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AN INTERIOR-POINT ALGORITHM FOR SIMPLICIAL CONE CONSTRAINED CONVEX QUADRATIC OPTIMIZATION

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ABSTRACT. In this paper, we are concerned with the numerical solution of simplicial cone constrained convex quadratic optimization (SCQO) problems. A reformulation of the K.K.T optimality conditions of SCQOs as an equivalent linear complementarity problem with $\mathcal{P}$-matrix ($\mathcal{P}$-LCP) is considered. Then, a feasible full-Newton step interior-point algorithm (IPA) is applied for solving SCQO via $\mathcal{P}$-LCP. For the completeness of the study, we prove that the proposed algorithm is well-defined and converges locally quadratic to an optimal of SCQOs. Moreover, we obtain the currently best well-known iteration bound for the algorithm with short-update method, namely, $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$. Finally, we present a various set of numerical results to show its efficiency.

1. INTRODUCTION

Consider the following simplicial cone constrained convex quadratic optimization SCQO:

$$\min_{x}\left[f(x)=\frac{1}{2}x^TQx+x^Tb+c\right] \text{ s.t. } x\in\mathcal{S}, \tag{1.1}$$

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where $Q \in \mathbb{R}^{n\times n}$ is a given symmetric positive definite matrix, $b \in \mathbb{R}^n, c \in \mathbb{R}^n$ and

$$\mathcal{S} = \left\{Ax \mid x \in \mathbb{R}^n_+\right\}$$

is the simplicial cone associated with the nonsingular matrix $A \in \mathbb{R}^{n\times n}$. Simplicial cone constrained convex quadratic optimization problems arise as an important problem in its own right, it has an important subclass of positively constrained convex quadratic programming, or equivalently the problem of projecting the point onto a simplicial cone (see [8]).

Since the path-breaking work of Karmarkar [15] for linear optimization (LO), several algorithmic variants of interior-point methods (IPMs) were developed for LO. The primal-dual path-following methods introduced by Kojima and al. [16] and Monteiro and al. [20] are the most attractive methods in IPMs. The latter is a powerful tool to solve a wide large of mathematical problems such as LO, (see [1, 11, 14, 16]), convex quadratic optimization (CQO) (see [2, 13], LCP (see [12, 17–19]), the linear semidefinite optimization (SDO) and the semidefnite linear complementarity problem (SDLCP) (see [6, 7, 9, 21]). Recently, Achache [4] presented a short-step feasible IPMs for solving monotone standard LCP. He showed that the algorithm enjoys the iteration bound, namely, $\mathcal{O}(\sqrt{n}\log(\frac{n}{\varepsilon}))$. Furthermore, he reported some numerical results which confirmed the efficiency of this algorithm.

In this paper, motivated by this work, we solve the SCQO by using IPMs and linear complementarity problems. First, across the Karish-Khun-Tucker (K.K.T) optimality conditions of SCQO (1.1), we reformulate it as an equivalent LCP. Further, we show that the corresponding LCP is a $\mathcal{P}$-LCP. Hence, due to Cottle et al. [10], the $\mathcal{P}$-LCP has a unique solution and so is the SCQO. Second, across the $\mathcal{P}$-LCP, we introduce a simple feasible short-step interior-point algorithm for solving the SCQO. In fact the latter uses at each iteration, only full-Newton steps with the advantage that no line search is required. For its well-definiteness and its local quadratic convergence to an optimal solution of SCQO, we suggest new appropriate defaults to ensure that the algorithm converges to the unique minimizer of SCQO (1.1). Moreover, the best known iteration bound, namely, $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$ is derived. Here, for its polynomial complexity, we have reconsidered the basic analysis used in [4] and other references and developed them to be suited for SCQOs.

Finally, some numerical results are provided to illustrate its efficiency for solving the SCQOs.

The outline of this paper is organized as follows. In section 2, the reformulation of SCQO as a $\mathcal{P}$-LCP is given. In section 3, a feasible full-Newton step primal-dual interior-point algorithm is proposed for solving SCQOs via the $\mathcal{P}$-LCP. In section 4, the complexity analysis and the currently best known iteration bound for short-step methods are established. In section 5, some numerical results are provided to show the efficiency of the proposed algorithm.

