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ON THE DIOPHANTINE EQUATION $\frac{1}{X}+\frac{3}{Y}+\frac{5}{Z}=\frac{3}{4}$

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ABSTRACT. In this paper, we show that the Diophantine equation $\frac{1}{x}+\frac{3}{y}+\frac{5}{z}=\frac{3}{4}$ has finitely many integral solutions in the positive integers x, y and z.

1. INTRODUCTION

In 1950, Erdos & Straus. [1] conjectured that for all integers $n \geq 2$, the rational number $\frac{4}{n}$ can be expressed as sum three unit fractions. Thus the conjecture formally states that for every integer $n \geq 4$, there exists positive integers x, y and z such that $\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

In 2011, Hari Kishan et. al. [2] discussed the Diophantine equations of second and higher degree of the form $3xy = n(x + y)$ and $3xyz = n(xy + yz + zx)$ etc.

In 2013 Rabago, J. F.T. & Tagle, R.P. [3] discussed the area and volume of a certain regular solid and the Diophantine equation $\frac{1}{2}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. Also in 2013, Sander, J. [4] discussed the Diophantine equation $\frac{p}{q}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ has finite number of positive integer solutions.

Recently in 2018, Bai T. [5] find the solutions of the Diophantine equation $\frac{1}{2}=\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ and generalizations of it. In 2019, Pakapongpun. [6] examined the

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Integral Solutions of the Diophantine Equation $\frac{1}{2} = \frac{1}{x} + \frac{2}{y} + \frac{3}{z}$ for positive integer x, y and z respectively.

In this paper, we show that integrals solutions of the Diophantine equation $\frac{1}{x} + \frac{3}{y} + \frac{5}{z} = \frac{3}{4}$.

2. Method of Analysis

In the Diophantine equation $\frac{1}{x} + \frac{3}{y} + \frac{5}{z} = \frac{3}{4}$, we have $x \geq 2, y \geq 5$ and $z \geq 7$, We are consider Three cases.

Case 1. If $x \leq y \leq z$ or $x \leq z \leq y$, then $\frac{1}{x} < \frac{3}{4}$. This implies that $x \geq 2$. Also $\frac{9}{x} \geq \frac{3}{4}$. We get $x \leq 12$. Thus $x = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$. We find the following cases:

$$x = 2 \implies \frac{3}{y} + \frac{5}{z} = \frac{1}{4} \tag{2.1}$$

$$x = 3 \implies \frac{3}{y} + \frac{5}{z} = \frac{5}{12} \tag{2.2}$$

$$x = 4 \implies \frac{3}{y} + \frac{5}{z} = \frac{1}{2} \tag{2.3}$$

$$x = 5 \implies \frac{3}{y} + \frac{5}{z} = \frac{11}{20} \tag{2.4}$$

$$x = 6 \implies \frac{3}{y} + \frac{5}{z} = \frac{7}{12} \tag{2.5}$$

$$x = 7 \implies \frac{3}{y} + \frac{5}{z} = \frac{17}{28} \tag{2.6}$$

$$x = 8 \implies \frac{3}{y} + \frac{5}{z} = \frac{5}{8} \tag{2.7}$$

$$x = 9 \implies \frac{3}{y} + \frac{5}{z} = \frac{23}{36} \tag{2.8}$$

$$x = 10 \implies \frac{3}{y} + \frac{5}{z} = \frac{13}{20} \tag{2.9}$$

$$x = 11 \implies \frac{3}{y} + \frac{5}{z} = \frac{29}{44} \tag{2.10}$$

$$x = 12 \implies \frac{3}{y} + \frac{5}{z} = \frac{2}{3} \tag{2.11}$$

In equations (2.1), (2.2), (2.3) and (2.7) we apply following the relations:

$$(y-12)(z-20)=240 \tag{2.12}$$

$$(5y-36)(z-12)=432 \tag{2.13}$$

$$(y-6)(z-10)=60 \tag{2.14}$$

$$(5y-24)(z-8)=192. \tag{2.15}$$

From equation (2.12) the possible cases are:

