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ON THE CLASS OF n-NORMAL OPERATORS AND MOORE-PENROSE INVERSE

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ABSTRACT. Let $T \in B(H)$ be a bounded linear operator on a complex Hilbert space H. For $n \in \mathbb{N}$, an operator $T \in B(H)$ is said to be n-normal if $T^nT^* = T^*T^n$. In this paper we investigate a necessary and sufficient condition for the n-normality of ST and TS, where $S, T \in B(H)$. As a consequence, we generalize Kaplansky theorem for normal operators to n-normal operators. Also, In this paper, we provide new characterizations of n-normal operators by certain conditions involving powers of Moore-Penrose inverse.

1. INTRODUCTION AND PRELIMINARIES

let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. For an arbitrary operator $T \in B(H)$, we denote by $R(T)$, $N(T)$ and T^* for the range, the null subspace and the adjoint operator of T. It is well known that for $T \in \mathcal{B}(\mathcal{H})$, there is a unique factorization $T = U|T|$, where $N(U) = N(T) = N(|T|)$, U is a partial isometry, i.e. $UU^*U = U$ and $|T| = (T^*T)^{\frac{1}{2}}$ is the modulus of T. This factorization is called the polar decomposition of T. An operator $T \in B(H)$ is said to be nilpotent if there exists $p \in \mathbb{N}$ such that $T^p = 0$,

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normal if $TT^* = T^*T$ and n-normal if $T^nT^* = T^*T^n$, where $n \in \mathbb{N}$. The class of n-normal operators was first introduced and studied by A.S. Jibril [12] as an extension of the class of normal operators. He showed that T is n-normal if and only if T^n is normal. From the definition, it is easily seen that this class contains nilpotent operators with nilpotency n. Also the class of n-normal operators has been studied by many authors, mainly Alzuraiqi and Patel [2], M. Cho and B. Nastovska [6] and Mosić and Djordjević [18].

Suppose that $T, S \in B(H)$. It is known that, in general, ST is not normal. Historical notes and several versions of the problem are investigated. Kaplansky showed that it may be possible that ST is normal while TS is not. Indeed, in [13] he showed that if S and ST are normal, then TS is normal if and only if T commutes with $|S|$. Later, Deutsch, Gibson and Schneider [8] proved that, if T and S are two complex square matrices, then ST and TS are normal if and only if $S^*ST = TSS^*$ and $STT^* = T^*TS$.

In section 2 of this paper, we focus on to study the n-ormality of ST and TS. Firstly, we present a new characterization of n-normal operators via the polar decomposition. In particular, we show that if the partial isometry factor U of the polar decomposition of the operator T is normal, then T is n-normal if and only if $(|T|U)^n = (U|T|)^n$. Afterwards, we generalize Kaplansky theorem for normal operators to n-normal operators. Also, we use the Fuglede-Putnam theorem to prove that both TS and ST are n-normal if and only if $S^*(ST)^n = (TS)^nS^*$ and $(ST)^nT^* = T^*(TS)^n$. Finally, we provide conditions related to the factors polar decomposition of S under which TS and ST becomes n-normal.

Now, we recall some notions that will be used in section 3 of this paper. For $T \in B(H)$, the Moore-Penrose inverse of T is the unique operator $T^+ \in \mathcal{B}(\mathcal{H})$ which satisfies:

$$TT^+T = T, \;\; T^+TT^+ = T^+, \;\; (TT^+)^* = TT^+, \;\; (T^+T)^* = T^+T.$$

It is well known that the Moore-Penrose inverse of T exists if and only if $\mathcal{R}(T)$ is closed. It is easy to see that $\mathcal{R}(T^+) = \mathcal{R}(T^*)$, TT^+ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(T)$ and that T^+T is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(T^*)$. The operator T is said to be EP operator, if $\mathcal{R}(T)$ is closed and $TT^+ = T^+T$. Clearly

$$T \text{ EP} \Longleftrightarrow \mathcal{R}(T) = \mathcal{R}(T^*) \Longleftrightarrow \mathcal{N}(T) = \mathcal{N}(T^*).$$

Obviously, every normal operator with closed range is EP but the converse is not true even in a finite dimensional space. For more details about on EP operators see [5, 9, 18].