The following notations are used throughout the paper. For a vector $x \in \mathbb{R}^n$, the Euclidean and maximum norms are denoted by $\|x\|$ and $\|x\|_\infty$, respectively. Given two vectors x and y in $\mathbb{R}^n$, $xy = (x_i y_i)_{1\leq i\leq n}$ denotes their coordinate-wise product and the same as for the vectors $x/y = (x_i/y_i)_{1\leq i\leq n}$ for $y \neq 0$, $\sqrt{x} = (\sqrt{x_i})_{1\leq i\leq n}$ and $x^{-1} = (1/x_i)_{1\leq i\leq n}$. The nonegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. For $x \in \mathbb{R}^n$, X denotes the diagonal matrix having the components of x as diagonal entries, i.e., X :=diag(x). The identity and the vector of all ones are denoted by I and e, respectively.

2. Reformulation of SCQO as an $\mathcal{P}$-LCP

In this section, some necessary definitions and theorems are required. A matrix M is positive definite if $x^T M x > 0$ for all nonzero $x \in \mathbb{R}^n$. $M \in \mathbb{R}^{n\times n}$ is called a $\mathcal{P}$-matrix if all its principal minors are positive. As a consequence, any positive definite matrix is a $\mathcal{P}$-matrix. Next, we define the standard LCP.

Definition 2.1. *The standard LCP consists to find vectors z, y in $\mathbb{R}^n$ such that*

$$z \geq 0,\ y \geq 0, \quad z = My + q, \quad z^T y = 0, \tag{2.1}$$

where $M \in \mathbb{R}^{n\times n}$ is a given matrix and $q \in \mathbb{R}^n$.

The following result was proved by Cottle, Pang and Stone [10], where any $\mathcal{P}$-LCP has a unique solution for every $q \in \mathbb{R}^n$.

Theorem 2.1. [*10, Theorem 3.3.7*] *A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}$-matrix if and only if the LCP has a unique solution for $q \in \mathbb{R}^n$. In this case the LCP is denoted by $\mathcal{P}$-LCP.*

Next task is to reformulate the SCQO (1.1) as a standard LCP. Starting from the definition of the simplicial cone $\mathcal{S}$ associated with the nonsingular matrix A, letting

$x = Ay$, then the problem (1.1) can be reformulated as the following convex quadratic optimization problem under positive constraints:

$$\min_{y} \left[f(y) = \frac{1}{2} y^T M y + y^T q + c \right] \text{ s.t. } y \in \mathbb{R}_+^n, \tag{2.2}$$

where

$$M = A^T Q A, \quad q = A^T b.$$

As the problem (2.2) is a continuous convex optimization and the constraints are positive then the optimality conditions of K.K.T are necessary and sufficient. Then, $y \in \mathbb{R}_+^n$ is an optimal solution of problem (2.2) if and only if there exists $z \in \mathbb{R}_+^n$ such that:

$$z = My + q, \quad z^T y = 0, \quad y \geq 0, \quad z \geq 0. \tag{2.3}$$

Due to (2.1), system (2.3) is only a standard LCP with $M = A^T Q A$ and $q = A^T b$.

Theorem 2.2. *Let $Q \in \mathbb{R}^{n \times n}$ be symmetric positive definite and A is nonsingular then $M = A^T Q A$ is a $\mathcal{P}$-matrix. Hence, the LCP (2.3) is a $\mathcal{P}$-LCP.*

Proof. Since $Q \in \mathbb{R}^{n \times n}$ is assumed to be symmetric positive definite and A is nonsingular then for all nonzero $v \in \mathbb{R}^n$, $v^T M v = v^T A^T Q A v = \|Qs\|^2 > 0$ where $s = Av \neq 0$, M is positive definite, therefore M is a $\mathcal{P}$-matrix and so the LCP (2.3) is a $\mathcal{P}$-LCP. By Theorem 2.1, the $\mathcal{P}$-LCP has a unique solution and so is the SCQO (1.1). □

Corollary 2.1. *The vector $x^\star = Ay^\star$ is the unique minimizer of SCQO if and only if the pair of vectors $(y^\star, z^\star)$ is the unique solution of $\mathcal{P}$-LCP (2.3).*

3. A FEASIBLE FULL-NEWTON STEP INTERIOR-POINT ALGORITHM FOR SCQO

In this section, we solve the SCQO (1.1) by the application of a feasible full-Newton step interior-point algorithm to the equivalent $\mathcal{P}$-LCP(2.3). To do so, we discuss first the notion of central-path of $\mathcal{P}$-LCP and the Newton search direction. Then the generic feasible interior-poin algorithm for SCQO (1.1) is presented. In the sequel, we assume that $\mathcal{P}$-LCP (2.3) satisfies the interior-point condition (IPC), i.e., there exists $y^0 > 0$ and $z^0 > 0$ such that $z^0 = My^0 + q$.