$$\begin{aligned}
&(y-12)=1,(z-20)=240;(y-12)=2,(z-20)=120\\
&(y-12)=3,(z-20)=80;(y-12)=4,(z-20)=60\\
&(y-12)=5,(z-20)=48;(y-12)=6,(z-20)=40\\
&(y-12)=8,(z-20)=30;(y-12)=10,(z-20)=24\\
&(y-12)=12,(z-20)=20;(y-12)=240,(z-20)=1\\
&(y-12)=120,(z-20)=2;(y-12)=80,(z-20)=3\\
&(y-12)=60,(z-20)=4;(y-12)=48,(z-20)=5\\
&(y-12)=40,(z-20)=6;(y-12)=30,(z-20)=8\\
&(y-12)=24,(z-20)=10;(y-12)=20,(z-20)=12.
\end{aligned}$$

thus we get the following solutions:

$$\begin{aligned}
(x,y,z)=&(2,13,260),(2,14,140),(2,15,100),(2,16,80),(2,17,68),(2,18,60),\\
&(2,20,50),(2,,22,44),(2,24,40),(2,252,21),(2,132,22),(2,92,23),\\
&(2,72,24),(2,60,25),(2,52,26),(2,42,28),(2,36,30),(2,32,32).
\end{aligned}$$

In the Same manner, in (2.13), we get $(x,y,z)=(3,8,120)$, $(3,9,60)$, $(3,36,15)$, $(3,18,20)$; in (2.14) we get $(x,y,z)=(4,7,70)$, $(4,8,40)$, $(4,9,40)$, $(4,10,25)$, $(4,11,22)$, $(4,12,20)$, $(4,16,16)$, $(4,18,15)$, $(4,66,11)$, $(4,36,12)$, $(4,26,13)$, $(4,21,12)$; and in (2.15) we get $(x,y,z)=(8,5,200)$, $(8,6,40)$, $(8,24,10)$, $(8,8,20)$.

Further, if $x=5$, then

$$\frac{3}{y}+\frac{5}{z}=\frac{11}{20}. \tag{2.16}$$

If $y \leq z$, then $\frac{11}{20} = \frac{3}{y} + \frac{5}{z} \leq \frac{8}{y}$, that is, $5 \leq y \leq 14$, only $y = 6, 10$ gives positive integer of z. Hence $(x, y, z) = (5, 6, 100), (5, 10, 20)$.

If $z \leq y$, then $\frac{11}{20} = \frac{3}{y} + \frac{5}{z} \leq \frac{8}{z}$ that is, $5 \leq z \leq 14$, only $z = 10$ give positive integer of y. Hence $(x, y, z) = (5, 60, 10)$.

'If $x = 6$, then

$$\frac{3}{y} + \frac{5}{z} = \frac{7}{12}. \tag{2.17}$$

If $y \leq z$, then $\frac{7}{12} = \frac{3}{y} + \frac{5}{z} \leq \frac{8}{y}$, that is, $6 \leq y \leq 13$, only $y = 6, 8, 9, 12$ gives positive integer of z. Hence $(x, y, z) = (6, 6, 60)(6, 8, 24), (6, 9, 20), (6, 12, 15)$.

If $z \leq y$, then $\frac{7}{12} = \frac{3}{y} + \frac{5}{z} \leq \frac{8}{z}$ that is, $6 \leq z \leq 13$, only $z = 9, 10, 12$ gives positive integer of y. Hence $(x, y, z) = (6, 108, 9), (6, 36, 10), (6, 18, 12)$.

If $x = 7$, then

$$\frac{3}{y} + \frac{5}{z} = \frac{17}{28}. \tag{2.18}$$

If $y \leq z$, then $\frac{17}{28} = \frac{3}{y} + \frac{5}{z} \leq \frac{8}{y}$, that is, $7 \leq y \leq 13$, only $y = 7, 12$ give positive integer of z. Hence $(x, y, z) = (7, 7, 28), (7, 12, 14)$.

If $z \leq y$, then $\frac{17}{28} = \frac{3}{y} + \frac{5}{z} \leq \frac{8}{z}$, that is, $7 \leq z \leq 13$, only $z = 10$ give positive integer of y. Hence $(x, y, z) = (7, 28, 10)$.

If $x = 9$, then

$$\frac{3}{y} + \frac{5}{z} = \frac{23}{36}. \tag{2.19}$$

If $y \leq z$, then $\frac{23}{36} = \frac{3}{y} + \frac{5}{z} \leq \frac{8}{y}$, that is, $9 \leq y \leq 12$, there is no positive integer of z. We do not solution of (2.19).