The ascent $a(T)$ and the descent $d(T)$ of T are given by

$$a(T) = \min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\},$$

and

$$d(T) = \min\{p \in \mathbb{N} : R(T^p) = R(T^{p+1})\}.$$

In both cases the infimum of the empty set is equal to ∞. If $a(T)$ and $d(T)$ are finite, they are equal and their common value is called the index of T and it is denoted by $ind(T)$ [11, Proposition 38.3].

The group inverse of $T \in \mathcal{B}(H)$ is the unique operator $T^{\#} \in \mathcal{B}(H)$ such that

$$TT^{\#}T = T \;\; T^{\#}TT^{\#} = T^{\#} \;\; TT^{\#} = T^{\#}T.$$

$T^{\#}$ exists if and only if $ind(T) \leq 1$ [14]. When $ind(T) = 0$, the group inverse reduces to the standard inverse, that is, $T^{\#} = T^{-1}$.

In section 3, firstly, we show that if T is n-normal operator with closed range, then $ind(T) \leq n$, and T can be written as a direct sum of an invertible n-normal operator and a nilpotent operator. Secondly, We prove that if T is Moore-Penrose invertible, then T is 2n-normal if and only if its Moore-Penrose inverse is too 2n-normal. But, this need not be true in case of (2n+1)-normal operators, as shown in example 3.5. In [18], Mosić and Djordjević presented a number of necessary and sufficient conditions for both Moore-penrose invertible and group invertible elements in rings with involution to be n-normal. Motivated by them, we will give some new equivalent conditions for the n-normaliy of Moore-Penrose invertible operators, by omitting the assumption of the group invertibility. Finally, we show that an n-normal operator is EP, if it is group invertible.

Now, we state some well-known results needed in the sequel.

Proposition 1.1. *[2, 12] Let $T \in B(H)$ and $n \in \mathbb{N}$. T is n-normal if and only if T^n is normal.*

Proposition 1.2. *[2] Let $T \in B(H)$ be n-normal. Then the following hold.*

(a) *T^* is n-normal.*

(b) *T^{-1} exists, then T^{-1} is n-normal.*

(c) *T^m is n-normal for any positive integer m.*
(d) *If $S \in B(H)$ is unitary equivalent to T, then S is n-normal.*

The well-known Fuglede-Putnam's Theorem is as follows:

Theorem 1.3. *[19] Let $T, S \in B(H)$ be normal operators and $X \in B(H)$. If $XS = TX$, then $XS^* = T^*X$.*

Lemma 1.4. *[17] For any $T \in B(H)$ with closed range, then the following are satisfied*

(a) $(T^*)^+ = (T^+)^*$.
(b) $(TT^*)^+ = (T^*)^+T^+$.
(c) $(T^*T)^+ = T^+(T^*)^+$.
(d) $T^+ = (T^*T)^+T^* = T^*(TT^*)^+$.
(e) $(T^*)^+ = T(T^*T)^+ = (TT^*)^+T$.
(f) *If $S \in B(H)$ such that $ST = TS$ and $ST^* = T^*S$, then $ST^+ = T^+S$.*

Lemma 1.5. *[4] If $T \in B(H)$ is group invertible, then*

(a) *T is EP if and only if $R(T) \subset R(T^*)$.*
(b) *If $S \in B(H)$, then $ST = TS$ if and only if $ST^\# = T^\#S$.*

2. On the n-normality of products of operators

Let $T = U|T|$ be the polar decomposition of an operator T. Then It is well known that T is normal is if and only if U is normal and U commutes with $|T|$ [10]. The following example shows that this need not be true in case of n-normal operators.

Example 2.1. *Let $T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \in \mathbb{C}^3$. Then $T^3 = 0$, which implies that T is 3-normal. The canonical polar decomposition of T is $T = U|T|$, where*

$$U = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \text{ and } |T| = \frac{1}{\sqrt{5}} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

it is easy to verify that $U^3 = \frac{1}{5\sqrt{5}}\begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ -4 & 2 & 0 \end{bmatrix}$ *is not normal. So* U *is not 3-normal and*

$$U|T| = T \neq \begin{bmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = |T|U.$$

Now, we present some new conditions under which certain operators are n-normal.

