ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **12** (2023), no.1, 107–113
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.12.1.7

# SOME BOUNDS FOR THE WIENER INDEX OF WEAK JOIN OF TWO GRAPHS

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ABSTRACT. In 2020, G. Suresh Singh and Manju V. N. [2] introduced the concept of weak join of two disjoint graphs with respect to the degrees of the given graphs and categorized them as homogeneous weak join, heterogeneous weak join and studied some of their properties. Homogeneous weak join can be splitted in to three cases namely, odd to odd degree weak join, even to even degree weak join, odd to odd and even to even degree weak join. Similarly, heterogeneous weak join includes the other three cases namely, odd to even degree weak join, even to odd degree weak join, odd to even and even to odd degree weak join. In this paper we try to find bounds for the Wiener index of odd to odd, even to even, odd to even and even to odd degree weak join of graphs.

## 1. INTRODUCTION

The Wiener index $W(G)$ of a graph $G$ is a distance based topological index defined as the sum of the distances between all pairs of vertices in $G$. In 1947, [1], Harold Wiener defined the Wiener index $W(G)$ of a graph with $n$ vertices $\{v_1 v_2, \ldots, v_n\}$ as the sum of the distances between the vertices of the graph $G$, $W(G) = \sum\limits_{i<j} d(v_i, v_j)$, where $d(v_i, v_j)$ denotes the distance between vertices $v_i$ and

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2020 *Mathematics Subject Classification.* 05C09, 05C76, 05C45.
*Key words and phrases.* Wiener index, odd to odd degree weak join, even to even degree weak join, odd to even degree weak join, even to odd degree weak join.
*Submitted:* 15.12.2022; *Accepted:* 30.12.2022; *Published:* 11.01.2023.

$v_j$. In 2019, Martin Knor et.al. [3] gave some results on the Wiener index of a graph. Throughout this paper we consider finite, undirected simple connected graphs.

The even to even degree weak join of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 +_w^{ee} G_2$ is the graph with vertex set $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$ together with the set of edges $uv$ where $u \in V_1$ and $v \in V_2$ having even degrees.

The odd to odd degree weak join of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 +_w^{oo} G_2$ is the graph with vertex set $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$ together with the set of edges $uv$ where $u \in V_1$ and $v \in V_2$ having odd degrees.

The even to odd degree weak join of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 +_w^{eo} G_2$ is the graph with vertex set $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$ together with the set of edges $uv$ where $u \in V_1$ is an even degree vertex and $v \in V_2$ is an odd degree vertex.

The odd to even degree weak join of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 +_w^{oe} G_2$ is the graph with vertex set $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$ together with the set of edges $uv$ where $u \in V_1$ is an odd degree vertex and $v \in V_2$ is an even degree vertex.

In further text, for the reasons of brevity, we will say even vertices instead of vertices of even degree and same for odd. For basic definitions and notations we refer to [4].

## 2. Upper Bound for the Wiener Index of Even to Even Degree Weak Join of Two Graphs

In this section we obtain an upper bound for the Wiener index of even to even degree weak join of two graphs. For that, we compute the distance between two vertices as follows. Let $G_1$ and $G_2$ be two graphs with vertex sets $\{u_i : i = 1, 2, \dots, n\}$ and $\{v_i : i = 1, 2, \dots, m\}$ respectively. Let $u_i$ be an odd vertex in $G_1$ and let,

$$\text{dist}_e^{G_1}(u_i) = \min_{u_k \in V(G_1)} \{d(u_i, u_k) : d_{G_1}(u_k) \text{ is even}\}$$

and $\text{dist}_e^{G_1} = \max\{\text{dist}_e^{G_1}(u_i)\}$. For an odd vertex $v_j$ in $G_2$, let

$$\text{dist}_e^{G_2}(v_j) = \min_{v_k \in V(G_2)} \{d(v_j, v_k) : d_{G_2}(v_k) \text{ is even}\}$$

and $\text{dist}_e^{G_2} = \max\{\text{dist}_e^{G_2}(v_j)\}$.