If $z \leq y$, then $\frac{23}{36} = \frac{3}{y} + \frac{5}{z} \leq \frac{8}{z}$ that is, $9 \leq z \leq 12$, only $z = 9$ give positive integer of y. Hence $(x, y, z) = (9, 36, 9)$.

If $x = 10$, then

$$\frac{3}{y} + \frac{5}{z} = \frac{13}{20}. \tag{2.20}$$

If $y \leq z$, then $\frac{13}{20} = \frac{3}{y} + \frac{5}{z} \leq \frac{8}{y}$, that is, $10 \leq y \leq 12$, there is no positive integer of z. We do not solution of (2.20).

If $z \leq y$, then $\frac{13}{20} = \frac{3}{y} + \frac{5}{z} \leq \frac{8}{z}$ that is, $10 \leq z \leq 12$, only $z = 10$ give positive integer of y. Hence $(x, y, z) = (10, 20, 10)$.

If $x=11$, then

$$\frac{3}{y}+\frac{5}{z}=\frac{29}{44}. \tag{2.21}$$

If $y\le z$, then $\frac{29}{44}=\frac{3}{y}+\frac{5}{z}\le\frac{8}{y}$, that is, $11\le y\le 12$, there is no positive integer of z. We do not solution of (2.21).

If $z\le y$, then $\frac{29}{44}=\frac{3}{y}+\frac{5}{z}\le\frac{8}{z}$, that is, $11\le z\le 12$, there is no positive integer of y. We do not solution of (2.21).

Case 2. If $y\le x\le z$ or $y\le z\le x$, then $\frac{3}{y}<\frac{3}{4}$. This implies that $y\ge 5$. Also $\frac{9}{y}\ge\frac{3}{4}$. We get $y\le 12$.

Thus $y=5,6,7,8,9,10,11,12$. We find the following cases:

$$y=5 \implies \frac{1}{x}+\frac{5}{z}=\frac{3}{20} \tag{2.22}$$

$$y=6 \implies \frac{1}{x}+\frac{5}{z}=\frac{1}{4} \tag{2.23}$$

$$y=7 \implies \frac{1}{x}+\frac{5}{z}=\frac{9}{28} \tag{2.24}$$

$$y=8 \implies \frac{1}{x}+\frac{5}{z}=\frac{3}{8} \tag{2.25}$$

$$y=9 \implies \frac{1}{x}+\frac{5}{z}=\frac{5}{12} \tag{2.26}$$

$$y=10 \implies \frac{1}{x}+\frac{5}{z}=\frac{9}{20} \tag{2.27}$$

$$y=11 \implies \frac{1}{x}+\frac{5}{z}=\frac{21}{44} \tag{2.28}$$

$$y=12 \implies \frac{1}{x}+\frac{5}{z}=\frac{1}{2}. \tag{2.29}$$

In equations (2.23), (2.26) and (2.29), we following the relations:

$$(x-4)(z-20)=80 \tag{2.30}$$

$$(5x-12)(z-12)=96 \tag{2.31}$$

$$(x-2)(z-10)=20. \tag{2.32}$$

From equation (2.30) the possible cases are:

$$(x-4)=1,(z-20)=80;(x-4)=2,(z-20)=40$$
$$(x-4)=4,(z-20)=20;(x-4)=5,(z-20)=16$$
$$(x-4)=8,(z-20)=10;(x-4)=10,(z-20)=8$$

$(x-4)=80,(z-20)=;(x-4)=40,(z-20)=2$
$(x-4)=20,(z-20)=4;(x-4)=16,(z-20)=5.$

Thus we get the following solutions:

$$(x,y,z)=(5,6,100),(6,6,60),(8,6,40),(9,6,36),(12,6,30),(14,6,28), \tag{2.33}$$

$$(84,6,21),(44,6,22),(24,6,24),(20,6,25). \tag{2.34}$$

In the same manner, in (2.31) we get $(x,y,z)=(3,9,44),(4,9,24),(12,9,14)$. In (2.32) we get $(x,y,z)=(3,12,30)$, $(4,12,20)$, $(6,12,15)$, $(7,12,14)$, $(12,12,12)$, $(22,12,11)$.