Proposition 2.2. *Let* n *be a positive integer and* $T \in B(H)$*, with polar decomposition* $T = U|T|$*. If* U *is normal, then the following statements are equivalent*

(1) T *is n-normal.*
(2) $(U|T|)^n = (|T|U)^n$.
(3) $UT^n = T^nU$.

Proof.
$(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ holds without the condition U is normal.

$(1) \Rightarrow (2)$. Assume that T is n-normal. Then $T^nT^* = T^*T^n$. That is $T^nT^*T = T^*TT^n$ and so $T^n|T| = |T|T^n$. It follows that

$$(U|T|)^n|T| = |T|(U|T|)^n = (|T|U)^n|T|.$$

Hence, $(U|T|)^n = (|T|U)^n$ on $\overline{R(|T|)}$. Since $N(|T|) = N(U)$, then

$$(U|T|)^n = (|T|U)^n = 0 \text{ on } N(|T|).$$

Therefore,

$$(U|T|)^n = (|T|U)^n \text{ on } H = \overline{R(|T|)} \oplus N(|T|).$$

$(2) \Rightarrow (3)$. Using the assumption (2), we obtain

$$UT^n = U(U|T|)^n = U(|T|U)^n = (U|T|)^nU = T^nU.$$

$(3) \Rightarrow (1)$. Assume that $UT^n = T^nU$. Since U is normal, by Fuglede-Putnam theorem we get $U^*T^n = T^nU^*$. It follows that

$$\begin{aligned} T^*T^n &= |T|U^*(U|T|)^n \\ &= |T|(U|T|)^nU^* \\ &= (|T|U)^n|T|U^* \\ &= (|T|U)^nT^*. \end{aligned}$$

On the other hand, since U^*U is an orthogonal projection onto $R(|T|)$, we have

$$\begin{aligned} T^nT^* &= (U|T|)^n|T|U^* \\ &= (U|T|)^nU^*U|T|U^* \\ &= U^*(U|T|)^nU|T|U^* \\ &= U^*U(|T|U)^n|T|U^*. \\ &= (|T|U)^nT^*. \end{aligned}$$

Hence, $T^*T^n = T^nT^*$. So that T is n-normal. □

In [13], Kaplansky showed that if $S, T \in B(H)$ such that S and ST are normal, then TS is normal if and only if T commutes with $|S|$. Kaplansky theorem's has been extended from normal operators to hyponormal operators [1], quasinormal operators [3] and D-normal matrices [7]. We have the following Kaplansky theorem's for n-normal operators.

Theorem 2.3. *Let $T, S \in B(H)$ such that S is normal and ST is n-normal. Then*

$$S^*ST = TS^*S \Longrightarrow TS \text{ is n-normal.}$$

Proof. Since S is normal, then by [10, Theorem 3] there exists a unitary operator U such that

$$S = U|S| = |S|U.$$

The assumption $S^*ST = TS^*S$ is equivalent to $(S^*S)^{\frac{1}{2}}T = T(S^*S)^{\frac{1}{2}}$. so that $|S|T = T|S|$. Consequently, we have

$$TS = TU|S| = |S|TU = U^*U|S|TU = U^*STU.$$

Therefore, TS is unitary equivalent to ST. Since ST is n-normal, according to Proposition 1.2 (d), we conclude that TS is also n-normal. □

Remark 2.4. *In Theorem 2.3, the reverse implication is false as shown by the following example.*

Example 2.5. *Let* $S = \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}$ *and* $T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ *in* $B(H \oplus H)$*, where* $P \geq 0$*. Then* S *is normal and by a simple computation, we have* $(ST)^2 = (TS)^2 = 0$*. Hence* ST *and* TS *are 2-normal, but*

$$S^*ST = \begin{pmatrix} 0 & 0 \\ P^2 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & P^2 \\ 0 & 0 \end{pmatrix} = TS^*S.$$

As a consequence of the previous theorem, we have the following result.

Corollary 2.6. *Let* $T, S \in B(H)$ *such that* S *is unitary. Then*

$$ST \text{ is n-normal} \Longleftrightarrow TS \text{ is n-normal.}$$

Proof. $\Longrightarrow$ Assume that ST is n-normal. Since S is unitary, then it is normal and $S^*ST = TS^*S = T$. Hence, by Theorm 2.3, ST is n-normal.