3.1. **The central-path for $\mathcal{P}$-LCP.** The basic idea of the path-following interior-point algorithm is to replace the second equation in (2.3), the so-called complementarity condition by the perturbed equation $zy = \mu e$ where $\mu > 0$. Hence, we obtain the following system of equations:

$$z = A^TQAy + A^Tb, \quad zy = \mu e, \quad y \geq 0, \quad z \geq 0. \tag{3.1}$$

By IPC assumption it is shown that for any $\mu > 0$ the parameterized system (3.1) has a unique solution denoted by $(y(\mu), z(\mu))$, which is called the $\mu-$center of $\mathcal{P}$-LCP. The set of μ -centers constructs the so-called central-path. Moreover, if μ tends to zero then the limit of central-path exists and converges to a solution of $\mathcal{P}$-LCP ([22]).

3.2. **The search direction for $\mathcal{P}$-LCP.** Applying Newton's method to system (3.1) for a given strictly feasible point (y, z) and the Newton search direction $(\Delta y, \Delta z)$ at this point is the unique solution of the system:

$$\begin{pmatrix} -A^TQA & I \\ Z & Y \end{pmatrix} \begin{pmatrix} \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} 0 \\ \mu e - yz \end{pmatrix}, \tag{3.2}$$

where Y:=diag(y), Z:=diag(z). By simple calculations, the system (3.2) can be written as follows:

$$\begin{cases} (A^TQA + Y^{-1}Z)\Delta y = Y^{-1}(\mu e - yz) \\ \Delta z = A^TQA\Delta y. \end{cases} \tag{3.3}$$

The system (3.3) is nonsingular since for any nonzero $v \in \mathbb{R}^n$, we have $v^T(A^TQA + Y^{-1}Z)v = v^TA^TQAv + v^TY^{-1}Zv > 0$ because $v^T(A^TQA)v > 0$ and $v^TY^{-1}Zv > 0$ for any nonzero $v \in \mathbb{R}^n$ and for all $y > 0$ and $z > 0$ (Y, Z are positive definite matrices), then the matrix $(A^TQA + Y^{-1}Z)$ is positive definite too and so it is nonsingular.

The new iterate is obtained by taking a full-Newton step as follows:

$$y_+ := y + \Delta y, \quad z_+ := z + \Delta z. \tag{3.4}$$

To simplify matters, we introduce the following vectors:

$$v = \sqrt{\frac{yz}{\mu}}, \quad d = \sqrt{\frac{y}{z}}. \tag{3.5}$$

So the scaled directions are given by

$$d_y = \frac{\Delta_y}{y}, \quad d_z = \frac{\Delta_z}{z}, \tag{3.6}$$

where $y > 0$, $z > 0$ and $\mu > 0$. Then we have:

$$\mu d_y d_z = \Delta y \Delta z \quad \text{and} \quad y\Delta z + z\Delta y = \mu v \left(d_y + d_z\right). \tag{3.7}$$

Hence the system defining Newton search directions can be written as:

$$\begin{cases} -\overline{M} d_y + d_z = 0 \\ \quad d_y + d_z = p_v \end{cases} \tag{3.8}$$

where $\overline{M} = DMD^{-1} = DA^TQAD^{-1}$, D:=diag(d), and

$$p_v = v^{-1} - v. \tag{3.9}$$

For the analysis of the algorithm and according to (3.9), we use a norm-based proximity measure $\delta\left(v\right)$ defined by:

$$\delta := \delta\left(yz; \mu\right) = \frac{1}{2}\|p_v\|. \tag{3.10}$$

Clearly, the value of $\delta\left(v\right)$ can be considered as a measure for the distance between the given pair (y, z) and the corresponding μ center $\left(y\left(\mu\right), z\left(\mu\right)\right)$ and we have:

$$\delta\left(v\right) = 0 \Longleftrightarrow v = e \Longleftrightarrow yz = \mu e.$$