If $y=5$, then

$$\frac{1}{x}+\frac{5}{z}=\frac{3}{20}. \tag{2.35}$$

If $x\leq z$, then $\frac{3}{20}=\frac{1}{x}+\frac{5}{z}\leq\frac{6}{x}$, that is, $5\leq x\leq 40$, only $x=7,8,10,12,15,20,40$ gives positive integer of z. Hence $(x,y,z)=(7,5,700)$, $(8,5,200)$, $(10,5,100)$, $(12,5,75)$, $(15,5,60)$, $(20,5,50)$, $(40,5,40)$.

If $z\leq x$, then $\frac{3}{20}=\frac{1}{x}+\frac{5}{z}\leq\frac{6}{z}$, that is, $5\leq z\leq 40$, only $z=34,35,36,40$ gives positive integer of x. Hence $(x,y,z)=(340,5,34)$, $(140,5,35)$, $(90,5,36)$, $(40,5,40)$.

If $y=7$, then

$$\frac{1}{x}+\frac{5}{z}=\frac{9}{28}. \tag{2.36}$$

If $x\leq z$, then $\frac{9}{28}=\frac{1}{x}+\frac{5}{z}\leq\frac{6}{x}$, that is, $7\leq x\leq 18$, only $x=7,12,14$ gives positive integer of z. Hence $(x,y,z)=(7,7,28)$, $(12,7,21)$, $(14,7,20)$.

If $z\leq x$, then $\frac{9}{28}=\frac{1}{x}+\frac{5}{z}\leq\frac{6}{z}$, that is, $7\leq z\leq 18$, only $z=16$ give positive integer of x. Hence $(x,y,z)=(112,7,16)$.

If $y=8$, then

$$\frac{1}{x}+\frac{5}{z}=\frac{3}{8}. \tag{2.37}$$

If $x\leq z$, then $\frac{3}{8}=\frac{1}{x}+\frac{5}{z}\leq\frac{6}{x}$, that is, $8\leq x\leq 16$, only $x=8,16$ gives positive integer of z. Hence $(x,y,z)=(8,8,16)$, $(16,8,16)$.

If $z\leq x$, then $\frac{3}{8}=\frac{1}{x}+\frac{5}{z}\leq\frac{6}{z}$, that is, $8\leq z\leq 16$, only $z=14,15,16$ gives positive integer of x. Hence $(x,y,z)=(56,8,14)$, $(24,8,15)$, $(16,8,16)$.

If $y=10$, then

$$\frac{1}{x}+\frac{5}{z}=\frac{9}{20}. \tag{2.38}$$

If $x\le z$, then $\frac{9}{20}=\frac{1}{x}+\frac{5}{z}\le\frac{6}{x}$, that is, $10\le x\le 13$, there is no positive integer of z. We do not solution of (2.36).

If $z\le x$, then $\frac{9}{20}=\frac{1}{x}+\frac{5}{z}\le\frac{6}{z}$, that is, $10\le z\le 13$,only $z=12$ give positive integer of x. Hence $(x,y,z)=(30,10,12)$.

If $y=11$, then

$$\frac{1}{x}+\frac{5}{z}=\frac{21}{44}. \tag{2.39}$$

If $x\le z$, then $\frac{21}{44}=\frac{1}{x}+\frac{5}{z}\le\frac{6}{x}$, that is, $11\le x\le 12$, there is no positive integer of z. We do not solution of (2.36).

If $z\le x$, then $\frac{21}{44}=\frac{1}{x}+\frac{5}{z}\le\frac{6}{z}$, that is, $11\le z\le 12$,only $z=11$ give positive integer of x. Hence $(x,y,z)=(44,11,11)$.

Case 3. If $z\le x\le z$ or $z\le y\le x$, then $\frac{5}{z}<\frac{3}{4}$. This implies that $z\ge 7$. Also $\frac{9}{z}\ge\frac{3}{4}$. We get $z\le 12$.