$\Longleftarrow$. Now we suppose that TS is n-normal. By Proposition 1.2 (a), S^*T^* is n-normal. Moreover, since S^* is unitaire, by the above implication T^*S^* is too n-normal. Therefore, ST is n-normal. □

The following result generalize Theorems 2 obtained for normal matrices in [8] to n-normal operators on an arbitrary Hilbert space.

Theorem 2.7. *Let* $T, S \in B(H)$ *and* $n \in \mathbb{N}$*. Then* ST *and* TS *are n-normal if and only if* $S^*(ST)^n = (TS)^nS^*$ *and* $(ST)^nT^* = T^*(TS)^n$*.*

Proof. Assume that ST and TS are n-normal. Then $(ST)^n$ and $(TS)^n$ are normal. Hence, From the hypothesis $(ST)^nS = S(TS)^n$ and $T(ST)^n = (TS)^nT$ and by Fuglede-Putnam Theorem, we get

$$S^*(ST)^n = (TS)^nS^* \text{ and } (ST)^nT^* = T^*(TS)^n.$$

Conversely, if $S^*(ST)^n = (TS)^nS^*$ and $(ST)^nT^* = T^*(TS)^n$, then multiplying the first equation by T^* and the second one by S^* we deduce that

$$(ST)^*(ST)^n = T^*(TS)^nS^* = (ST)^n(ST)^*,$$

and

$$(TS)^n(TS)^* = S^*(ST)^nT^* = (TS)^*(TS)^n.$$

Therefore, ST and TS are n-normal. □

Theorem 2.8. *Let $T, S \in B(H)$ and let $S = U|S|$ be the polar decompositions of S with U is unitary. For $n \in \mathbb{N}$. The following properties hold*

(1) *If TU is normal and $(ST)^nU = U(TS)^n$, then ST and TS are n-normal.*
(2) *If ST and TS are n-normal, then $(ST)^nU = U(TS)^n$.*

Proof.
(1) Suppose that TU is normal and $(ST)^nU = U(TS)^n$.

In case $n = 1$, since U is unitary, then the assumption $STU = UTS$ is equivalent to $|S|TU = TU|S|$. It follows from [8, Theorem 3], that ST and TS are normal.

Now, in case $n \geq 2$, we observe that

$$\begin{aligned}
(ST)^nU = U(TS)^n &\iff ST(ST)^{n-1}U = UT(ST)^{n-1}S \\
&\iff U|S|T(ST)^{n-1}U = UT(ST)^{n-1}U|S| \\
&\iff U^*U|S|T(ST)^{n-1}U = U^*UT(ST)^{n-1}U|S| \\
&\iff |S|T(ST)^{n-1}U = T(ST)^{n-1}U|S| \qquad (2.1)\\
&\iff |S|^2T(ST)^{n-1}U = T(ST)^{n-1}U|S|^2. \qquad (2.2)
\end{aligned}$$

Consequently,

$$\begin{aligned}
(TS)^nS^* &= T(ST)^{n-1}SS^* \\
&= T(ST)^{n-1}U|S|^2U^* \\
&= |S|^2T(ST)^{n-1}UU^* \quad (by\,(2.2)) \\
&= |S|^2T(ST)^{n-1} \\
&= |S|U^*U|S|T(ST)^{n-1} \\
&= S^*(ST)^n.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}|S|(TS)^{n-1}TU &= |S|T(ST)^{n-1}U\\ &= T(ST)^{n-1}U|S| \quad (by\,(2.1))\\ &= (TS)^n\\ &= TU|S|(TS)^{n-1}.\end{aligned}$$

Since TU is normal, then by Fuglede-Putnam theorem we obtain

$$(TU)^*|S|(TS)^{n-1} = |S|(TS)^{n-1}(TU)^*.$$

From this equality, it follows

$$\begin{aligned}(ST)^nT^* &= S(TS)^{n-1}TT^*\\ &= U|S|(TS)^{n-1}TUU^*T^*\\ &= U|S|(TS)^{n-1}(TU)^*TU \quad (TU\ is\ normal)\\ &= U(TU)^*|S|(TS)^{n-1}TU\\ &= UU^*T^*|S|(TS)^{n-1}TU\\ &= T^*TU|S|(TS)^{n-1}\\ &= T^*(TS)^n.\end{aligned}$$

Hence, by Theorem 2.7, ST and TS are n-normal.