3.3. **The algorithm.** Let $\epsilon > 0$ be a given tolerance and $\theta \in]0, 1[$ the update parameter (default $\theta = \frac{1}{\sqrt{3n}}$), the algorithm starts with a strictly feasible initial point (y^0, z^0) such that $\delta(y^0 z^0; \mu_0) \leq \tau$ where $0 < \tau < 1$. Determining the search directions $(\Delta y, \Delta z)$, the algorithm produces a new iterate $(y_+, z_+) = (y + \Delta y, z + \Delta z)$. Then, it updates the barrier parameter μ to $(1 - \theta)\mu$ and solves the Newton system. This procedure is repeated until the stopping criterion $y_+^T z_+ \leq \epsilon$ is satisfied.

The generic feasible full-Newton step interior-point algorithm for SCQOs is presented in Figure 1.

Input:
An accuracy parameter $\epsilon > 0$;
A threshold parameter $0 < \tau < 1$ (default $\tau = \sqrt{\frac{3}{7}}$);
A barrier update parameter $0 < \theta < 1$ (default $\theta = \frac{1}{\sqrt{3n}}$);
A strictly feasible point (y^0, z^0) and $\mu_0 = \frac{1}{2}$ s.t. $\delta(y^0 z^0, \mu_0) \leq \tau$;
begin
$y := y^0$, $z := z^0$, $\mu := \mu_0$;
While $n\mu \geq \epsilon$ **do**
Solve system (3.3) to obtain $(\Delta y, \Delta z)$;
Update $y := y + \Delta y$; $z := z + \Delta z$;
$\mu := (1-\theta)\mu$;
end while
end.

Fig 1. Algorithm 3.3

4. COMPLEXITY ANALYSIS

In this section, we will show under our new defaults $\tau = \sqrt{\frac{3}{7}}$ and $\theta = \frac{1}{\sqrt{3n}}$ that Algorithm 3.3 solves the SCQOs in polynomial and ensures the locally quadratic convergence of the Newton process through the algorithm. Our analysis is straightforward to monotone LCPs ([4]).

We first quote the following technical lemma which will be used later.

Lemma 4.1. *[7, Lemma 2.2.1] Let $\delta > 0$ and (d_y, d_z) be a solution of system (3.8). Then, we have*

$$0 \leq d_y^T d_z \leq 2\delta^2, \tag{4.1}$$

and

$$\|d_y d_z\|_\infty \leq \delta^2, \quad \|d_y d_z\| \leq \sqrt{2}\,\delta^2. \tag{4.2}$$

In the following lemma, we show that the feasibility of the full-Newton step when the proximity $\delta\,(yz, \mu) < 1$.

Lemma 4.2. *[7, Lemma 2.2.3] Let $\delta = \delta\left(yz, \mu\right) < 1$.Then $y_+ > 0$ and $z_+ > 0$,which means that the full-Newton step is strictly feasible.*

For convenience, we may write

$$v_+ = \sqrt{\frac{y_+ z_+}{\mu}}.$$

The next lemma shows the influence of the full-Newton step on the proximity measure.

Lemma 4.3. *[7, Lemma 2.2.4] If $\delta < 1$. Then*

$$\delta_+ := \delta\left(v_+; \mu\right) \leq \frac{\delta^2}{\sqrt{2\left(1 - \delta^2\right)}}.$$

In addition, if $\delta \leq \sqrt{\frac{3}{7}}$, thus $\delta_+ \leq \delta^2$ which means the full-Newton step converges locally quadratically through the algorithm.

In the following lemma, we obtain the upper bound of a duality gap after a full-Newton step.

Lemma 4.4. *Let $\delta \leq \sqrt{\frac{3}{7}}$ and suppose that the vectors y_+ and z_+ are obtained by using a full-Newton step, thus $y_+ = y + \Delta y$ and $z_+ = z + \Delta z$ we have*

$$y_+^T z_+ \leq 2\mu n. \tag{4.3}$$

Proof. Using (3.6) and (3.7) we have

$$\begin{aligned} y_+ z_+ &= (y + \Delta y)(z + \Delta z) \\ &= yz + y\Delta y + z\Delta z + \Delta y \Delta z \\ &= \mu\left(e + d_y d_z\right), \end{aligned}$$

then

$$y_+^T z_+ = \mu e^T\left(e + d_y^T d_z\right) = \mu\left(n + d_y^T d_z\right).$$

Next, let $\delta \leq \sqrt{\frac{3}{7}}$ then $\delta^2 \leq \frac{3}{7}$, using (4.1) we deduce that

$$y_+^T z_+ \leq \mu\left(n + \frac{6}{7}\right).$$

But as $n + \frac{6}{7} \leq 2n, \forall n \geq 1,$ this gives the required result. □

In the following theorem, we investigate the influence on the proximity measure of Newton process followed by a step along the central-path.

