ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **12** (2023), no.1, 115–136
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.12.1.8

ADOMIAN DECOMPOSITION METHOD FOR SOLVING SPATIALLY INHOMOGENUOUS POPULATION BALANCE EQUATION

Imane Achour[1] and Abdelghani Bellagoun

ABSTRACT. In this article, Adomian Method is applied for spatially inhomogeneous Population Balance Equation incorporating breakage and coalescence processes in both batch and continuous flow systems. It should be Note that the analytical solutions obtained using this technique are not available in the literature.

1. INTRODUCTION

Energy and mass balances are essential tools in process modeling to describe the changes that occur during the physical and chemical reactions. An additional balance for particulate processes is generally referred to as the population balance. Population balances may be considered as an old subject that has its origin in the Boltzmann equation more a century ago [26].

Particulate processes are the processes by which two particles or more undergo changes in its physical properties. they are modeled by equations called Population

[1]*corresponding author*
2020 *Mathematics Subject Classification.* 74H10, 35C10, 74G10, 74H05.
Key words and phrases. Population Balance Equation, Adomian Decomposition Method, Breakage process, Coalescence process.
Submitted: 21.12.2022; *Accepted:* 06.01.2022; *Published:* 13.01.2023.

balance equations which are partial integro-differential equations. These equations are widely used in engineering applications such as precipitation, crystallization, atmospheric physics, pharmaceutical production, aerosol formation, colloid chemistry, polymerization and emulsions and many others [9, 14, 24, 27, 28].

The population balance equations use a particle number density distribution $f(t, v, z)$ defined in a phase space in order to balance a population of particles. It describes the dynamic evolution of the density function. This evolution depends on two kinds of interactions, the first kind is the interaction between particles and called particle-particle interactions while the second is the interaction of particles and the continuous flow field in which these particles are dispersed and this type of interaction namely particle-continuous phase interactions. These PBEs are characterized by a source term, this source term accounts for different mechanisms in which particles of a specific state can either appear or disappear from the system. these mechanisms are instantaneous compared to a system scale such as particle breakage , aggregation, growth and nucleation [26]. However, in this work only breakage and coalescence processes will be considered. Such examples of breakage process are The size reduction of solid materials and the growth of a population of bacteries in which reproduction occurs by binary cell division .

In literature, there are several numerical methods that are used to solve these equations such as method of moments, method of finite elements, method of successive approximations, pivot technique, method of weighted residuals [20–23, 25].

At the beginning of the 1980's, the American mathematician George Adomian presented a powerful technique. this method can solve linear or nonlinear, deterministic or stochastic operator equations, including algebraic, ordinary differential, partial differential, integral, integro-differential, delay differential.... it allows solution without discretizing the equation or even approximating the operators by such schemes as linearization or perturbation [1]. When the solution exists, it is found as a rapidly converging series such that the variables are not discretized.

Recently, The Adomian method has attained high interest in the area of series solutions in particular. It is basically decomposing the solution in the form of a series, also the nonlinear operator into a series. The components of the series solution are obtained by a specific kind of polynomials called "Adomian's polynomials" recursively.

Many authors compared this technique with other analytical and numerical techniques. For instance, in [34] A.M. Wazwaz compared the method with the power series solutions, also N. Bellomo, D. Sarafyan and M. A. Golberg with the Picard method of successive iterations [6] [15], as well as J. Y. Edwards, J. A. Roberts, and N. J. Ford with the Runge-Kutta [13] and others.

It is important to note that the decomposition method does not require discretization of variables. By compare it with other method, it is not much affected by computational round-off and there is no necessity of large computer memory and time. usually, The solution obtained using the Adomian method in the form of series converging to the solution in a closed form which is reliable and effective.

In recent works, the decomposition method has been applied to solve PBEs for pure breakage for both batch and continuous systems [16], and for coalescence but only for the batch system [17]. In [19], the Adomian and the variational iteration methods have applied to particle breakage equation for both batch and continuous flow systems. Furthermore, the variational iteration and projection methods have used to solve certain spatially distributed population balance equation [18].

It should be mentioned that most of papers dealing with solving this type of nonlinear equation use numerical methods to solve them. The purpose of this paper is to solve a class of such equations analytically. Our objective is to derive exact solutions of spatially inhomogenuous PBEs that incorporate breakage, coalescence in batch and continuous systems.

This article is organized as follows. In section 2, we begin with a general formulation of the PBE. In section 3, a general description of the method of analysis which is the Adomian decomposition method is given with some convergence remarks. In section 4, the efficiency of the ADM is demonstrated by several examples. In the last section 5, we make some conclusions.

2. Formulation of Population Balance Equation

In this paper, we consider the following one-dimensional spatially inhomogeneous population balance equation (PBE). It describes the time-space evolution of the particle number distribution function $f(t,v,z)$ under the simultaneous affect of breakage and coalescence processes in a continuous flow system [26, 29]. The PBE reads

$$\underbrace{\frac{\partial\left(f(t,v,z)\right)}{\partial t}}_{\text{the accumulation term}}+\underbrace{\frac{\partial(U_d f(t,v,z))}{\partial z}}_{\text{convection in physical space}}-\underbrace{\frac{\partial^2(D_d f(t,v,z))}{\partial z^2}}_{\text{diffusion in physical space}}=\frac{1}{\theta}\left(f^{feed}(v)-f(t,v,z)\right) + \underbrace{\phi(f,t,v,z)}_{\text{the source term}} \tag{2.1}$$

This equation needs to be supplemented with appropriate initial or boundary conditions.

Here it is assumed that the number of particles $f(t,v,z)$ depends on time $t \geq 0$, an internal and external coordinates. The internal coordinate refer to the properties attached to each individual particle, an example of internal coordinate is size (i.e., volume, diameter or mass), whereas the external coordinates refer to the spatial distribution of the particles in the particulate system.