Using these notations, we can write the distance between any two vertices $u_i$ and $v_j$ in $G_1 +_w^{ee} G_2$ as follows,

$$d(u_i, v_j) = \begin{cases} 1 & \text{if } u_i \text{ and } v_j \text{ are even vertices} \\ 1 + \text{dist}_e^{G_1}(u_i) & \text{if } u_i \text{ is an odd vertex and } v_j \text{ is an even vertex} \\ 1 + \text{dist}_e^{G_2}(v_j) & \text{if } u_i \text{ is an even vertex and } v_j \text{ is an odd vertex} \\ 1 + \text{dist}_e^{G_1}(u_i) & \\ \quad + \text{dist}_e^{G_2}(v_j) & \text{if } u_i \text{ and } v_j \text{ are odd vertices} \end{cases}$$

The next theorem gives an upper bound for $W(G_1 +_w^{ee} G_2)$.

**Theorem 2.1.** *Let $G_1$ and $G_2$ be two graphs of order $n$ and $m$ respectively. Let $n_o, m_o$ and $n_e, m_e$ be the number of odd and even vertices in $G_1$ and $G_2$ respectively with $n_e, m_e$ are non zeroes. Then,*

$$W(G_1 +_w^{ee} G_2) \leq W(G_1) + W(G_2) + mn + \textit{dist}_e^{G_1} m n_o + \textit{dist}_e^{G_2} n m_o.$$

*Proof.* Let $V_e(G_i)$ denote the set of all even vertices in $G_i$ and $V_o(G_i)$ be the set of all odd vertices in $G_i$, $i = 1, 2$. Then,

$$\begin{aligned} W(G_1 +_w^{ee} G_2) &= W(G_1) + W(G_2) + \sum_{\substack{u_i \in V_e(G_1) \\ v_j \in V_e(G_2)}} d(u_i, v_j) + \sum_{\substack{u_i \in V_e(G_1) \\ v_j \in V_o(G_2)}} d(u_i, v_j) \\ &\quad + \sum_{\substack{u_i \in V_o(G_1) \\ v_j \in V_e(G_2)}} d(u_i, v_j) + \sum_{\substack{u_i \in V_o(G_1) \\ v_j \in V_o(G_2)}} d(u_i, v_j) \\ &= W(G_1) + W(G_2) + \sum_{\substack{u_i \in V_e(G_1) \\ v_j \in V_e(G_2)}} 1 + \sum_{\substack{u_i \in V_e(G_1) \\ v_j \in V_o(G_2)}} (1 + \text{dist}_e^{G_2}(v_j)) \\ &\quad + \sum_{\substack{u_i \in V_o(G_1) \\ v_j \in V_e(G_2)}} (1 + \text{dist}_e^{G_1}(u_j)) + \sum_{\substack{u_i \in V_o(G_1) \\ v_j \in V_o(G_2)}} (1 + \text{dist}_e^{G_1}(u_i) + \text{dist}_e^{G_2}(v_j)) \\ &\leq W(G_1) + W(G_2) + mn + (n_e m_o + n_o m_o)\text{dist}_e^{G_2} \\ &\quad + (n_o m_e + n_o m_o)\text{dist}_e^{G_1} \\ &\leq W(G_1) + W(G_2) + mn + \text{dist}_e^{G_1} m n_o + \text{dist}_e^{G_2} n m_o. \end{aligned}$$

$\square$

**Remark 2.1.** *If either $G_1$ or $G_2$ has no even degree vertices, then $W(G_1 +_w^{ee} G_2) = \infty$.*

The next corollary give a special case for which equality holds in Theorem 2.1 .

**Corollary 2.1.** *Let $G_1$ and $G_2$ be two non Eulerian graphs of order atleast 3. Then, equality hold in Theorem 2.1 if both graphs $G_1$ and $G_2$ contain Eulerian trail in which no two odd vertices are adjacent.*

*Proof.* Let $|V(G_1)| = n$ and $|V(G_2)| = m$. Since $G_1$ and $G_2$ contain Eulerian trail, then atleast one of them contain exactly two odd degree vertices. If both $G_1$ and $G_2$ contains exactly two non adjacent odd vertices.Then,

$$\begin{aligned} W(G_1 +_w^{ee} G_2) &= W(G_1) + W(G_2) + (n-2)(m-2) + 2 \cdot (n-2) \cdot 2 \\ &\quad + 2 \cdot (m-2) \cdot 2 \\ &= W(G_1) + W(G_2) + mn + 2(m+n). \end{aligned}$$

