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APPLICATION OF THE GENERALISED XLINDLEY MODEL FOR RELIABILITY DATA

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ABSTRACT. In this study, we presented a new two-parameter model termed the Generalized XLindely distribution. This novel model is a combination of the exponential and the Two-Parameter lindley distributions. After exploring the statistical characterization of this model, we estimated its parameters using the maximum likelihood method and the Maximum Product of Spacings Method. The approximate confidence interval, based on a normal approximation is additionally calculated. We applied our model to real lifetime data sets to demonstrate its validity, and it was discovered that our distribution fits significantly better than other current distributions.

1. INTRODUCTION

In overall, the objective of establishing new distributions is to develop flexible mathematical models that can handle lifetime data. This flexibility can be obtained simply by combining or generalizing several distributions. Several authors showed their interest in this type of work.

To deal with lifetime data, Lindley [6] presented a distribution that carries his name and is identified by a combination of exponential and gamma distributions.

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In the present, the researchers present many extended forms of this distribution. For example, in 2009, Zakerzadeh and Dolati [13] introduced a generalization of this distribution namely a three-parameter generalization of the Lindley distribution and studied various properties of this new distribution. Also, Zakerzadeh and Dolati provided numerical examples to show the flexibility of this model. Generalized Poisson Lindley [7] given by Mahmoudi and Zakerzadeh, Shanker and Shukla [8] have conducted a thorough comparison of the three-parameter generalized gamma distribution and the generalized Lindley distribution.

In 2015, Chouia and Zeghdoudi [4] proposed the one-parameter model with probability density function (pdf) and cumulative distribution function (CDF) that are respectively given by

$$f(y;\alpha) = \frac{\alpha^2(2+\alpha+y)}{(1+\alpha)}\exp(-\alpha y), \quad \alpha > 0, y > 0, \tag{1.1}$$

$$F(y;\alpha) = 1 - (1 + \frac{\alpha y}{(1+\alpha)^2})\exp(-\alpha y), \quad \alpha > 0, y > 0; \tag{1.2}$$

This distribution combines the Exponentiel distribution and the Lindley distribution with the same parameter α and mixing proportion $p = \frac{\alpha}{\alpha+1}$. We developed our new model based on this concept by combining the exponential and Two-parameter lindley distributions [10], whose pdfs are given by:

$$g_1(x;\lambda) = \lambda \exp\left(-\lambda x\right), x > 0, \lambda > 0. \tag{1.3}$$

$$g_2(x;\lambda,\gamma) = \frac{\lambda^2\left(1+\gamma x\right)}{\lambda+\gamma}\exp\left(-\lambda x\right), x > 0, \lambda > 0, \gamma > 0. \tag{1.4}$$

The rest of the research paper includes six important sections. Firstly, it introduces Generalized XLindley distribution with its pdf, CDF, releability, and the hazard function. Secondly, it presents the statistical properties of this distribution as moments and stochastic orderings. The maximum likelihood estimator, the maximum Product of Spacings Method and the asymptotic confidence interval of the unidentified parameters of the GXLindley distribution are raised in section 4. Section 5 includes an application to sets of real data. Finally, section 6 draws a conclusion.

2. GXL DISTRIBUTION

Let X_1 and X_2 be two independent random variables distributed according to exponential with parameter λ and Two-Parameter lindley with parameters λ and γ. Consider the random variable $Z = X_1$ with probability $p = \frac{\lambda}{\lambda+\gamma}$, and $Z = X_2$ with probability $1-p = \frac{\gamma}{\lambda+\gamma}$, the pdf of the random variable Z can be shown as a mixture as follows:

$$f(z,\Theta) = pg_1(x) + (1-p)g_2(x). \tag{2.1}$$

By using the pdfs given in equations (1.3) and (1.4). We define the pdf for the new distribution namely Generalized XLindley, by substituting in equation (2.1). This pdf is

$$f(z;\Theta) = \frac{\lambda^2\left(\lambda+2\gamma+\gamma^2 z\right)}{\left(\lambda+\gamma\right)^2}\exp\left(-\lambda z\right), z>0, \lambda>0, \gamma>0, \tag{2.2}$$

where Θ is parameter vector (λ,γ).

This distribution contains the XLindley distribution as a particular case wher $\gamma = 1$, if $\gamma = 0$, the equation (2.2) reduces to the pdf of the Two-Parameter Lindley distribution.