To prove (2), we also consider two cases:

Case (i): $n = 1$. Since ST and TS are normal, by [8, Theorem 3], $|S|TU = TU|S|$. multiplying this equality by U from the left side, we get $STU = UTS$.

Case (ii): $n \geq 2$. By (2.2), it is enough to prove that $|S|^2T(ST)^{n-1}U = T(ST)^{n-1}U|S|^2$. Sinse ST and TS are n-normal, using again the Theorem 2.7, we obtain

$$\begin{aligned}|S|^2T(ST)^{n-1}U &= S^*ST(ST)^{n-1}U\\ &= S^*(ST)^nU\\ &= (TS)^nS^*U\\ &= T(ST)^{n-1}SS^*U\\ &= T(ST)^{n-1}|S^*|^2U\\ &= T(ST)^{n-1}U|S|^2 \quad (\text{since } |S^*|U = U|S|).\end{aligned}$$

□

3. Moore-Penrose inverse and n-normal operators

It is well known that the ascent and descent of a normal operator with closed range are finite. It is equally true for an n-normal operator with closed range.

Theorem 3.1. *Let $T \in B(H)$ be an n-normal. Then the following properties hold:*

(1) $a(t) \leq n$.

(2) *If T has a closed range, then*

  (a) $ind(T) \leq n$.

  (b) *$R(T^k)$ is closed for all $k \geq n$ and T has the following matrix representation $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$ with respect to the orthogonal sum $H = R(T^n) \oplus N(T^n)$, where $T_1 = T \setminus R(T^n)$ is an invertible n-normal operator and T_2 is a nilpotent operator with nilpotency n.*

Proof.

(1) To prove that $a(t) \leq n$ it will suffice to show that $N(T^{n+1}) \subset N(T^n)$. As the converse inclusion is obvious it follows that $N(T^n) = N(T^{n+1})$, which implies that $N(T^k) = N(T^{k+1})$, for $k \geq n$. Let $x \in N(T^{n+1})$. Then $T^{n+1}(x) = 0$ and so we get

$$T^n(x) \in N(T) \cap R(T^n) \subset N(T^n) \cap R(T^n) \subset N(T^n) \cap \overline{R(T^n)}.$$

Since T is n-normal, T^n is normal. Therefore, $N(T^n) = N((T^n)^*)$. Consequently,

$$T^n(x) \in N((T^n)^*) \cap \overline{R(T^n)} = \{0\}.$$

Hence, $x \in N(T^n)$.

(2) Now, suppose that $R(T)$ is closed.

(a) If $n = 1$, then T is normal with clsed range and so $ind(T) \leq 1$.

If $n \geq 2$, then $(T^*)^n T = T(T^*)^n$ and $R(T) = R(TT^*)$, since T is n-normal with closed range. It follows that

$$R((T^n)^*) = (T^*)^{n-1} R(T^*) = (T^*)^{n-1} R(T^*T) = R((T^*)^n T) = R(T(T^*)^n).$$

Then, $R((T^n)^*) \subset R(T)$. Since TT^+ is the orthogonal projection onto $R(T)$, therefore $TT^+(T^n)^* = (T^n)^*$ and so $T^{n+1}T^+ = T^n$. Thus, $R(T^n) \subset R(T^{n+1})$. Also, since the converse inclusion is obvious it follows that $R(T^n) = R(T^{n+1})$, which implies that $N(T^k) = N(T^{k+1})$, for $k \geq n$. So, we get $d(T) \leq n$ and thus, $ind(T) \leq n$ by [11, Proposition 38.3].

(b) Finally, since $ind(T) \leq n$, according to [16, Corollary 2.2], we deduce that $R(T^k)$ is closed for all $k \geq n$ and

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : R(T^n) \oplus N(T^n) \to R(T^n) \oplus N(T^n),$$

where $T_1 = T \setminus R(T^n)$ is an invertible operator and T_2 is a nilpotent operator. Then we have

$$T^n = \begin{bmatrix} T_1^n & 0 \\ 0 & 0 \end{bmatrix} : R(T^n) \oplus N(T^n) \to R(T^n) \oplus N(T^n).$$

Since T^n is normal, then T_1^n is also normal. Therefore, T_1 is n-normal. □

The following exemple shows that if T is an n-normal operator such that $R(T^n)$ is closed, then $R(T)$ need not be closed.

