \psi\,(t-s)\,\mathcal{D}_s^H K ds.$$

Let $\mathbb{L}_{1,2}^H$ the space of process K defined on $(\Omega, \mathcal{G}, \mathbb{P})$ and has a value in $\mathcal{Q}$ such that

$$\mathbb{E}\left(||K||_T^2 + \int_0^T \int_0^T \left|\mathcal{D}_s^H K_t\right|^2 dsdt\right) < \infty.$$

The following propositions are well famous now.

Proposition 2.1. *Let $K \in \mathbb{L}_{1,2}^H$, then the integral $\int_0^T K_s dB_s^H$ exists in $\mathbb{L}^2\,(\Omega, \mathcal{G}, \mathbb{P})$. Moreover, we have*

$$\mathbb{E}\left(\int_0^T K_s dB_s^H\right) = 0,$$

and

$$\mathbb{E}\left(\int_0^T K_s dB_s^H\right)^2 = \mathbb{E}\left(||K||_T^2 + \int_0^T \int_0^T \mathcal{D}_s^H K_t \mathcal{D}_t^H K_s dsdt\right).$$

The following proposition given in ([8, Theorem 11.1]) would be helpful below.

Proposition 2.2. *Let $\vartheta_j\,(s)\,,\ \theta_j\,(s)$ be in $\mathbb{D}^{1,2}$ and $\mathbb{E}\int_0^T \left(|\vartheta_j\,(s)|^2 + |\theta_j\,(s)|^2\right) ds < \infty$, where $j = 1, 2$.Assume that $\mathcal{D}_t^H \vartheta_1\,(s)$ and $\mathcal{D}_t^H \vartheta_2\,(s)$ are continuously differentiable with respect to $(s,t) \in [0,T]$ for almost all $\omega \in \Omega$, suppose also that $\mathbb{E}\int_0^T \int_0^T \left|\mathcal{D}_t^H \vartheta_j\,(s)\right|^2 dsdt < \infty$. Denote*

$$X_i\,(t) = \int_0^t \theta_j\,(s)\,ds + \int_0^t \vartheta_j\,(s)\,dB_s^H,\ 0 \leq t \leq T.$$

Then

$$\begin{aligned} X_1\,(t)\,X_2\,(t) = {} & \int_0^t X_1\,(s)\,\theta_2\,(s)\,ds + \int_0^t X_1\,(s)\,\vartheta_2\,(s)\,dB_s^H + \int_0^t X_2\,(s)\,\theta_1\,(s)\,ds \\ & + \int_0^t X_2\,(s)\,\vartheta_1\,(s)\,dB_s^H + \int_0^t \mathcal{D}_s^H X_1\,(s)\,\theta_2\,(s)\,ds + \int_0^t \mathcal{D}_s^H X_2\,(s)\,\theta_1\,(s)\,ds. \end{aligned}$$

Let's end this section by presenting an Itô formula for the non-convergence type integral (see [8, Theorem 10.3]).

Proposition 2.3. *Let ϑ, θ: $[0,T] \to \mathbb{R}$ be two non-stochastic measurable functions. If*

$$X_t = X_0 + \int_0^t \theta_s ds + \int_0^t \vartheta_s dB_s^H,\ 0 \leq t \leq T,$$

where X_0 *is a constant and* $K \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$*, then we have* $\forall t \in [0,T]$,

$$K(t, X_t) = K(0, X_0) + \int_0^t \frac{\partial K}{\partial s}(s, X_s)\,ds + \int_0^t \frac{\partial K}{\partial x}(s, X_s)\,dx$$
$$+ \frac{1}{2}\int_0^t \frac{\partial^2 K}{\partial x^2}(s, X_s)\left(\frac{d}{ds}\,||\vartheta||_s^2\,ds\right),$$

where

$$\tfrac{d}{ds}\,||\vartheta||_s^2 = \tfrac{d}{ds}\int_0^s\int_0^s \psi(r-u)\,\vartheta_r\vartheta_u\,dr\,du = 2\vartheta_s\int_0^s \phi(r-s)\,\vartheta_r\,dr.$$

By the preliminary results mentioned previously, we are now able to study our main results.

3. Fractional BSDE with weak monotonocity coefficient

In this section, we study the results of the existence and uniqueness solution to the weak monotone backward SDEs associated to fBm. We use the approximation technic to proof it.

Assume that

- κ_0 is a constant.
- θ, ϑ: $[0,T] \to \mathbb{R}$ are two measurables non-stochastic functions, ϑ is differentiable and such that $\vartheta(t) \neq 0$ $\forall t \in [0,T]$. Note that, since $||\vartheta||_t^2 = H(2H-1)\int_0^t\int_0^t |r-u|^{2H-2}\,\vartheta(r)\,\vartheta(u)\,dr\,du$, we have $\frac{d}{dt}\left(||\vartheta||_t^2\right) = \vartheta(t)\,\hat{\vartheta}(t) \geq 0$, with $\hat{\vartheta}(t) = \int_0^t \psi(t-u)\,\vartheta(u)\,du$.