The CDF of the GXL distribution is given by

$$F(z;\Theta) = 1-\left[1+\frac{\lambda\gamma^2}{\left(\lambda+\gamma\right)^2}z\right]\exp\left(-\lambda z\right), z>0, \lambda>0, \gamma>0. \tag{2.3}$$

The reliability and hasard rate functions of the GXL distribution have been respectively showed as

$$S(z;\Theta) = 1+\frac{\lambda\gamma^2}{\left(\lambda+\gamma\right)^2}z\exp\left(-\lambda z\right), z>0, \lambda>0, \gamma>0, \tag{2.4}$$

$$H(z;\Theta) = \frac{\lambda^2\left(2\gamma+\gamma^2 z\right)}{\left(\lambda+\gamma\right)^2+\lambda\gamma^2 z}, z>0, \lambda>0, \gamma>0. \tag{2.5}$$

Figures 1 demonstrate certain plots of the GXL distribution for the values specified for λ and γ.

It is necessary to find the maximum of equation (2.2) in order to identify the mode of the pdf's GXL distribution. As a result, the point at which the first derivative of the function (2.2) is equal to zero should be found. The first derivative is

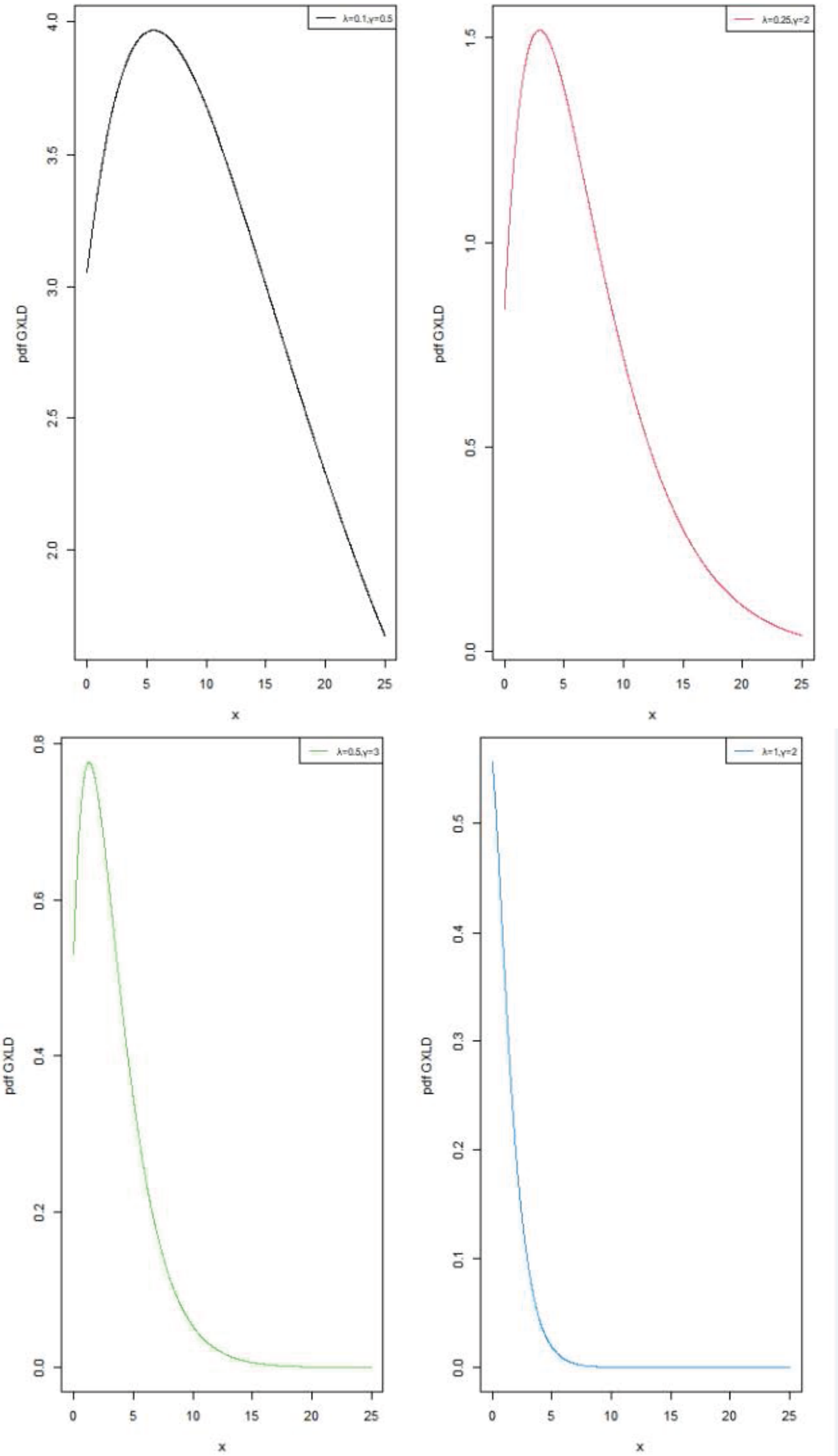

FIGURE 1. GXL pdf plots for different parameter values

provided by:

$$\frac{\partial}{\partial x}f(z;\Theta)=\left(\frac{\lambda}{\lambda+\gamma}\right)^{2}\left[\gamma^{2}(1-\lambda z)-\lambda^{2}+2\lambda\right]\exp\left(-\lambda z\right), z>0,\lambda>0,\gamma>0,$$

if we take $\frac{\partial}{\partial x}f(z;\Theta)=0$, so

$$Mo=\frac{\gamma^{2}-\lambda^{2}-2\lambda\gamma}{\lambda\gamma^{2}}.$$

The pdf of the GXL distribution is:

(i) increasing in z for $z\in[0,Mo]$;
(ii) decreasing in z for $z\in[Mo,\infty]$.

3. Statistical Properties of the GXL Distribution

3.1. **Moments and related measures.** The rth moment of the GXL distribution has been calculated as

$$\mu_{r}^{'}=\frac{\lambda\left[\left(\gamma^{2}\left(r+1\right)+\lambda^{2}+2\lambda\gamma\right)r!\right]}{\lambda^{r}\left(\lambda+\gamma\right)^{2}}, r=1,2,\ldots. \tag{3.1}$$

Using $r=1,2,3$ and 4 in (3.1) the first four moments about the origin are gained as

$$\mu_{1}^{'}=1+\left(\frac{\gamma}{\lambda+\gamma}\right)^{2},$$

$$\mu_{2}^{'}=\frac{2}{\lambda}+\frac{4\gamma^{2}}{\lambda\left(\lambda+\gamma\right)^{2}},$$

$$\mu_{3}^{'}=\frac{6}{\lambda^{2}}+\frac{18\gamma^{2}}{\lambda^{2}\left(\lambda+\gamma\right)^{2}},$$

$$\mu_{4}^{'}=\frac{24}{\lambda^{3}}+\frac{96\gamma^{2}}{\lambda^{3}\left(\lambda+\gamma\right)^{2}}.$$

It is simple to demonstrate that for $\gamma=1$, the distribution's moments about the origin decrease to the respective moments of the XLindley distribution. Furthermore, the distribution is positively skewed because the mean is always higher than the mode. As a result, the mean and variance of the proposed distribution can be calculated as

$$E(z)=\frac{\left(\lambda+\gamma\right)^{2}+\gamma^{2}}{\left(\lambda+\gamma\right)^{2}},$$

$$V(z)=\frac{(\lambda+\gamma)^2(2-\lambda)\left[2\gamma^2+(\lambda+\gamma)^2\right]-\lambda\gamma^4}{\lambda(\lambda+\gamma)^4}.$$

The skewness and kurtosis measures can now be computed from the following expressions

$$\text{Skewness}=\frac{\mu_3^{'}-3\mu_1^{'}\mu_2^{'}+2\mu_1^{'3}}{(\mu_2^{'}-\mu_1^{'})^3}$$

$$\text{Kurtosis}=\frac{\mu_4^{'}-4\mu_1^{'}\mu_3^{'}+6\mu_2^{'}\mu_1^{'2}-3\mu_1^{'4}}{(\mu_2^{'}-\mu_1^{'})^2}$$

The mean (M), variance (Var), coefficient of variation (CV), skewness (SK) and kurtosis (KU) of the GXL distribution for various values of $\Theta=(\lambda,\gamma)$ are listed in Table 1.

TABLE 1. Mean, Variance, coefficient of variation, skewness and kurtosis of the GXLD

(λ,γ)	(0.25,0.5)	(0.5,1.2)	(0.75,2)	(1,0.5)
M	1.1111	1.0865	1.0743	1.4444
Var	6.8819	3.0417	1.6115	0.3580
CV	2.3610	1.6051	1.1815	0.4142
SK	7.0899	5.6981	6.3773	65.3519
KU	46.8457	27.9326	28.4220	520.0951

3.2. **Stochastic ordering.** The use of stochastic orderings of non-negative continuous random variables to assess comparative behavior is a useful technique. The variable Z is stated to be greater than the variable Y in the case when Z and Y are independent with CDFs F_Z and F_Y, respectively, in the following contexts:

- Stochastic order $(Z\leq_s Y)$ if $F_Z(z)\geq F_Y(z)\;\forall\, z$,
- Mean residual life order $(Z\leq_{mrl} Y)$ if $m_Z(z)\geq m_Y(z)\;\forall\, z$,
- Hazard rate order $(Z\leq_{hr} Y)$ if $h_Z(z)\geq h_Y(z)\;\forall\, z$,
- Likelihood ratio order $(Z\leq_{lr} Y)$ if $\frac{f_Z(z)}{f_y(z)}$ is an decreasing function of z.