□

**Remark 2.2.** *From Corollary 2.1, we have the following*

(i) $W(P_n +_w^{ee} P_m) = \binom{n+1}{3} + \binom{m+1}{3} + mn + 2(m+n)$

(ii) $W(P_n +_w^{ee} C_m) = \begin{cases} \frac{m^3}{8} + \binom{n+1}{3} + m(n+2) & \text{if } m \text{ is even}, \\ \frac{m(m^2-1)}{8} + \binom{n+1}{3} + m(n+2) & \text{if } m \text{ is odd}. \end{cases}$

## 3. Upper Bound for the Wiener Index of Odd to Odd Degree Weak Join of Two Graphs

In this section, we find an upper bound for the Wiener index of odd to odd degree weak join of two graphs. For that we have the following notations. Let $G_1$ and $G_2$ be two graphs with vertex sets $\{u_i : i = 1, 2, \ldots, n\}$ and $\{v_i : i = 1, 2, \ldots, m\}$ respectively. Let $u_i$ be an even vertex in $G_1$ and let,

$$\text{dist}_o^{G_1}(u_i) = \min_{u_k \in V(G_1)} \{d(u_i, u_k) : d_{G_1}(u_k) \text{ is odd}\}$$

and $\text{dist}_o^{G_1} = \max\{\text{dist}_o^{G_1}(u_i)\}$. For an even vertex $v_j$ in $G_2$, let

$$\text{dist}_o^{G_2}(v_j) = \min_{v_k \in V(G_2)} \{d(v_j, v_k) : d_{G_2}(v_k) \text{ is odd}\}$$

and $\text{dist}_o^{G_2} = \max\{\text{dist}_o^{G_2}(v_j)\}$.

Using these notations, we can write the distance between any two vertices $u_i$ and $v_j$ in $G_1 +_w^{oo} G_2$ as follows,

$$d(u_i, v_j) = \begin{cases} 1 & \text{if } u_i \text{ and } v_j \text{ are odd vertices} \\ 1 + \text{dist}_o^{G_1}(u_i) & \text{if } u_i \text{ is an even vertex and } v_j \text{ is an odd vertex} \\ 1 + \text{dist}_o^{G_2}(v_j) & \text{if } u_i \text{ is an odd vertex and } v_j \text{ is an even vertex} \\ 1 + \text{dist}_o^{G_1}(u_i) & \\ \quad +\text{dist}_o^{G_2}(v_j) & \text{if } u_i \text{ and } v_j \text{ are even vertices} \end{cases}.$$

The next theorem gives an upper bound for $W(G_1 +_w^{oo} G_2)$.

**Theorem 3.1.** *Let $G_1$ and $G_2$ be two graphs of order $n$ and $m$ respectively. Let $n_o, m_o$ and $n_e, m_e$ be the number of odd and even vertices in $G_1$ and $G_2$ respectively with $n_o, m_o$ are non zeroes. Then,*

$$W(G_1 +_w^{oo} G_2) \leq W(G_1) + W(G_2) + mn + \textit{dist}_o^{G_1} mn_e + \textit{dist}_o^{G_2} nm_e.$$

*Proof.* The proof is similar to the proof of Theorem 2.1. □

**Remark 3.1.** *If either $G_1$ or $G_2$ has no odd degree vertices, then $W(G_1 +_w^{oo} G_2) = \infty$.*

## 4. Upper Bound for the Wiener Index of Heterogeneous Weak Join of Two Graphs

In this section we try to find out the upper bounds for the Wiener index of heterogeneous weak joins of two graphs. Firstly we consider even to odd weak join. We can write the distance between any two vertices $u_i$ and $v_j$ in $G_1 +_w^{eo} G_2$ as follows,

$$d(u_i, v_j) = \begin{cases} 1 & \text{if } u_i \text{ is an even vertex and } v_j \text{ is an odd vertex} \\ 1 + \text{dist}_e^{G_1}(u_i) & \text{if } u_i \text{ and } v_j \text{ are odd vertices} \\ 1 + \text{dist}_o^{G_2}(v_j) & \text{if } u_i \text{ and } v_j \text{ are even vertices} \\ 1 + \text{dist}_e^{G_1}(u_i) & \\ \quad +\text{dist}_o^{G_2}(v_j) & \text{if } u_i \text{ is an odd vertex and } v_j \text{ is an even vertex} \end{cases}.$$