In the following, let $(\kappa_t)_{0\leq t\leq T}$ be a solution of the following stochastic differential equation associated by fBm:

$$\begin{cases} d\kappa_t = \theta(t)\,dt + \vartheta(t)\,dB_t^H, \\ \kappa_0 = \pi_{0.} \end{cases}$$

Now, we Introduce the following backward SDE driven by fBm

$$\begin{cases} -dY_t = g(t, \kappa_t, Y_t, Z_t)dt - Z_t dB_t^H, \\ Y_T = \xi. \end{cases}$$

Throughout this paper, for $\delta > 0$, we will use the following spaces of processes

- Let $\mathbb{L}^2(\Omega, \mathcal{G}_T, \mathbb{P})$ is the space of $\mathcal{G}_T$-measurable non-deterministic variables ξ: $\Omega \to \mathbb{R}$ with $\mathbb{E}\left(e^{\delta T}|\xi|^2\right) < \infty$.

- We denote by $\mathcal{C}^{1,2}_{pol}\left([0,T]\times\mathbb{R}\right)$ the space of all $\mathcal{C}^{1,2}$-applications over $[0,T]\times\mathbb{R}$, such that it and its derivatives have polynomial growth.
- $\mathcal{V}_{[0,T]}=\left\{Y=\psi\left(\cdot,\kappa\right);\ \psi\in\mathcal{C}^{1,2}_{pol}\left([0,T]\times\mathbb{R}\right)\right\}$, $\frac{d\psi}{dt}$is bounded, $t\in[0,T]$.
- The completion of $\mathcal{V}_{[0,T]}$ is noted by $\tilde{\mathcal{V}}^{H}_{[0,T]}$ and endowed with the following norm:

$$||Y||_v=\left(\mathbb{E}\int_0^T e^{\delta t}t^{2H-1}\left|Y_t\right|^2dt\right)^{\frac{1}{2}}=\left(\mathbb{E}\int_0^T e^{\delta t}t^{2H-1}\left|\psi\left(t,\kappa_t\right)\right|^2dt\right)^{\frac{1}{2}}.$$

Definition 3.1. *A solution of the equation* (3.1) *is a pair of processes* $(Y,Z)\in\tilde{\mathcal{V}}^{\frac{1}{2}}_{[0,T]}\times\tilde{\mathcal{V}}^{H}_{[0,T]}$ *and satisfies the equation* (3.1).

Also, we have the following proposition. It's proof follows the same way as in [1, Proposition 3.6].

Proposition 3.1. *Assume the pair* $(Y_t,Z_t)_{t\in[0,T]}$ *is a solution of the BSDE driven by fBm* (3.1). *Then:*

(a) *We have the following non-deterministic representation*

$$\mathcal{D}^H_tY_t=\frac{\hat{\sigma}(t)}{\sigma(t)}Z_t,\qquad 0\leq t\leq T.$$

(b) *There is a strictly positive constant* M *such that*

$$\frac{t^{2H-1}}{M}\leq\frac{\hat{\sigma}(t)}{\sigma(t)}\leq\frac{M}{t^{2H-1}},\qquad 0\leq t\leq T.$$

Now, we suppose that the coefficient g satisfy the following assumptions **(H)**:

(H1.1) For any fixed t, $g(t,\cdot,\cdot,\cdot)$ is continuous.

(H1.2) There exist a constant $\gamma\in[0,1)$ and strictly non-negative constant C such that

$$|g(t,\varrho,\chi,z)|\leq C\left(1+|\varrho|^\gamma+|\chi|^\gamma+|\varsigma|^\gamma\right).$$

(H1.3) There exists a nondecreasing and concave function $\rho\left(\cdot\right):\mathbb{R}_+\to\mathbb{R}_+$ with $\rho\left(0\right)=0$, $\rho\left(v\right)>0$ for $v>0$ and $\int_{0+}\frac{dv}{\rho(v)}=+\infty$ such that $d\mathbb{P}\times dt-a.e.$

$$\left(\chi-\acute{\chi}\right)\left(g(t,\varrho,\chi,\varsigma)-g(t,\varrho,\acute{\chi},\varsigma)\right)\leq\rho\left(|\chi-\acute{\chi}|^2\right),$$

for all $(\varrho,\chi,\acute{\chi},\varsigma)\in\mathbb{R}^4$.