Example 3.2. *Le $T : \ell^2(\mathbb{N}) \to\to \ell^2(\mathbb{N})$ be the unilateral weighted shift defined by*

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, 0, \frac{x_3}{3}, 0, \ldots).$$

Then T is 2-normal, since $T^2 = 0$. Hence $R(T^2)$ is closed but $R(T)$ is not closed.

It was proved in Proposition 1.2, (b) that if T is an invertible n-normal operator, then T^{-1} is also n-normal. But, what happens if we replace the inverse by the Moore-Penrose inverse? In order to give the answer to this question, we need the following lemma.

Lemma 3.3. *Let $T \in B(H)$ with closed range. If T is a nilpotent operator with nilpotency 2, then T^+ is also nilpotent with nilpotency 2.*

Proof. Let $T = U|T|$ be the polar decomposition of T, where U is a partial isometry with $R(U) = \overline{R(T)}$ and $N(U) = N(T)$. Since $T^2 = 0$, $R(T) \subset N(T)$. It follows that

$$R(U) = \overline{R(T)} \subset N(T) = N(U).$$

Hence, $U^2 = 0$ and so $(U^*)^2 = 0$. On the other hand, $T^+ = |T|^+U^* = U^*|T^+|$. Therefore, $(T^+)^2 = |T|(U^*)^2|T^+| = 0$, which completes the proof. □

Proposition 3.4. *Let $T \in B(H)$ with closed range. Then T is 2n-normal if and only if T^+ is 2n-normal. In this case $(T^{2n})^+ = (T^+)^{2n}$.*

Proof. By Proposition 1.2 (c), it is suficient to consider the case that T is 2-normal. According to theorem 2.1, T has the following operator matrix

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : R(T^2) \oplus N(T^2) \to R(T^2) \oplus N(T^2),$$

where T_1 is invertible, T_1^2 is normal and $T_2^2 = 0$. Since $R(T)$ is closed, we get

$$T^+ = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & T_2^+ \end{bmatrix} \text{ and } (T^2)^+ = \begin{bmatrix} T_1^{-2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore by Lemma 3.3

$$(T^+)^2 = \begin{bmatrix} T_1^{-2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $(T^2)^+ = (T^+)^2$. and T^+ is 2-normal, since the normality of T_1^2 implies the normality of T^2. Conversely, If T^+ is 2-normal, then $(T^+)^+ = T$ is also 2-normal. Therefore, the proof is complete. □

Proposition 3.4 is not necessarily true if T is (2n+1)-normal. Indeed if we take the example 2.1, T is 3-normal. Now, by simple calculations we have

$$T^+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } (T^+)^3 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

But as $(T^+)^3$ is not normal, T^+ is not 3-normal.

The next result provide some equivalent conditions for an operator in $B(H)$ with closed range to be n-normal. the condition(4) was etablished by Mosić and Djordjević for both Moore-penrose invertible and group invertible elements in rings with involution [18, Lemma 3.1].

Theorem 3.5. *Let $T \in B(H)$ with closed range and $n \in \mathbb{N}$. Then the following statements are equivalent*

(1) *T is n-normal.*
(2) *$T^nT^*T = T^*TT^n$ and $T^nTT^* = TT^*T^n$.*
(3) *$T^n(T^*T)^+ = (T^*T)^+T^n$ and $T^n(TT^*)^+ = (TT^*)^+T^n$.*
(4) *$T^nT^+ = T^+T^n$ and $(T^*)^nT^+ = T^+(T^*)^n$.*

Proof.

$(1) \Rightarrow (2)$. Since T is n-normal, then $T^nT^* = T^*T^n$. So, we can easily verify that the statement (2) holds.