Thus $z=7,8,9,10,11,12$. We find the following cases:

$$z=7 \implies \frac{1}{x}+\frac{3}{y}=\frac{1}{28} \tag{2.40}$$

$$z=8 \implies \frac{1}{x}+\frac{3}{y}=\frac{1}{8} \tag{2.41}$$

$$z=9 \implies \frac{1}{x}+\frac{3}{y}=\frac{7}{36} \tag{2.42}$$

$$z=10 \implies \frac{1}{x}+\frac{3}{y}=\frac{1}{4} \tag{2.43}$$

$$z=11 \implies \frac{1}{x}+\frac{3}{y}=\frac{13}{44} \tag{2.44}$$

$$z=12 \implies \frac{1}{x}+\frac{3}{y}=\frac{1}{3}. \tag{2.45}$$

In equations (2.38), (2.39), (2.41) and (2.43), we following the relations:

$$(x-28)(y-84)=2352 \tag{2.46}$$

$$(x-8)(y-24)=192 \tag{2.47}$$

(2.48) $$(x-4)(y-12)=48$$

(2.49) $$(x-3)(y-9)=27.$$

From equation (2.44) the possible cases are:

$$(x-28)=1,(y-84)=2352;(x-28)=2,(y-84)=1176$$
$$(x-28)=3,(y-84)=784;(x-28)=4,(y-84)=588$$
$$(x-28)=6,(y-84)=392;(x-28)=7,(y-84)=336$$
$$(x-28)=12,(y-84)=196;(x-28)=2352,(y-84)=1$$
$$(x-28)=1176,(y-84)=2;(x-28)=784,(y-84)=3$$
$$(x-28)=588,(y-84)=4;(x-28)=392,(y-84)=6$$
$$(x-28)=336,(y-84)=7;(x-28)=196,(y-84)=12.$$

Thus we get the following solutions: $(x,y,z) = (29,2436,7)$, $(30,1260,7)$, $(31,868,7)$, $(32,672,7)$, $(34,476,7)$, $(35,420,7)$, $(40,280,7)$, $(2380,85,7)$, $(1204,86,7)$, $(812,87,7)$, $(616,88,7)$, $(420,90,7)$, $(364,91,7)$, $(224,96,7)$.

In the same manner, in (2.45), we get $(x,y,z) = (9,216,8)$, $(10,120,8)$, $(11,88,8)$, $(12,72,8)$, $(14,56,8)$, $(16,48,8)$, $(20,40,8)$, $(200,25,8)$, $(104,26,8)$, $(72,27,8)$, $(56,28,8)$, $(40,30,8)$, $(32,32,8)$, $(24,36,8)$. In (2.46), we get $(x,y,z) = (5,60,10)$, $(6,36,10)$, $(7,28,10)$, $(8,24,10)$, $(10,20,10)$, $(12,18,10)$, $(16,16,10)$, $(52,13,10)$, $(28,14,10)$, $(20,15,10)$. In (2.47), we get $(x,y,z) = (4,36,12)$, $(6,18,12)$, $(30,10,12)$.

If $z=9$, then

(2.50) $$\frac{1}{x}+\frac{3}{y}=\frac{7}{36}.$$

If $x \le y$, then $\frac{7}{36}=\frac{1}{x}+\frac{3}{y}\le\frac{4}{x}$, that is, $9\le x\le 20$, only $x=9,12$ gives positive integer of y. Hence $(x,y,z)=(9,36,9)$, $(12,27,9)$.

If $y \le x$, then $\frac{7}{36}=\frac{1}{x}+\frac{3}{y}\le\frac{4}{y}$, that is, $9\le y\le 20$,only $y=16,18$ gives positive integer of x. Hence $(x,y,z)=(144,16,9)$, $(36,18,9)$.

If $z=11$, then

(2.51) $$\frac{1}{x}+\frac{3}{y}=\frac{13}{44}.$$

If $x \le y$, then $\frac{13}{44}=\frac{1}{x}+\frac{3}{y}\le\frac{4}{x}$, that is, $11\le x\le 13$, there is no positive integer of y. We do not solution of (2.49).

If $y \leq x$, then $\frac{13}{44}=\frac{1}{x}+\frac{3}{y} \leq \frac{4}{y}$, that is, $11 \leq y \leq 13$,only $y=11,12$ gives positive integer of x. Hence $(x,y,z)=(44,11,11)$, $(22,21,11)$.

3. CONCLUSION

In this research, has been discussed the Diophantine equations $\frac{1}{x}+\frac{3}{y}+\frac{5}{z}=\frac{3}{4}$ for each positive integers x, y and z respectively, We Show that has only finite number of solutions in the above process.

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