(H1.4) There exists a positive constant C such that for any $(\varrho,\acute{\varrho},\varsigma,\acute{\varsigma})\in\mathbb{R}^4$

$$|g(t,\varrho,\chi,\varsigma)-g(t,\acute{\varrho},\chi,\acute{\varsigma})|\leq C\left(|\varrho-\acute{\varrho}|+|\varsigma-\acute{\varsigma}|\right).$$

When the assumption **(H1.1) and (H1.2)** are satisfied, we can define the family of semi norms $(\Phi_n(g))_n$

$$\Phi_n(g) = \left(\mathbb{E}\int_0^T \sup_{|\varrho|,|\chi|,|\varsigma|\leq n} |g(t,\varrho,\chi,\varsigma)|^2 dt\right)^{\frac{1}{2}}.$$

Remark 3.1. *Since $|\eta|^\gamma \leq (1+|\eta|)$ for $\gamma \in [0,1)$ then assumption **(H1.2)** implies that*

$$|g(t,\varrho,\chi,z)| \leq C(4+|\varrho|+|\chi|+|\varsigma|).$$

Now, we mention an estimate for the distance between two solution of F-BSDE (3.1). This estimate is very important to study the existence and uniqueness of the solution, we then consider the our main result.

Proposition 3.2. *We assume ξ^1, $\xi^2 \in \mathbb{L}^2(\Omega,\mathcal{G}_T,\mathbb{P})$ and g_1, g_2 satisfy the hypothesis **(H)**. Let for $j=\{1,2\}$, (Y^j,Z^j) belong in $\mathcal{B}^2 = \tilde{\mathcal{V}}^{\frac{1}{2}}_{[0,T]} \times \tilde{\mathcal{V}}^H_{[0,T]}$ has a unique solution of the following BSDEs driven by fBm*

$$Y_t^j = \xi^j + \int_t^T g(u,\kappa_u,Y_u^j,Z_u^j)du - \int_t^T Z_u^j dB_u^H, \qquad t\in[0,T].$$

Then, we find

$$\begin{aligned}&\mathbb{E}\left(e^{\delta t}\left|Y_t^1-Y_t^2\right|^2 + \int_t^T e^{\delta u}\left|Y_u^1-Y_u^2\right|^2 du + \int_t^T e^{\delta u}u^{2H-1}\left|Z_u^1-Z_u^2\right|^2 du\right)\\ &\leq C\left(\mathbb{E}e^{\delta T}\left|\xi^1-\xi^2\right|^2 + \frac{C}{N^{2(1-\gamma)}} + \rho_N^2(g_1-g_2)\right).\end{aligned} \tag{3.1}$$

Proof. Applying Itô's formula to $\left|Y_t^j\right|^2$, we get

$$\left|Y_t^j\right|^2 = |\xi^j|^2 + 2\int_t^T Y_u^j g(u,\kappa_u,Y_u^j,Z_u^j)du - 2\int_t^T Y_u^j Z_u^j dB_u^H - 2\int_t^T \mathcal{D}_u^H Y_u^j Z_u^j du.$$

Using the integration by part formula, we get

$$\begin{aligned}e^{\delta t}\left|Y_t^j\right|^2 + \delta\int_t^T e^{\delta u}\left|Y_u^j\right|^2 du &= e^{\delta T}\left|\xi^j\right|^2 + 2\int_t^T e^{\delta u}Y_u^j g(u,\kappa_u,Y_u^j,Z_u^j)du\\ &\quad - 2\int_t^T e^{\delta u}Y_u^j Z_u^j dB_u^H - 2\int_t^T e^{\delta u}\mathcal{D}_u^H Y_u^j Z_u^j du.\end{aligned}$$

By hypothesis **(H1.2)**, Young's inequality and **Remark 3.1**, we have

$$\begin{aligned}2\int_t^T e^{\delta u}Y_u^j g(u,\kappa_u,Y_u^j,Z_u^j)du &\le 2C\int_t^T e^{\delta u}Y_u^j\left(4+|\kappa_u|+\left|Y_u^j\right|+\left|Z_u^j\right|\right)du\\ &\le C+\int_t^T e^{\delta u}\left(1+C^2+2C+\frac{MC^2}{u^{2H-1}}\right)\left|Y_u^j\right|^2du\\ &+\int_t^T e^{\delta u}\left|\kappa_u\right|^2du+\frac{1}{M}\int_t^T e^{\delta u}u^{2H-1}\left|Z_u^j\right|^2du.\end{aligned}$$

Therefore, using Proposition 3.1, we can write

$$\begin{aligned}&\mathbb{E}\left(e^{\delta t}\left|Y_t^j\right|^2+\delta\int_t^T e^{\delta u}\left|Y_u^j\right|^2du+\tfrac{2}{M}\int_t^T e^{\delta u}u^{2H-1}\left|Z_u^j\right|^2du\right)\\ &\le \mathbb{E}e^{\delta T}\left|\xi^j\right|^2+C+\mathbb{E}\left(\int_t^T e^{\delta u}\left(1+C^2+2C+\tfrac{MC^2}{u^{2H-1}}\right)\left|Y_u^j\right|^2du\right)\\ &+\mathbb{E}\left(\int_t^T e^{\delta u}\left|\kappa_u\right|^2du+\tfrac{1}{M}\int_t^T e^{\delta u}u^{2H-1}\left|Z_u^j\right|^2du\right).\end{aligned}\tag{3.2}$$