$(2) \Rightarrow (3)$. Assume that (2) holds. Since T^*T and TT^* are both self-adjoint, then $(T^*T)^{\#} = (T^*T)^+$ and $(TT^*)^{\#} = (TT^*)^+$. Hence from Lemma 1.5 (b), we deduce

$$T^n(T^*T)^+ = (T^*T)^+T^n \text{ and } T^n(TT^*)^+ = (TT^*)^+T^n.$$

$(3) \Rightarrow (4)$. First, we show that $(T^*)^nT^+ = T^+(T^*)^n$. Since $(T^*)^+ = T(T^*T)^+$, the commutativity T^n with $(T^*T)^+$ implies that $T^n(T^*)^+ = (T^*)^+T^n$. Thus $(T^*)^nT^+ = T^+(T^*)^n$, by taking an adjoint.

Now, we show that $T^nT^+ = T^+T^n$. By Lemma 1.4, we have

$$\begin{aligned} T^nT^+ &= T^n(T^*T)^+T^* \\ &= (T^*T)^+T^nT^* \\ &= T^+(T^*)^+T^nT^* \\ &= T^+T^n(T^*)^+T^* \\ &= T^+T^n(TT^+)^* \\ &= T^+T^nTT^+. \end{aligned}$$

Moreover, from the hypothesis $T^n(TT^*)^+ = (TT^*)^+T^n$ and Lemma 1.4, we get

$$\begin{aligned} T^+T^n &= T^*(TT^*)^+T^n \\ &= T^*T^n(TT^*)^+ \\ &= T^*T^n(T^*)^+T^+ \\ &= T^*(T^*)^+T^nT^+ \\ &= (T^+T)^*T^nT^+ \\ &= T^+TT^nT^+. \end{aligned}$$

Finally, $T^nT^+ = T^+T^n$, since $T^+T^nTT^+ = T^+TT^nT^+$.

$(4) \Rightarrow (1)$. Assume that (4) holds. Hence $T^n(T^*)^+ = (T^*)^+T^n$ and $T^n((T^*)^+)^* = ((T^*)^+)^*T^n$. According to Lemma 1.4 (f), we find $T^nT^* = T^*T^n$, since $((T^*)^+)^+ = T^*$. Thus, T is n-normal. □

Proposition 3.6. *Let $T \in B(H)$ and $n \in \mathbb{N}$. If $R(T^n)$ is closed, then the following statements are equivalent*

(1) *T is n-normal.*
(2) *$(T^n)^+T = T(T^n)^+$ and $(T^n)^+T^* = T^*(T^n)^+$.*
(3) *T^n is EP and $(T^n)^+T^* = T^*(T^n)^+$.*

Proof.
(1) $\Rightarrow$ (2). Suppose that T is n-normal. Then T^n is normal and so $(T^n)^+ = (T^n)^\#$. Since $T^nT = TT^n$ and $T^nT^* = T^*T^n$, by Lemma 1.4 (f), we deduce $(T^n)^+T = T(T^n)^+$ and $(T^n)^+T^* = T^*(T^n)^+$.
(2) $\Rightarrow$ (3). From the assumption $(T^n)^+T = T(T^n)^+$, it follows $(T^n)^+T^n = T^n(T^n)^+$. Hence, T is EP.
(3) $\Rightarrow$ (1). Asume that (3) holds. Then $(T^n)^+$ is also EP and so

$$((T^n)^+)^\# = (T^n)^+)^+ = T^n.$$

Therefore, from the assumption $(T^n)^+T^* = T^*(T^n)^+$ and by Lemma 1.5 (b), we obtain $T^nT^* = T^*T^n$. Thus T is n-normal. ☐

Remark 3.7. *It is well known that every normal operator with closed range is EP. Hence one might expect that there is a relationship between n-normal operators and EP operators. But in the example 2.1, T is 3-normal but it is not EP, because*

$$TT^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T^+T.$$

Now, we provide a condition under which an n-normal operator becomes EP.

Proposition 3.8. *Let $T \in B(H)$ be an n-normal operator. If T is group invertible, then T is EP.*

Proof. If T is group invertible, then $R(T)$ is closed and $d(T) \leq 1$. This implies $R(T) = R(T^n)$. On the other hand, since T is n-normal, T^n is normal and so $N(T^n) = N((T^n)^*)$. By taking the orthogonal complements in this equality and since $R(T^n)$ and $R((T^n)^*)$ are closed, we deduce that

$$R(T^n) = R((T^n)^*) \subset R(T^*).$$

Hence, $R(T) \subset R(T^*)$ and according to Lemma 1.5 (a), we conclude that T is EP. $\square$

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