Theorem 4.1. *Let* $\delta \leq \sqrt{\frac{3}{7}}$ *and* $\mu_+ = (1-\theta)\mu$, *where* $0 < \theta < 1$. *Then*

$$\delta^2(y_+ z_+; \mu_+) \leq \frac{9}{56} + \frac{\theta^2\left(n + \frac{6}{7}\right)}{4(1-\theta)} + \frac{15}{56}\theta.$$

In addition, if $\theta = \frac{1}{\sqrt{3n}}$ *and* $n \geq 3$, *then* $\delta(y_+ z_+, \mu_+) \leq \sqrt{\frac{3}{7}}$.

Proof. We have

$$\begin{aligned}
&4\delta^2(y_+ z_+; \mu_+) \\
&= \| \sqrt{1-\theta}\, v_+^{-1} - \frac{1}{\sqrt{1-\theta}} v_+ \|^2 \\
&= \|\sqrt{1-\theta}\left(v_+^{-1} - v_+\right) - \frac{\theta}{\sqrt{1-\theta}} v_+\|^2 \\
&= (1-\theta)\|v_+^{-1} - v_+\|^2 + \frac{\theta^2}{1-\theta}\|v_+\|^2 - 2\theta\left(v_+^{-1} - v_+\right)^T v_+ \\
&= (1-\theta)\| v_+^{-1} - v_+ \|^2 + \frac{\theta^2}{1-\theta}\|v_+\|^2 - 2\theta\left(v_+^{-1}\right)^T v_+ + 2\theta v_+^T v_+ \\
&= 4\delta_+^2(1-\theta) + \frac{\theta^2}{1-\theta}\|v_+\|^2 - 2\theta n + 2\theta\|v_+\|^2.
\end{aligned}$$

Because $\left(v_+^{-1}\right)^T v_+ = n$ and $v_+^T v_+ = \| v_+ \|^2$ and according Lemma 4.4 we get

$$\| v_+\|^2 = \frac{1}{\mu} y_+^T z_+ \leq \left(n + \frac{6}{7}\right),$$

which implies that

$$\delta^2(y_+ z_+; \mu_+) \leq (1-\theta)\delta_+^2 + \frac{\theta^2\left(n + \frac{6}{7}\right)}{4(1-\theta)} + \frac{3\theta}{7}.$$

As $\delta \leq \sqrt{\frac{3}{7}}$ Lemma 4.3 implies that $\delta_+^2 \leq \frac{\delta^2}{\sqrt{2(1-\delta^2)}} = \frac{9}{56}$. Therefore, after some simplifications we obtain

$$\delta^2(y_+ z_+; \mu_+) \leq \frac{9}{56} + \frac{\theta^2\left(n + \frac{6}{7}\right)}{4(1-\theta)} + \frac{15}{56}\theta.$$

Let $\theta = \frac{1}{\sqrt{3n}}$ then $\theta^2 = \frac{1}{3n}$, this imply that

$$\delta^2\left(y_+z_+;\mu_+\right) \leq \frac{9}{56} + \frac{\frac{n+\frac{6}{7}}{3n}}{4\left(1-\theta\right)} + \frac{15}{56}\theta.$$

And since $\frac{n+\frac{6}{7}}{3n} \leq \frac{3}{7}$ for $n \geq 3$ then

$$\delta^2\left(y_+z_+;\mu_+\right) \leq \frac{9}{56} + \frac{3}{28\left(1-\theta\right)} + \frac{15}{56}\theta.$$

For $n \geq 3, \theta \in \left[0, \frac{1}{3}\right]$, we consider the following function:

$$f\left(\theta\right) = \frac{9}{56} + \frac{15}{56}\theta + \frac{3}{28\left(1-\theta\right)}.$$