By Gronwall's inequality, we find for $M>1$

$$\begin{aligned}&\mathbb{E}\left(e^{\delta t}\left|Y_t^j\right|^2\right)\\ &\le\Theta\left(t,T,C,M,\xi^j\right)\times\exp\left(\left(1+C^2+2C\right)T+MC^2\tfrac{T^{2-2H}-t^{2-2H}}{2-2H}\right)<\infty.\end{aligned}$$

And by (3.3) also, one has

$$\mathbb{E}\int_t^T e^{\delta u}u^{2H-1}\left|Z_u^j\right|^2du\le\Theta\left(t,T,C,M,H,\xi^j\right)<\infty.$$

Now, we set

$$B=\left\{(\omega,u)\,,\;\;|Y_u^1|+|Y_u^2|+|Z_u^1|+|Z_u^2|+|\kappa_u|\ge N\right\},\,B^c=\Omega\setminus B.$$

Again by $(a)\,,(b)$ in Proposition 3.1 and Itô's formula, we obtain

$$\begin{aligned}&\mathbb{E}\left(e^{\delta t}\left|Y_t^1-Y_t^2\right|^2+\delta\int_t^T e^{\delta u}\left|Y_u^1-Y_u^2\right|^2du\right)\\ &+\tfrac{2}{M}\mathbb{E}\left(\int_t^T e^{\delta u}u^{2H-1}\left|Z_u^1-Z_u^2\right|^2du\right)\\ &\le\mathbb{E}\left(e^{\delta T}\left|\xi^1-\xi^2\right|^2\right)+I^1+I^2+I^3+I^4,\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}I^1&=2\mathbb{E}\int_t^T e^{\delta u}\left(Y_u^1-Y_u^2\right)\left(g_1(u,\kappa_u,Y_u^1,Z_u^1)-g_2(u,\kappa_u,Y_u^2,Z_u^2)\right)1_B du,\\ I^2&=2\mathbb{E}\int_t^T e^{\delta u}\left(Y_u^1-Y_u^2\right)\left(g_1(u,\kappa_u,Y_u^1,Z_u^1)-g_1(u,\kappa_u,Y_u^2,Z_u^1)\right)1_{B^c}du,\\ I^3&=2\mathbb{E}\int_t^T e^{\delta u}\left(Y_u^1-Y_u^2\right)\left(g_1(u,\kappa_u,Y_u^2,Z_u^1)-g_1(u,\kappa_u,Y_u^2,Z_u^2)\right)1_{B^c}du,\\ I^4&=2\mathbb{E}\int_t^T e^{\delta u}\left(Y_u^1-Y_u^2\right)\left(g_1(u,\kappa_u,Y_u^2,Z_u^2)-g_2(u,\kappa_u,Y_u^2,Z_u^2)\right)1_{B^c}du.\end{aligned}$$

We now need to estimate I^1, I^2, I^3 and I^4.

It follows from the assumption **(H1.2)** and $(\Sigma + \Upsilon + \Psi + \Lambda)^2 \leq 4(\Sigma^2 + \Upsilon^2 + \Psi^2 + \Lambda^2)$ that $\forall \epsilon > 0$

$$\begin{aligned} I^1 &\leq \epsilon^2 \mathbb{E}\left(\int_t^T e^{\delta u} \left|(Y_u^1 - Y_u^2)\right|^2 1_B du\right) \\ &+ \tfrac{8C}{\epsilon^2}\mathbb{E}\left(\int_t^T e^{\delta u}\left(1 + |\kappa_u|^{2\gamma} + |Y_u^1|^{2\gamma} + |Z_u^1|^{2\gamma}\right) 1_B du\right) \\ &+ \tfrac{8C}{\epsilon^2}\mathbb{E}\left(\int_t^T e^{\delta u}\left(1 + |\kappa_u|^{2\gamma} + |Y_u^2|^{2\gamma} + |Z_u^2|^{2\gamma}\right) 1_B du\right). \end{aligned} \tag{3.4}$$

By inequality (3.3), Hölder's inequality and Chebyshev's inequality, then

$$I^1 \leq \epsilon^2 \mathbb{E}\left(\int_t^T |Y_u^1 - Y_u^2|^2 du\right) + \tfrac{C}{N^{2(1-\gamma)}}. \tag{3.5}$$

The assumption **(H1.3)** give

$$I^2 \leq 2\mathbb{E}\left(\int_t^T \rho\left(e^{\delta u}|Y_u^1 - Y_u^2|^2\right) du\right). \tag{3.6}$$

Due to the assumption **(H1.4)** and Young's inequality that $\forall \epsilon > 0$,

$$\begin{aligned} I^3 &\leq \epsilon^2 C^2 \mathbb{E}\left(\int_t^T e^{\delta u}\frac{1}{u^{2H-1}}\left|Y_u^1 - Y_u^2\right|^2 du\right) \\ &+ \frac{1}{\epsilon^2}\mathbb{E}\left(\int_t^T e^{\delta u} u^{2H-1}\left|Z_u^1 - Z_u^2\right|^2 du\right). \end{aligned} \tag{3.7}$$