Remark 3.1. *The next consequences (see [11]) are particularly remarkable:*

$$\underbrace{(\textit{Likelihood ratio order}\Rightarrow\textit{Hasard rate order}\Rightarrow\textit{Mean residual life order})}$$

$$\Downarrow$$

Stochastic order

Theorem 3.1. *Let Z and Y be two independent random variables with parameters (λ_1, γ_1) and (λ_2, γ_2) respectively, that follow the GXL distribution. If $\lambda_1 = \lambda_2$, then $Z \leq_{lr} Y$ and hence $(Z \leq_{hr} Y)$, $Z \leq_{mrl} Y$, and $Z \leq_s Y$.*

Proof. We have, for all $z > 0$,

$$\frac{f_Z(z)}{f_y(z)} = \left(\frac{\lambda_1}{\lambda_2}\right)^2 \left(\frac{\lambda_2+\gamma_2}{\lambda_1+\gamma_1}\right)^2 \left(\frac{\lambda_1+2\gamma_1+\gamma_1^2 z}{\lambda_2+2\gamma_2+\gamma_2^2 z}\right) \exp\left\{-(\lambda_1-\lambda_2)z\right\}.$$

Now,

$$\log\frac{f_Z(z)}{f_y(z)} = 2\log\left(\frac{\lambda_1}{\lambda_2}\right) + 2\log\left(\frac{\lambda_2+\gamma_2}{\lambda_1+\gamma_1}\right) + \log\left(\lambda_1+2\gamma_1+\gamma_1^2 z\right)$$
$$- \log\left(\lambda_2+2\gamma_2+\gamma_2^2 z\right) + (\lambda_2-\lambda_1)\, z.$$

Thus,

$$\frac{d}{dz}\log\frac{f_Z(z)}{f_y(z)} = \frac{\gamma_1^2}{\lambda_1+2\gamma_1+\gamma_1^2 z} - \frac{\gamma_2^2}{\lambda_2+2\gamma_2+\gamma_2^2 z} + (\lambda_2-\lambda_1). \tag{3.2}$$

Clearly, we have two different cases:

(i) If $\lambda_1 = \lambda_2$ and $\gamma_1 \leq \gamma_2$, then (2.2) is negative. This means that $Z \leq_{lr} Y$ and hence $(Z \leq_{hr} Y)$, $Z \leq_{mrl} Y$, and $Z \leq_s Y$.

(ii) If $\gamma_1 = \gamma_2$ and $\lambda_1 \leq \lambda_2$,then (2.2) is negative. This means that $Z \leq_{lr} Y$ and hence $(Z \leq_{hr} Y)$, $Z \leq_{mrl} Y$, and $Z \leq_s Y$.

□

4. Methods of Estimation for the Distribution Parameters

4.1. **Maximum Likelihood Estimators.** Let $Z_1, Z_2, \ldots, Z_m$ be m independent and identical random variables again from the GXL distribution with parameters λ and γ. For establishing the MLE of Θ, we now have likelihood function based on observed sample $\underset{\sim}{z} = (z_1, z_2, \ldots, z_m)$ provided by

$$\ell(z;\Theta) = \left(\frac{\lambda}{\lambda+\gamma}\right)^{2m} \prod_{k=1}^{m}\left[\lambda+2\gamma+\gamma^2 z\right] \exp(-\lambda\sum_{k=1}^{m} z_k).$$

The log likelihood function is calculated as

$$\log\ell(z;\Theta) = 2m\log\lambda - 2m\log(\lambda+\gamma) + \sum_{k=1}^{m}\left(\lambda+2\gamma+\gamma^2 z_k\right) - \lambda\sum_{k=1}^{m} z_k.$$

The partial derivatives of $\log \ell(z;\Theta)$ with respect to the proposed model parameters λ and γ are respectively given by

$$\frac{\partial \log \ell(z;\Theta)}{\partial \lambda} = \frac{2m}{\lambda} - \frac{2nm}{\lambda+\gamma} + \sum_{k=1}^{m}\left(\frac{1}{\lambda+2\gamma+\gamma^2 z_k}\right) - \sum_{k=1}^{m} z_k$$

$$\frac{\partial \log \ell(z;\Theta)}{\partial \gamma} = \frac{2m}{\lambda+\gamma} + \sum_{k=1}^{m}\left(\frac{2+2\gamma}{\lambda+2\gamma+\gamma^2 z_k}\right)$$

It is frequently more convenient to solve these equations numerically using nonlinear optimization methods. The need for numerical approaches like the Newton-Raphson method, the Monte Carlo method, the BB method, and others is because the explicit formula for the maximum likelihood estimator of the parameter is not close. So, in our case, the R software and specifically the package "maxLik" can be used to optimize the log-likelihood and obtain the MLE's.