As

$$f^{'}\left(\theta\right) = \frac{1}{28\left(1-\theta\right)^2} + \frac{15}{56} > 0,$$

so f is continuous and monotone increasing on $\left[0, \frac{1}{3}\right]$. Consequently

$$f\left(\theta\right) \leq f\left(\frac{1}{3}\right) \simeq 0.4107 \leq \frac{3}{7}, \text{ for all } \theta \in \left[0, \frac{1}{3}\right].$$

Then, after the barrier parameter is update to $\mu_+ = \left(1-\theta\right)\mu$ with $\theta = \frac{1}{\sqrt{3n}}$ and if $\delta \leq \sqrt{\frac{3}{7}}$, we get $\delta\left(y_+z_+;\mu_+\right) \leq \sqrt{\frac{3}{7}}$. This completes the proof . □

A consequence of Theorem 4.1 is that under our defaults the algorithm is well-defined since the conditions $y_+ > 0, z_+ > 0$ and $\delta\left(y_+z_+;\mu_+\right) \leq \sqrt{\frac{3}{7}}$ hold through the algorithm.

4.1. **Iteration bound.** In the following lemma, we derive the upper bound for the total number of iterations produced by the algorithm.

Lemma 4.5. *Suppose that y^0 and z^0 are strictly feasible starting point such that $\delta\left(y^0z^0;\mu_+\right) \leq \sqrt{\frac{3}{7}}$ for each $\mu_0 > 0$. Moreover, let y^k and z^k be the vectors obtained after k iterations. Then the inequality $\left(y^k\right)^T z^k \leq \epsilon$ is satisfied if*

$$k \geq \frac{1}{\theta}\log\left(\frac{2n\mu_0}{\epsilon}\right).$$

Proof. From (4.1), it follows that:

$$\left(y^k\right)^T z^k \leq 2n\mu_k = 2n\left(1-\theta\right)^k\mu_0.$$

Then the inequality $\left(y^k\right)^T z^k \leq \epsilon$ holds if $2n\left(1-\theta\right)^k \mu_0 \leq \epsilon$. We take logarithms, so we may write

$$k \log\left(1-\theta\right) \leq \log \epsilon - \log\left(2n\mu_0\right).$$

We know that $-\log\left(1-\theta\right) \geq \theta$ for $0 \leq \theta \leq 1$. So the inequality holds only if

$$k\theta \geq \log \epsilon - \log\left(2n\mu_0\right) = \log\left(\frac{2n\mu_0}{\epsilon}\right).$$

This completes the proof. □

We end this subsection with a theorem that gives the iteration bound of the algorithm .

Theorem 4.2. *Using defaults $\theta = \frac{1}{\sqrt{3n}}$ and $\mu_0 = \frac{1}{2}$, we obtain that the algorithm given in Fig 1 at most requires at $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ iterations for getting an $\epsilon-$ approximated solution of $\mathcal{P}$-LCP.*

Proof. Let $\theta = \frac{1}{\sqrt{3n}}$ and $\mu_0 = \frac{1}{2}$, by using Lemma 4.5 , the result follows. □

5. Numerical results

In this section, we present some numerical problems of different sizes for testing the effectiveness of the Algorithm 3.3, each example is followed by a table containing the computational results obtained by the algorithm. All programs were implemented in MATLAB R2016a on a personal PC with 1.40 GHZ AMD E1-2500 APU Radeon(TM) HD Graphic, 8 GB memory and Windows 10 operating system. In the implementation, we use $\epsilon = 10^{-6}$, different values of the barrier parameter μ_0, $\theta = \frac{1}{\sqrt{3n}}$ and we use some constant values of θ in order to improve the performances of our algorithm. Here, the starting point and the unique solution of $\mathcal{P}$-LCP are denoted by (y^0, z^0) and $(y^\star, z^\star)$ respectively with $x^\star = Ay^\star$ the unique solution of SCOO. The number of iterations required and the time executed by the algorithm are denoted by "Iter" and "CPU" respectively. For a comparison we implement the algorithm (3.1) in [4] with $\theta = \frac{1}{2\sqrt{n}}$.