Finally by inequality $2ab \leq a^2 + b^2$, we have

$$I^4 \leq \mathbb{E}\left(\int_t^T e^{\delta u}|Y_u^1 - Y_u^2|^2 du\right) + \Phi_N^2\left(g_1 - g_2\right). \tag{3.8}$$

Combining (3.4) and $(3.6) - (3.9)$, we conclude that

$$\begin{aligned} &\mathbb{E}\left(e^{\delta t}|Y_t^1 - Y_t^2|^2 + \delta\int_t^T e^{\delta u}|Y_u^1 - Y_u^2|^2 du + \left(\tfrac{2}{M} - \tfrac{1}{\epsilon^2}\right)\int_t^T e^{\delta u}u^{2H-1}|Z_u^1 - Z_u^2|^2 du\right) \\ &\leq \mathbb{E}\left(e^{\delta T}|\xi^1 - \xi^2|^2\right) + \tfrac{C}{N^{2(1-\gamma)}} + \rho_N^2\left(g_1 - g_2\right) \\ &+\mathbb{E}\int_t^T \left(2\rho\left(e^{\delta u}|Y_u^1 - Y_u^2|^2\right) + \left(1 + \epsilon^2 + \tfrac{\epsilon^2 C^2}{u^{2H-1}}\right)e^{\delta u}|Y_u^1 - Y_u^2|^2\right) du, \end{aligned}$$

Now, for $M > 0$ choosing $\epsilon > 0$ such that $\frac{2}{M} - \frac{1}{\epsilon^2} > 1$ and $C_{\epsilon, C, H}(u) = 1 + \epsilon^2 + \frac{\epsilon^2 C^2}{u^{2H-1}} > 2$ for $u \in [t, T]$, we get

$$\begin{aligned} &\mathbb{E}\left(e^{\delta t}|Y_t^1 - Y_t^2|^2 + \int_t^T e^{\delta u}u^{2H-1}|Z_u^1 - Z_u^2|^2 du\right) \\ &\leq \mathbb{E}\left(e^{\delta T}|\xi^1 - \xi^2|^2\right) + \tfrac{C}{N^{2(1-\gamma)}} + \rho_N^2\left(g_1 - g_2\right) \\ &+\mathbb{E}\int_t^T C_{\epsilon, C, H}(u)\left(\rho\left(e^{\delta u}|Y_u^1 - Y_u^2|^2\right) + e^{\delta u}|Y_u^1 - Y_u^2|^2\right) du. \end{aligned}$$

Taking $\bar{\rho}(v) = \rho(v) + v$, for $v > 0$, such that $\int_{0+} \frac{dv}{\bar{\rho}(v)} = +\infty$, from Fubini's theorem and Jensen's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(e^{\delta t}\left|Y_t^1 - Y_t^2\right|^2 + \int_t^T e^{\delta u} u^{2H-1}\left|Z_u^1 - Z_u^2\right|^2 du\right) \\
&\leq \mathbb{E}\left(e^{\delta T}\left|\xi^1 - \xi^2\right|^2\right) + \frac{C}{N^{2(1-\gamma)}} + \rho_N^2\left(g_1 - g_2\right) \\
&+ \int_t^T C_{\epsilon,\,C,\,H}(u)\,\bar{\rho}\left(\mathbb{E}\left(e^{\delta u}\left|Y_u^1 - Y_u^2\right|^2 + \int_u^T e^{\delta u} r^{2H-1}\left|Z_r^1 - Z_r^2\right|^2 dr\right)\right) du,
\end{aligned}$$

using Bihari's inequality, wz conclude that

$$\begin{aligned}
&\mathbb{E}\left(e^{\delta t}\left|Y_t^1 - Y_t^2\right|^2 + \int_t^T e^{\delta u} u^{2H-1}\left|Z_u^1 - Z_u^2\right|^2 du\right) \\
&\leq C_{\epsilon,\,C,\,H,\,T}\left(\mathbb{E}\left(e^{\delta T}\left|\xi^1 - \xi^2\right|^2\right) + \frac{C}{N^{2(1-\gamma)}} + \rho_N^2\left(g_1 - g_2\right)\right).
\end{aligned}$$

Therefore, the proof of **Proposition 3.2** is completey. □

Before to state the proof of our main result, we first give the following technical approximation lemma.

Lemma 3.1. *Let $g : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ be a measurable function satisfies assumptions **(H)**. Then there exists the sequence of function g_n such that*

(i) $\forall t \in [0,T]$, $g_n(t,\cdot,\cdot,\cdot)$ *is a continuous.*

(ii) *Convergence* : $\forall N$, $\Phi_N(g_n, g) \underset{n\to\infty}{\to} 0$ *a.s..*

(iii) $\forall\, n$, g_n *is locally weak monotone in* χ *i.e., for any* n, N *such that* $n \geq N$, *we get*

$$(\chi - \acute{\chi})\left(g_n(t,\varrho,\chi,\varsigma) - g_n(t,\varrho,\acute{\chi},\varsigma)\right) \leq \rho\left(|\chi - \acute{\chi}|^2\right).$$

for any ϱ, χ, $\acute{\chi}$, ς *satisfaying* $|\varrho| \leq N$, $|\chi| \leq N$, $|\acute{\chi}| \leq N$, $|\varsigma| \leq N$, *where* $\rho(\cdot)$ *is the same function in assumption* ***(H1.3)***.