The corresponding Fisher information matrix that we observed is provided by

$$I(\Theta) = \begin{pmatrix} \frac{\partial^2 \log \ell(\Theta;z)}{\partial \lambda^2} & \frac{\partial^2 \log \ell(\Theta;z)}{\partial \lambda \partial \gamma} \\ \frac{\partial^2 \log \ell(\Theta;z)}{\partial \gamma \partial \lambda} & \frac{\partial^2 \log \ell(\Theta;z)}{\partial \gamma^2} \end{pmatrix}.$$

When m is large enough, the distribution of the adjacent random vector next to $\left(\hat{\lambda},\hat{\gamma}\right)$ can be approximated by a two-dimensional normal distribution of mean vector (λ,γ) and covariance matrix $I\left(\hat{\lambda},\hat{\gamma}\right)^{-1}$. We can create asymptotic confidence intervals for λ and γ by indicating $V_{\hat{\lambda}}$and $V_{\hat{\gamma}}$, the diagonal elements of this matrix. Indeed, the asymptotic confidence intervals (CIs) of λ and γ at the level $100\,(1-\tau)\,\%$ are respectively provided by

$$CI_\lambda = \left[\hat{\lambda} - z_{\tau/2}\sqrt{V_{\hat{\lambda}}}, \hat{\alpha} + z_{\tau/2}\sqrt{V_{\hat{\gamma}}}\right],$$
$$CI_\gamma = \left[\hat{\gamma} - z_{\tau/2}\sqrt{V_{\hat{\gamma}}}, \hat{\gamma} + z_{\tau/2}\sqrt{V_{\hat{\gamma}}}\right],$$

where $z_{\tau/2}$ is the upper $(\tau/2)\,th$ percentile of the standard normal distribution

4.2. **Maximum Product of Spacings Method.** In this part, we are using an alternative way to the MLE method known as the maximum product of spacings (MPS). This technique is used to estimate the parameters of continuous univariate models and was created by Cheng and Amin [3]. This method is also used for censored application by Almetwally and Almongy [1] and Alshenawy et al. [2].

Let $Z_1, Z_2, \ldots, Z_m$ be a random sample of size m, the GXL distribution's uniform spacing can also be described by

$$D_j(\Theta) = \int_{z_{(j-1)}}^{z_{(j)}} f(z;\Theta)\, dz; j = 1, \ldots, m+1,$$

where, $F\left(z_{(0)};\Theta\right) = 0$ and $F\left(z_{(m+1)};\Theta\right) = 1$, we take the geometric mean of the differences $D_j(\Theta)$ as

$$\bar{D} = \sqrt[m+1]{\prod_{j=1}^{m+1} D_j(\Theta)}.$$

The geometric mean $\bar{D}$ of the differences is maximized to obtain the maximum product spacing estimators (MPS) $\hat{\lambda}$ and $\hat{\gamma}$ of λ and γ and taking the logarithm of the previous expression, we now have

$$\log \bar{D} = \frac{1}{m+1} \sum_{j=1}^{m+1} \log \left[\left(1 + \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j-1)}\right) \exp\left\{-\lambda z_{(j-1)}\right\} - \left(1 + \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j)}\right) \exp\left\{-\lambda z_{(j)}\right\} \right].$$

The MPS estimators $\hat{\lambda}$ and $\hat{\gamma}$ can be calculated by solving the following non linear equations concurrently

$$\frac{\partial \log \bar{D}}{\partial \lambda} = \frac{1}{m+1} \sum_{j=1}^{m+1} \left[\frac{z_{(j-1)} \left(\frac{\gamma^2(\gamma-\lambda)}{(\lambda+\gamma)^3} - \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j-1)} - 1\right) \exp\left\{-\lambda z_{(j-1)}\right\}}{\left(1 + \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j-1)}\right) \exp\left\{-\lambda z_{(j-1)}\right\} - \left(1 + \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j)}\right) \exp\left\{-\lambda z_{(j)}\right\}} - \frac{z_{(j)} \left(\frac{\gamma^2(\gamma-\lambda)}{(\lambda+\gamma)^3} - \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j)} - 1\right) \exp\left\{-\lambda z_{(j)}\right\}}{\left(1 + \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j-1)}\right) \exp\left\{-\lambda z_{(j-1)}\right\} - \left(1 + \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j)}\right) \exp\left\{-\lambda z_{(j)}\right\}} \right],$$