Example 1. *Consider the SCQO problem with*

$$b = \left(-1, -4, 4, -2, 1, 10, 4, 0, 5, -11\right)^T,$$

$$Q = \begin{bmatrix} 6 & 0.5 & 6 & 1 & 3 & 2 & -2 & 0 & 0 & 4 \\ 0.5 & 8.25 & -3.5 & 1 & -3.5 & 2 & 1.5 & -2.5 & -6 & -4.5 \\ 6 & -3.5 & 38 & -1.5 & 7 & -6 & -1 & 2.5 & 16 & 3 \\ 1 & 1 & -1.5 & 8.25 & -2 & 2 & -1.5 & 0. & 0 & -6 \\ 3 & -3.5 & 7 & -2 & 11 & -4 & -1 & -0.5 & 0 & -5 \\ 2 & 2 & -6 & 2 & -4 & 8 & -4 & 0 & -2.5 & 8 \\ -2 & 1.5 & -1 & -1.5 & -1 & -4 & 7 & -4 & 1 & -4 \\ 0 & -2.5 & 2.5 & 0 & -0.5 & 0 & -4 & 7.25 & -0.5 & 4 \\ 0 & -6 & 16 & 0 & 0 & -2.5 & 1 & -0.5 & 16.25 & 9.5 \\ 4 & -4.5 & 3 & -6 & -5 & 8 & -4 & 4 & 9.5 & 41 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -1 & -2 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -2 & 0 & 3 & 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -2 & 0 & 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 & -2 & 0 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 & -1 & 2 & 0 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 & -1 & -1 & -2 & 0 & 3 & 3 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & 0 & 3 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & 0 \end{bmatrix}.$$

The unique solution of the $\mathcal{P}$-LCP is given by

$$y^\star = (0, 0.09, 0, 0, 0.0549, 0, 0, 0, 0, 0,)^T,$$
$$z^\star = (4.3634, 0, 1.5624, 5.5552, 0, 19.9944, 9.3423, 69.6119, 86.007, 48.1573)^T.$$

Then the unique minimizer of this problem is:

$$x^\star = Ay^\star$$
$$= (0.27, 0.1646, -0.0154, 0.0746, -0.09, -0.1998, -0.1449, \dots, -0.1449)^T.$$

The obtained number of iterations and the elapsed time via specified different values of μ_0 and θ are summarized in Table 1:

TABLE 1. The numerical results, after the algorithm reaches $n\mu \leq 10^{-6}$

$\mu_0 \longrightarrow$	0.5		0.05		0.005		0.0005	
$\theta \downarrow$	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*
0.7	13	0.05229	11	0.04637	9	0.04747	8	0.04486
0.5	23	0.05135	19	0.04978	16	0.04944	13	0.04537
$\frac{1}{\sqrt{3n}}$	77	0.06364	66	0.05992	54	0.05507	43	0.05120

Example 2. *Consider the SCQO problem with*

$$Q = \begin{bmatrix} 3 & 1 & \dots & 0 \\ 1 & 3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & \dots & n \\ 0 & 1 & \ddots & n-1 \\ \vdots & \ddots & \ddots & 2 \\ 0 & \dots & 0 & 1 \end{bmatrix},$$

$b = (A^T)^{-1}(e - A^T QAe)$. *The strictly feasible initial point taken in the algorithm is :*

$$y^0 = \frac{1}{2}(1, 1, \dots, 1)^T > 0, \quad z^0 = My^0 + q > 0.$$

A solution of $\mathcal{P}$-LCP is given by:

$$y^\star = (0, 1.2154, 1.0615, 1, \dots, 1)^T, \quad z^\star = (0.1846, 0, 0, \dots, 0).$$

For example if $n = 10$, then the unique minimizer of this problem is

$$x^\star = Ay^\star = (54.6154, 45.3385, 36.0615, 28, 21, 15, 10, 6, 3, 1)^T.$$

The details of obtained numerical results with different values of μ_0 and θ with the size $n = 10$, are presented in table 2:

TABLE 2. The numerical results, after the algorithm reaches $n\mu \leq 10^{-6}$

$\mu_0 \longrightarrow$	0.5		0.05		0.005		0.0005	
$\theta \downarrow$	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*
$\frac{1}{2\sqrt{n}}$	99	0.0345	77	0.0271	63	0.0229	50	0.0177
$\frac{1}{\sqrt{3n}}$	77	0.0227	66	0.0213	54	0.0169	43	0.0128

The details of obtained numerical results with different sizes of n, relaxed values of μ_0 and $\theta = \frac{1}{\sqrt{3n}}$ are presented in Table 3 .