(iv) *There is a constant* $\gamma \in [0,1)$ *and a constant* $C > 0$ *such that for every* $(t,\varrho,\chi,\varsigma) \in [0,T] \times \mathbb{R}^3$, *the sequence of function* g_n *satisfy*

$$|g_n(t,\varrho,\chi,\varsigma)| \leq C\left(1 + |\varrho|^\gamma + |\chi|^\gamma + |\varsigma|^\gamma\right).$$

(v) $\forall\, n$, g_n *is Lipschitz in* ς *and* ϱ *i.e., there is a constant* $C_n > 0$ *such that*

$$|g_n(t,\varrho,\chi,\varsigma) - g_n(t,\acute{\varrho},\chi,\acute{\varsigma})| \leq C_n\left(|\varrho - \acute{\varrho}| + |\varsigma - \acute{\varsigma}|\right).$$

(vi) $\forall\, n$, g_n *is Lipschitz in* χ *i.e., there is a constant* $A_n > 0$ *such that*

$$|g_n(t,\varrho,\chi,\varsigma) - g_n(t,\varrho,\acute{\chi},\varsigma)| \leq A_n\,|\chi - \acute{\chi}|.$$

(vii) *For any* n, N *with* $n \geq N$, g_n *is locally Lipschitz in* ϱ *and* ς, *i.e., for any* ϱ, $\acute{\varrho}$, χ, ς *and* $\acute{\varsigma}$ *satisfying* $|\varrho| \leq N$, $|\acute{\varrho}| \leq N$, $|\chi| \leq N$, $|\acute{\varsigma}| \leq N$, $|\varsigma| \leq N$, *then*

$$|g_n(t,\varrho,\chi,\varsigma) - g_n(t,\acute{\varrho},\chi,\acute{\varsigma})| \leq C\left(|\varrho-\acute{\varrho}| + |\varsigma-\acute{\varsigma}|\right),$$

where C *is the same constant in assumption* ***(H1.4).***

Proof. See, [16, Lemma 2.1]. □

We are now ready to state our main result.

Theorem 3.1. *Suppose* $\xi \in \mathbb{L}^2(\Omega, \mathcal{G}_T, \mathbb{P})$ *and* g *satisfies the hypothesis* ***(H)***. *Then, the pair* (Y,Z) *belong in* $\mathcal{B}^2$ *with* $\mathcal{B}^2 = \tilde{\mathcal{V}}^{\frac{1}{2}}_{[0,T]} \times \tilde{\mathcal{V}}^{H}_{[0,T]}$ *is a unique solution of BSDE driven by fBm* (3.1).

Proof. Choosing $g_1 = g_2$ and $\xi^1 = \xi^2$ and going $N \to \infty$ in inequality (3.2), we can get the uniqueness result. Let g_n a sequence of function related to g by **Lemma** 3.1. Then g_n satisfies (v) and (vi) in **Lemma 3.1** for each n. So, according to the result of Hu and Peng [9], the pair $(Y^n_\cdot, Z^n_\cdot)$ belongs in $\mathcal{B}^2$, for each $n \geq N$ is the unique solution to the following BSDE driven by fBm

$$Y^n_t = \xi + \int_t^T g_n(u,\kappa_u,Y^n_u,Z^n_u)du - \int_t^T Z^n_u dB^H_u, \qquad t \in [0,T].$$

Appling Itô's formula to $e^{\delta t}|Y^n_t|^2$, for $M>1$, we have

$$\mathbb{E}\left(e^{\delta t}|Y^n_t|^2 + \delta\int_t^T e^{\delta u}|Y^n_u|^2du + \frac{1}{M}\int_t^T e^{\delta u}u^{2H-1}|Z^n_u|^2du\right) \leq \tilde{\Theta}(t,T,C,M,\xi) < \infty.$$

Like the proof of **Proposition 3.2**, show that, for $M>2$ and p, q large enough

$$\begin{aligned}&\mathbb{E}\left(e^{\delta t}|Y^p_t - Y^q_t|^2 + \int_t^T e^{\delta u}|Y^p_u - Y^q_u|^2du + \int_t^T e^{\delta u}u^{2H-1}|Z^p_u - Z^q_u|^2du\right)\\ &\leq K\left(\frac{C}{N^{2(1-\gamma)}} + \Phi^2_N(g_p - g_q)\right).\end{aligned}$$