$$\frac{\partial \log \bar{D}}{\partial \gamma} = \frac{1}{m+1} \sum_{j=1}^{m+1} \left[\frac{z_{(j-1)} \left(\frac{2\lambda\gamma(\gamma+\lambda)(1-\gamma)}{(\lambda+\gamma)^3}\right) \exp\left\{-\lambda z_{(j-1)}\right\}}{\left(1 + \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j-1)}\right) \exp\left\{-\lambda z_{(j-1)}\right\} - \left(1 + \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j)}\right) \exp\left\{-\lambda z_{(j)}\right\}}\right.$$

$$- \frac{z_{(j)} \left(\frac{2\lambda\gamma(\gamma+\lambda)(1-\gamma)}{(\lambda+\gamma)^3} \right) \exp\left\{-\lambda z_{(j)}\right\}}{\left(1 + \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j-1)}\right) \exp\left\{-\lambda z_{(j-1)}\right\} - \left(1 + \frac{\lambda\gamma^2}{(\lambda+\gamma)^2} z_{(j)}\right) \exp\left\{-\lambda z_{(j)}\right\}} \Bigg]$$

5. Application of Real Data Analysis

This section demonstrates the utility of the GXL distribution for two real data sets. The proposed model is especially in comparison to other competing models, these: Lognormal distribution (LND), Weibull distribution (WD) [12], Generalization of Two-Parameter Lindley Distribution (GTPLD) [9] and Generalized Lindley distribution (GLD) [13], the densities associated with those distributions are offered by the following functions (for $x > 0$):

$$f_{LND}(z;\lambda,\gamma) = \frac{1}{\sqrt{2\pi}\gamma z} \exp\left\{-\frac{1}{2}(\frac{\log z - \lambda}{\gamma})^2\right\}, z > 0, \lambda > 0, \gamma > 0,$$

$$f_{WD}(z;\lambda,\gamma) = \lambda\gamma z^{\gamma-1} \exp\left(-\lambda z^{\gamma}\right), z > 0, \lambda > 0, \gamma > 0,$$

$$f_{GTPLD}(z;\lambda,\gamma,\zeta) = \frac{\gamma\lambda^2}{\zeta\lambda+1} z^{\gamma-1}(\zeta + z^{\gamma}) \exp\left(-\lambda z\right), z > 0, \lambda > 0, \gamma > 0, \zeta > 0,$$

$$f_{GLD}(z;\lambda,\gamma,\zeta) = \frac{\lambda^{\gamma+1}}{\lambda+\zeta} \frac{z^{\gamma-1}}{\Gamma(\gamma+1)} (\gamma + \zeta z) \exp\left(-\lambda z^{\gamma}\right), z > 0, \lambda > 0, \gamma > 0, \zeta > 0.$$

The unknown parameters of the preceding pdf's are all non-negative real values. We take the following criteria when comparing distributions:

- The Akaike Information Criterion (AIC),
- Hannan-Quinn Information $(HQIC)$,
- Bayesian Information Criterion (BIC), and
- Consistent Akaike Information Criterion $(CAIC)$.

These statistics are presented by

$$AIC = -2L + 2J,$$

$$BIC = -2L + J\log(m),$$

$$CAIC = -2L + \frac{2Jm}{(m-J-1)},$$

$$HQIC = -2L + 2J\log[\log(m)],$$

where L represents the MLE's log-likelihood function, J the number of above model parameters and m the sample size.

The model with smaller values for each of these statistics may be selected as the best fit for the data. The R program was used to obtain all of the results.

Tables 2 and 4 display the MLEs of the parameters, whereas Tables 3 and 5 compare the GXL model with the above distributions. With all fitted models based on two real data sets, the new proposed model has the smallest AIC, BIC, $CAIC$, and $HQIC$ statistics. As a result, it can be selected as the best model between them. Figures 2 and 3 represent the estimated pdf, estimated CDF and the P-P plot for the two real data sets from Lawless (2003), pp: 204 and 263 [5].

5.1. **Application 1.** The first data represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test, this data are: 1.4 5.1 6.3 10.8 12.1 18.5 19.7 22.2 23.0 30.6 37.3 46.3 53.9 59.8 and 66.2. The results of application 1 are shown in Table 2, Table 3 and Figure 2.

TABLE 2. MLEs for the real data 1.

Estimates				
Model		$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\zeta}$
GXLD	MLE	0.0622	0.3403	-
	SE	0.0172	0.6091	-
LND	MLE	2.9305	1.0252	-
	SE	0.2647	0.1871	-
WD	MLE	0.0119	1.3058	-
	SE	0.0112	0.2492	-
GTPLD	MLE	0.0247	1.2027	54.9930
	SD	0.0388	0.3371	214.2735
GLD	MLE	0.0641	1.2025	0.0832
	SD	0.0213	0.8131	0.2706

5.2. **Application 2.** The second set of data represents the number of cycles to failure for 25 100-cm specimens of yarn, tested at a particular strain level, this data are: 15 20 38 42 61 76 86 98 121 146 149 157 175 176 180 180 198 220 224 251 264 282 321 325 653. The results of application 1 are shown in Table 4, Table 5 and Figure 3:

TABLE 3. AIC, BIC, CAIC, and HQIC statistics for the real data 1.