**Theorem 4.1.** *Let $G_1$ and $G_2$ be two graphs of order $n$ and $m$ respectively. Let $n_o, m_o$ and $n_e, m_e$ be the number of odd and even vertices in $G_1$ and $G_2$ respectively with $n_o, m_o$ and $n_e, m_e$ are non zeroes. Then,*

$$W(G_1 +_w^{eo} G_2) \leq W(G_1) + W(G_2) + mn + \mathit{dist}_e^{G_1} mn_o + \mathit{dist}_o^{G_2} nm_e.$$

*Proof.* Let $V_e(G_i)$ denote the set of all even vertices in $G_i$ and $V_o(G_i)$ be the set of all odd vertices in $G_i$, $i = 1, 2$. Then,

$$\begin{aligned} W(G_1 +_w^{eo} G_2) &= W(G_1) + W(G_2) + \sum_{\substack{u_i \in V_e(G_1) \\ v_j \in V_e(G_2)}} d(u_i, v_j) + \sum_{\substack{u_i \in V_e(G_1) \\ v_j \in V_o(G_2)}} d(u_i, v_j) \\ &\quad + \sum_{\substack{u_i \in V_o(G_1) \\ v_j \in V_e(G_2)}} d(u_i, v_j) + \sum_{\substack{u_i \in V_o(G_1) \\ v_j \in V_o(G_2)}} d(u_i, v_j) \\ &= W(G_1) + W(G_2) + \sum_{\substack{u_i \in V_e(G_1) \\ v_j \in V_e(G_2)}} (1 + \text{dist}_o^{G_2}(v_j)) + \sum_{\substack{u_i \in V_e(G_1) \\ v_j \in V_o(G_2)}} 1 \\ &\quad + \sum_{\substack{u_i \in V_o(G_1) \\ v_j \in V_e(G_2)}} (1 + \text{dist}_e^{G_1}(u_i) + \text{dist}_o^{G_2}(v_j)) + \sum_{\substack{u_i \in V_o(G_1) \\ v_j \in V_o(G_2)}} (1 + \text{dist}_e^{G_1}(u_i)) \\ W(G_1 +_w^{eo} G_2) &\leq W(G_1) + W(G_2) + mn + \text{dist}_e^{G_1} mn_o + \text{dist}_o^{G_2} nm_e. \end{aligned}$$

□

In case of odd to even degree weak join, we can write the distance between any two vertices $u_i$ and $v_j$ in $G_1 +_w^{oe} G_2$ as follows,

$$d(u_i, v_j) = \begin{cases} 1 & \text{if } u_i \text{ is an odd vertex and } v_j \text{ is an even vertex} \\ 1 + \text{dist}_e^{G_2}(v_j) & \text{if } u_i \text{ and } v_j \text{ are odd vertices} \\ 1 + \text{dist}_o^{G_1}(u_i) & \text{if } u_i \text{ and } v_j \text{ are even vertices} \\ 1 + \text{dist}_e^{G_2}(v_j) + \text{dist}_o^{G_1}(u_i) & \text{if } u_i \text{ is an even vertex and } v_j \text{ is an odd vertex} \end{cases}$$

**Theorem 4.2.** *Let $G_1$ and $G_2$ be two graphs of order $n$ and $m$ respectively. Let $n_o, m_o$ and $n_e, m_e$ be the number of odd and even vertices in $G_1$ and $G_2$ respectively with $n_o, m_o$ and $n_e, m_e$ are non zeroes. Then,*

$$W(G_1 +_w^{oe} G_2) \leq W(G_1) + W(G_2) + mn + \mathit{dist}_o^{G_1} mn_e + \mathit{dist}_e^{G_2} nm_o.$$

*Proof.* The proof is similar to the proof of Theorem 4.1. □

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