Now going to the limit successively on p, q and N, we conclude that $(Y^n_\cdot, Z^n_\cdot)$ is a Cauchy sequence in $\mathcal{B}^2$. Hence, there exists a pair of processes $(Y_\cdot, Z_\cdot)$ such that

$$\mathbb{E}\left(e^{\delta t}|Y^n_t - Y_t|^2 + \int_t^T e^{\delta u}|Y^n_u - Y_u|^2du + \int_t^T e^{\delta u}u^{2H-1}|Z^n_u - Z_u|^2du\right) \to 0,$$

as $n \to \infty$. In order to verify that $(Y_\cdot, Z_\cdot)$ is a solution of the fractional BSDE (3.1), we just have to prove that

$$g_n(u,\kappa_u,Y^n_u,Z^n_u) \to g(u,\kappa_u,Y_u,Z_u), \text{ as } n \to \infty. \tag{3.9}$$

Set

$$B_n = \{(\omega, u), \ \ |Y_u^n| + |Y_u| + |Z_u^n| + |Z_u| + |\kappa_u| \geq N\}, \ B_n^c = \Omega \setminus B_n.$$

Then

$$\mathbb{E}\left(\int_t^T |g_n(u, \kappa_u, Y_u^n, Z_u^n) - g(u, \kappa_u, Y_u, Z_u)|^2 \, du\right) \leq 2I_1^n + 4I_2^n + 4I_3^n + 2I_4^n,$$

where

$$\begin{aligned}
I_1^n &= \mathbb{E}\left(\int_t^T |g_n(u, \kappa_u, Y_u^n, Z_u^n) - g(u, \kappa_u, Y_u^n, Z_u)|^2 \, 1_{B_n} du\right), \\
I_2^n &= \mathbb{E}\left(\int_t^T |g_n(u, \kappa_u, Y_u^n, Z_u^n) - g_n(u, \kappa_u, Y_u^n, Z_u)|^2 \, 1_{B_n^c} du\right), \\
I_3^n &= \mathbb{E}\left(\int_t^T |g_n(u, \kappa_u, Y_u^n, Z_u) - g(u, \kappa_u, Y_u^n, Z_u)|^2 \, 1_{B_n^c} du\right), \\
I_4^n &= \mathbb{E}\left(\int_t^T |g(u, \kappa_u, Y_u^n, Z_u) - g(u, \kappa_u, Y_u, Z_u)|^2 \, du\right).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\mathbb{E}\left(\int_t^T |g_n(u, \kappa_u, Y_u^n, Z_u^n) - g(u, \kappa_u, Y_u, Z_u)|^2 \, du\right) \\
&\leq \frac{C}{N^{2(1-\gamma)}} + C \sup_{t \leq u \leq T}\left(\frac{1}{u^{2H-1}}\right) \mathbb{E}\left(\int_t^T e^{\delta u} u^{2H-1} |Z_u^n - Z_u|^2 \, du\right) \qquad (3.10) \\
&\quad + 4\Phi_N^2 (g_n - g) \\
&\quad + 2\mathbb{E}\left(\int_t^T |g(u, \kappa_u, Y_u^n, Z_u) - g(u, \kappa_u, Y_u, Z_u)|^2 \, du\right).
\end{aligned}$$

Since

$$\mathbb{E}\int_0^T e^{\delta u} u^{2H-1} |Z_u^n - Z_u|^2 \, du \to 0, \text{ as } n \to \infty,$$

there is a sub-sequence of Y^n, denoted by Y^n, such that $Y^n \to Y$ $a.e.$, $a.s.$. Therefore, the continuity of g in y and Lebesgue's dominated convergence theorem give that

$$\lim_{n\to\infty} \mathbb{E}\left(\int_t^T |g(u, \kappa_u, Y_u^n, Z_u) - g(u, \kappa_u, Y_u, Z_u)|^2 \, ds\right) = 0.$$

Then, passing to the limit in inequality (3.11) respectively on n and N, we prove that (3.10) holds.

It remains to show that the pair (Y, Z) satisfies equation (3.1) on the interval $[0, T]$. We have for any $t \in [t_k, T]$,

$$Y_t^n - \xi - \int_t^T g_n(u, \kappa_u, Y_u^n, Z_u^n) du \underset{n\to\infty}{\to} Y_t - \xi - \int_t^T g(u, \kappa_u, Y_u, Z_u) du, \text{ in } \mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}).$$

And $Z_t^n\mathbf{1}_{[t,T]} \to Z_t\mathbf{1}_{[t,T]}$ in $\mathbb{L}^2(\Omega, \mathcal{G}, \mathcal{Q})$. Discussing as in the proof of Theorem 23 in [12] we prove that (Y, Z) satisfy BSDE (3.1) on $[t_k, T]$. By repeating the above method in finite steps to obtain a solution to the fractional BSDE (3.1) on $[t_{k-1}, t_k]$, $[t_{k-2}, t_{k-1}]$, ..., and then on $[0, T]$. Therefore, the proof of **Theorem 3.1** is complete. □

4. Comparison theorem

In this section we study a comparison theorem for one fractional BSDEs of the following form:

$$Y_t^i = \xi^i + \int_t^T g^i(u, \kappa_u, Y_u^i, Z_u^i)du - \int_t^T Z_u^i dB_u^H, 0 \leq t \leq T. \tag{4.1}$$

where for any $i \in \{1, 2\}$, $g^i : \Omega \times [0, T] \times \mathbb{R}^3 \to \mathbb{R}$. We assume in addition that

$$(\mathbf{H1.5}) \begin{cases} \xi^1 \leq \xi^2, \\ f^1\left(s, \kappa, y^2, z^2\right) \leq f^2\left(s, \kappa, y^2, z^2\right), \\ \forall \left(s,\ \kappa,\ y^2,\ z^2\right) \in [0, T] \times \mathbb{R}^3. \end{cases}$$

We have the following theorem.