	Statistics			
Model	AIC	BIC	CAIC	HQIC
GXLD	132.21	133.631	133.215	132.200
LND	135.23	136.646	136.23	135.214
WD	132.04	133.456	133.04	132.024
GTPLD	133.94	136.064	136.121	133.917
GLD	134.16	136.284	136.341	134.137

TABLE 4. MLEs for the real data 2.

Estimates				
Model		$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\zeta}$
GXLD	MLE	0.0107	0.2595	-
	SE	0.0019	0.8183	-
LND	MLE	4.8795	0.8743	-
	SE	0.1746	0.1235	-
WD	MLE	0.0025	1.1480	-
	SE	0.0006	0.0589	-
GTPLD	MLE	0.0085	1.0372	19.0247
	SD	0.0075	0.1443	61.3271
GLD	MLE	0.0101	0.8186	3.9740
	SD	0.0030	0.4858	63.1287

TABLE 5. AIC, BIC, CAIC, and HQIC statistics for the real data 2.

	Statistics			
Model	AIC	BIC	CAIC	HQIC
GXLD	309.74	312.179	310.287	310.418
LND	312.16	314.597	312.705	312.836
WD	310.56	312.997	311.105	311.236
GTPLD	310.89	314.536	312.022	311.894
GLD	310.88	313.004	313.061	310.857

6. CONCLUSION

The interest, in the present work, is given to a two-parameter model, namely the GXL distribution. The latter is a combination of the exponential and the Two-Parameter lindley distributions. The mathematical expression of its probability

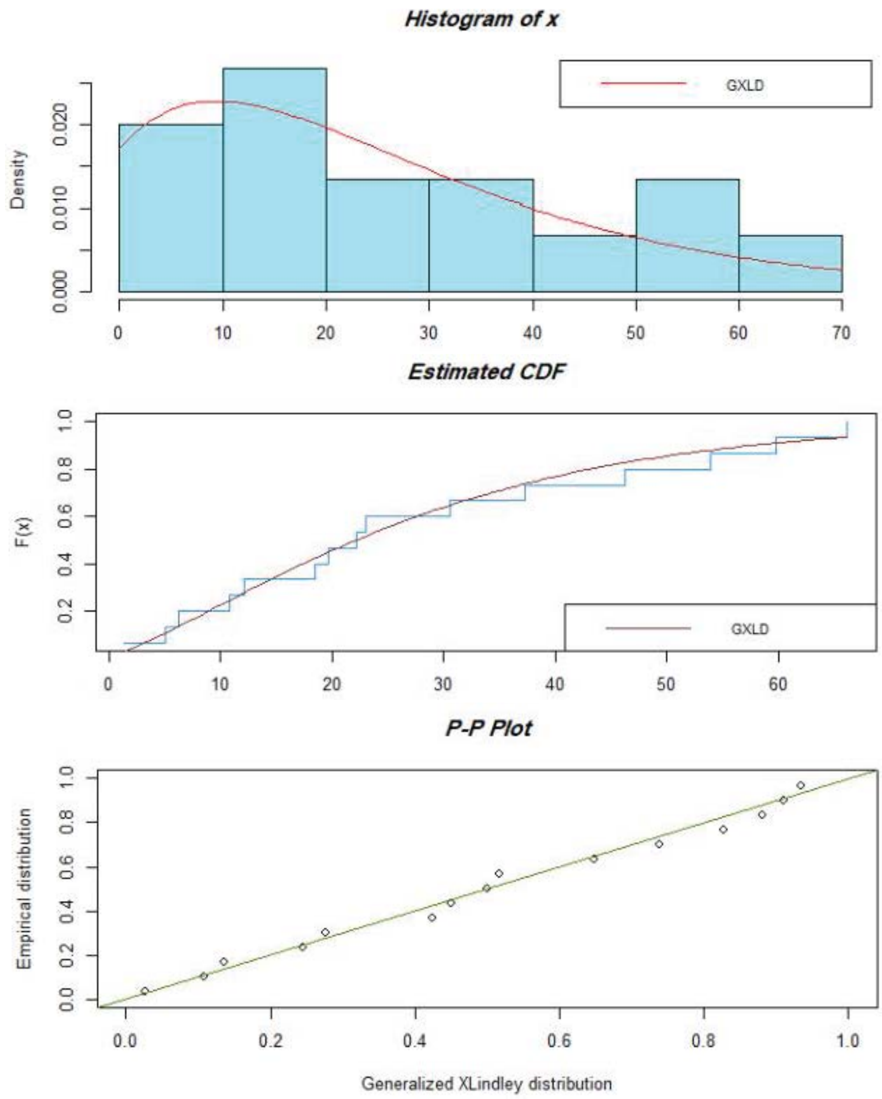

FIGURE 2. Estimated pdf, CDF and P-P plot for the failure times of the 15 electronic components.