TABLE 3. The numerical results, after the algorithm reaches $n\mu \leq 10^{-6}$

$\mu_0 \longrightarrow$	0.5		0.05		0.005		0.0005	
size $n \downarrow$	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
20	117	0.0559	100	0.0483	84	0.0446	67	0.0255
50	200	0.2569	173	0.1967	146	0.1554	119	0.1336
100	299	2.0932	260	1.8765	221	1.4960	182	1.3049
200	442	13.141	387	11.439	332	10.012	277	8.350
1000	624	39.152	552	35.992	481	33.301	409	30.751

Example 3. *Consider the SCQO problem with*

$$Q = \begin{bmatrix} 4 & -1 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 4 & \ddots & \ddots & \vdots & 0 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 4 & -1 & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -1 & 4 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 5 & -2 & 0.25 & \cdots & 0.5 \\ 0 & 1 & \ddots & \ddots & \vdots & -2 & 5 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 0.25 & \ddots & \ddots & \ddots & 0.25 \\ \vdots & \ddots & \ddots & 1 & 0 & \vdots & \ddots & \ddots & 5 & -2 \\ 0 & \cdots & 0 & 0 & 1 & 0.25 & \cdots & 0.25 & -2 & 5 \end{bmatrix},$$

$$A = \begin{bmatrix} -2 & -1 & 0.5 & \cdots & 0.5 & 1 & 0 & 0 & \cdots & 0 \\ 4 & -2 & \ddots & \ddots & \vdots & 0 & 1 & \ddots & \ddots & \vdots \\ 3 & \ddots & \ddots & \ddots & 0.5 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & -1 & \vdots & \ddots & \ddots & 1 & 0 \\ 3 & \cdots & 3 & 4 & -2 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & 0.1 & \cdots & 0 & -2 & -1 & 0.5 & \cdots & 0.5 \\ 0 & -1 & \ddots & \ddots & \vdots & 4 & -2 & \ddots & \ddots & \vdots \\ 0.1 & \ddots & \ddots & \ddots & 0.1 & 3 & \ddots & \ddots & \ddots & 0.5 \\ \vdots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & -2 & -1 \\ 0 & \cdots & 3 & 0.1 & -1 & 3 & \cdots & 3 & 4 & -2 \end{bmatrix},$$

and

$$b = -2QAe.$$

The strictly feasible initial point taken in the algorithm is:

$$y^0 = (5, 5, \ldots, 5)^T, \quad z^0 = My^0 + q.$$

The solution of $\mathcal{P}$-LCP is given by

$$y^\star = (2, 2, \ldots, 2)^T, \quad z^\star = (0, 0, \ldots, 0)^T.$$

So the optimum of this problem is computed via $x^\star = Ay^\star$. The numerical results with different size of n are summarized in next Table.

TABLE 4. The numerical results, after the algorithm reaches $n\mu \leq 10^{-6}$

$\mu_0 \longrightarrow$	*0.5*		*0.05*		*0.005*		*0.0005*	
size $n \downarrow$	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*	*Iter*	*CPU*
10	*13*	*0.0347*	*11*	*0.0342*	*9*	*0.0335*	*8*	*0.0315*
50	*15*	*0.0517*	*13*	*0.0492*	*11*	*0.0465*	*9*	*0.0449*
100	*15*	*0.1255*	*13*	*0.1082*	*11*	*0.1011*	*9*	*0.0944*
200	*16*	*0.5258*	*14*	*0.3967*	*12*	*0.3443*	*10*	*0.2927*
1000	*17*	*35.252*	*15*	*27.392*	*13*	*26.955*	*11*	*23.156*
1600	*18*	*176.126*	*16*	*158.173*	*14*	*98.691*	*12*	*80.873*

6. CONCLUSION

In this paper, a convex quadratic programming problem under simplicial cone constraints is studied, and via its K.K.T optimality conditions is transformed into a $\mathcal{P}$-LCP. For its numerical solution, a feasible full-step primal-dual path following interior-point algorithm is proposed. First for the sake of benefit of readers we have reconsider the analysis of certain authors, and we make it suited for SCQOs. Here, we suggested new defaults such as $\delta \leq \sqrt{\frac{3}{7}}$ and $\theta = \frac{1}{\sqrt{3n}}$ for its well-definiteness and its convergence to the unique minimizer of SCQOs. Further, its best iteration bound is derived. The obtained numerical results illustrate that the algorithm is efficient and valid to solve the SCQO problems. An interesting topic of research in the future is to solve the SCQOs by introducing the active set methods.

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