Theorem 4.1. *Suppose that* (ξ^1, f^1) *and* (ξ^2, f^2) *satisfy* ***(H1.1)-(H1.5)***. *If* (Y_s^i, Z_s^i), $i = 1, 2$ *are solutions to Eq.* (4.1)*, then we have*

$$\forall t \in [0, T], \qquad Y^1 \leq Y^2, \qquad \mathbb{P} - a.s.$$

Proof. Let us define $\Delta Y_t = Y_t^2 - Y_t^1$, $\Delta Z_t = Z_t^2 - Z_t^1$, $\Delta\xi = \xi^2 - \xi^1$ and

$$\Delta f\left(t, \kappa_t, \Delta Y_t, \Delta Z_t\right) = f^2\left(t, \kappa_t, \Delta Y_t + Y_t^1, \Delta Z_t + Z_t^1\right) - f^1(t, \kappa_t, Y_t^1, Z_t^1).$$

It follows that $(\Delta Y_t, \Delta Z_t)_{t\in[0,T]}$ satisfies the fractional BSDE

$$\Delta Y_t = \Delta\xi + \int_t^T \Delta f\left(u, \kappa_u, \Delta Y_u, \Delta Z_u\right) du - \int_t^T \Delta Z_u dB_u^H, 0 \leq t \leq T.$$

Applying Itô-Tanaka's formula to $\left|\Delta Y_t^-\right|^2$, we obtain

$$\begin{aligned} &\mathbb{E}\left(\left|\Delta Y_t^-\right|^2 + \tfrac{2}{M}\int_t^T \mathbf{1}_{\{\Delta Y_u<0\}} u^{2H-1}\left|\Delta Z_u\right|^2 du\right) \\ &\leq \mathbb{E}\left(\Delta\xi^-\right) - 2\mathbb{E}\int_t^T \mathbf{1}_{\{\Delta Y_u<0\}}\Delta Y_u^- \Delta f\left(u, \kappa_u, \Delta Y_u, \Delta Z_u\right) du. \end{aligned}$$

Since $f^2(u,\kappa_u,Y_u^2,Z_u^2)-f^1(u,\kappa_u,Y_u^2,Z_u^2)\geq 0$ and $\Delta\xi=\xi^1-\xi^2\geq 0$, we have

$$\mathbb{E}\left(\left|\Delta Y_t^-\right|^2+\frac{2}{M}\int_t^T \mathbf{1}_{\{\Delta Y_u<0\}}u^{2H-1}\left|\Delta Z_u\right|^2 du\right)$$
$$\leq 2\mathbb{E}\int_t^T \Delta Y_u^-\left(f^1(u,\kappa_u,Y_u^2,Z_u^2)-f^1(u,\kappa_u,Y_u^1,Z_u^1)\right)du.$$

From **(H1.3)**, **(H1.4)** and Young's inequality, we have

$$2\Delta Y_u^- f^1\left(u,\kappa_u,Y_u^2,Z_u^2\right)-f^1(u,\kappa_u,Y_u^1,Z_u^1)$$
$$\leq C_{M,H}(u)\,\bar{\rho}\left(\left|\Delta Y_u^-\right|^2\right)+\frac{u^{2H-1}}{M}\left|\Delta Z_u\right|^2.$$

where $C_{M,H}(u)=\frac{M}{u^{2H-1}}>2$ and $\bar{\rho}(v)=\rho(v)+v$ for $v>0$ such that $\int_{0+}\frac{dv}{\bar{\rho}(v)}=+\infty$. Then

$$\mathbb{E}\left(\left|\Delta Y_t^-\right|^2\right)\leq \mathbb{E}\int_t^T C_{M,H}(u)\,\bar{\rho}\left(\left|\Delta Y_u^-\right|^2\right)du,$$

using Fubini's theorem and Jensen's inequality, we get

$$\mathbb{E}\left(\left|\Delta Y_t^-\right|^2\right)\leq \int_t^T C_{M,H}(u)\,\bar{\rho}\left(\mathbb{E}\left|\Delta Y_u^-\right|^2\right)du.$$

Bihari's inequality implies that $\Delta Y_t=Y_t^2-Y_t^1\geq 0$ $\mathbb{P}-a.s.$ for all $t\in[0,T]$. □

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