density is workable. Consequently, this enables us to determine its different statistical properties. The method of maximum likelihood and maximum product of spacings are employed to estimate the parameters. With the support of the maximum likelihood estimators, the asymptotic confidence intervals for model parameters are also obtained. The study of two real data sets shows the applicability of this new model (GXL distribution). This application demonstrates that it has the

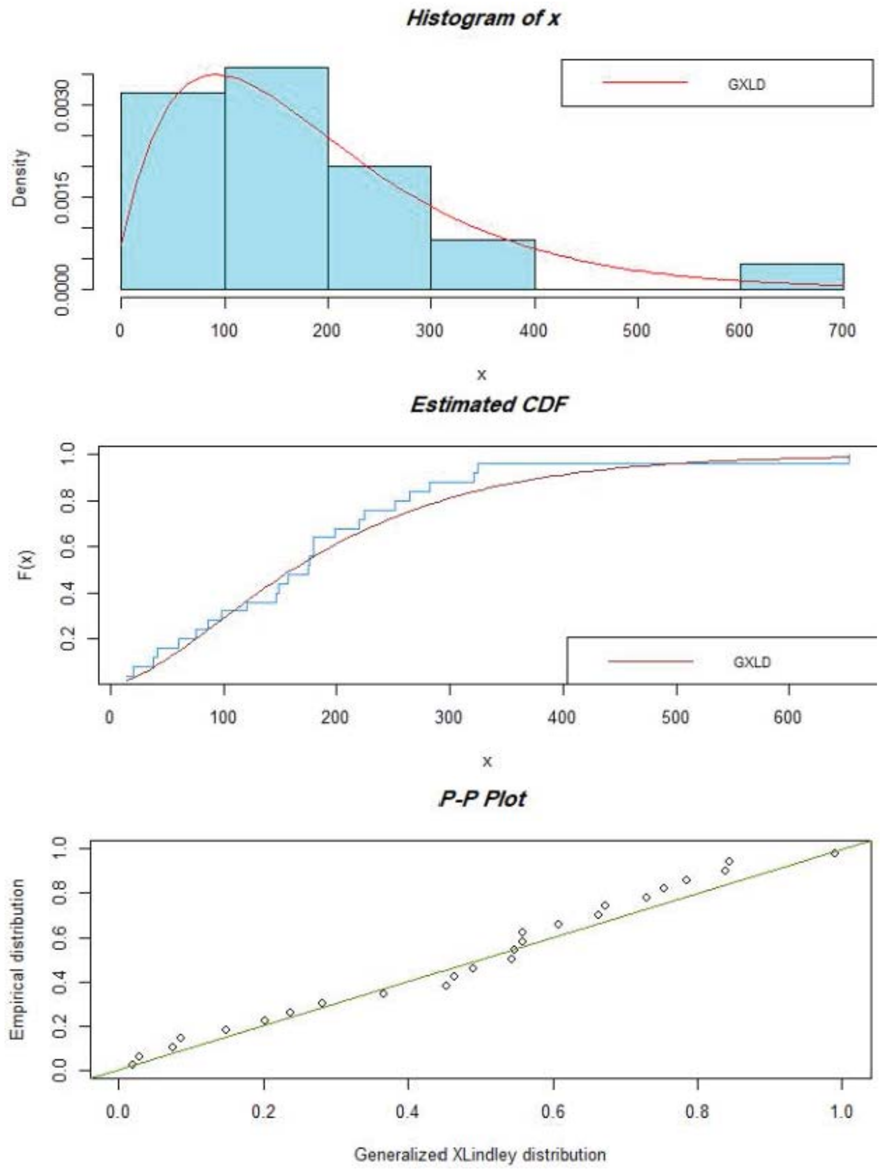

FIGURE 3. Estimated pdf, CDF and P-P plot for the number of cycles to failure for 25 100-cm specimens of yarn.

potential to significantly affect other commonly used statistical models in terms of fit. The suggested distribution could be regarded as a proven alternative to other distributions such as GL distribution, GG distribution,. . .

Finally, we anticipate that our suggested model will find widespread application for real data in several fields for example medicine, engineering, and social sciences.

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