The left hand side of 2.1 is devoted to the time and space evolution of the particle number density: it includes a convection term with the particle velocity U_d and a diffusion term including a diffusion rate D_d. In the right hand side, the first term is the net bulk flow into the vessel. The source term $\rho(f,t,v,z)$ represents the contribution to $f(t,v,z)$ of the change in the number of particles. due to particle breakage and coalescence [32, 33]:

$$\begin{aligned}\rho(f,t,v,z) &= -\Gamma(v)f(t,v,z)+\int_v^{+\infty}\beta(v,v')\Gamma(v')f(t,v',z)\,dv' \\ &-\int_0^{+\infty}\omega(v,v')f(t,v',z)f(t,v,z)\,dv' \\ &+\frac{1}{2}\int_0^{v}\omega(v-v',v')f(t,v-v',z)f(t,v',z)\,dv',\end{aligned} \tag{2.2}$$

where $\Gamma(v)$ and $\omega(v,v')$ are the breakage and coalescence frequencies, respectively, $\beta(v,v')\,dv'$ is the number of daughter droplets having volume in the range from v to $v+dv$ formed upon breakage of the droplet of volume v'. In the source term $\rho(f,t,v,z)$, the first two terms represent the particle formation and loss due to breakage succeeded by two terms represent particle formation and loss due to coalescence.

The following is the list of relevant combinations of processes for which the continuous PBE has been solved analytically.

Case study I. PBE with convection term in batch system

$$\frac{\partial f(t,v,z)}{\partial t}+U_d\frac{\partial f(t,v,z)}{\partial z}=-\Gamma(v)f(t,v,z)+\int_v^{+\infty}\beta(v,v')\Gamma(v')f(t,v',z)\,dv' \tag{2.3}$$

Case study II. PBE with diffusion term in batch system

$$\frac{\partial f(t,v,z)}{\partial t}=D_d\frac{\partial^2(f(t,v,z))}{\partial z^2}-\Gamma(v)f(t,v,z)+\int_v^{+\infty}\beta(v,v')\Gamma(v')f(t,v',z)\,dv' \tag{2.4}$$

Case study III. PBE with convection term in continuous system

$$\begin{aligned}\frac{\partial f(t,v,z)}{\partial t}+U_d\frac{\partial f(t,v,z)}{\partial z}&=\frac{1}{\theta}\left(f^{feed}(v)-f(t,v,z)\right)-\Gamma(v)f(t,v,z)\\&+\int_v^{+\infty}\beta(v,v')\Gamma(v')f(t,v',z)\,dv'\end{aligned} \tag{2.5}$$

Case study IV. PBE with convection term (Pure Coalescence)

$$\begin{aligned}\frac{\partial f(t,v,z)}{\partial t}+U_d\frac{\partial(f(t,v,z))}{\partial z}&=\frac{1}{2}\int_0^{v}\omega(v-v',v')f(t,v-v',z)f(t,v',z)\,dv'\\&-\int_0^{+\infty}\omega(v,v')f(t,v',z)f(t,v,z)\,dv'\end{aligned} \tag{2.6}$$

3. Method of analysis and some convergence remarks

Let us review the main principles of the Adomian decomposition method by considering the nonlinear differential equation ([2]):

$$Lf+Rf+N(f)=g, \tag{3.1}$$

where

- L is the linear operator to be inverted and usually taken as the highest-ordered partial derivative. its inverse operator is defined by:

(3.2)
$$L^{-1}(.) = \underbrace{\int\int\ldots\int}_{n-fold}(.)\,dt\ldots\,dt\,dt;$$

- R is the linear remainder operator;
- N is the nonlinear operator;
- g is a given function;
- f is the unknown function.

Transforming the equation (3.1) to the canonical form, we have

(3.3)
$$f = f_0 - L^{-1}(Rf + N(f))$$

According to the problem if it is an initial or boundary value problem, equation (3.3) is to be determined by the appropriate initial or boundary condition. Using the ADM the unknown function f can be expressed by the sum

(3.4)
$$f = \sum_{m=0}^{+\infty} f_m.$$

The nonlinear term $N(f)$ has the Adomian polynomials representation

(3.5)
$$N(f) = \sum_{m=0}^{+\infty} A_m(u_0, u_1, \ldots, u_m),$$

where the A_n can calculated for a wide class of nonlinearities and are defined implicitly by the classical formula [2]

(3.6)
$$A_m = \frac{1}{m!}\frac{d^m}{d\epsilon^m}[N(\sum_{i=0}^{+\infty}\epsilon^i f_i)]_{\epsilon=0}, \quad m \geq 0.$$

The components of the series solution (3.4) are determined by the standard Adomian recursion scheme:

(3.7)
$$\begin{cases} f_0 = L^{-1}(g) + \psi \\ f_{m+1} = -L^{-1}(Rf_m + A_m), \quad m \geq 0. \end{cases}$$

Note Here that ψ is the solution of the linear homogenous equation

(3.8)
$$L\psi = 0.$$

All the components of the (3.4) are determined. So the solution constructed as

$$\lim_{n\to+\infty} \varphi_n = f, \tag{3.9}$$

where $\{\varphi_n\}_{n\geq 0}$ is a rapidly convergent sequence of analytic approximants

$$\varphi_n = \sum_{m=0}^{n-1} f_m. \tag{3.10}$$

As example in the last case study which is nonlinear case, we consider

$$L = \frac{\partial}{\partial z}, \quad R = \frac{\partial}{\partial t}, \tag{3.11}$$

whereas the nonlinear term can be written as the sum of two nonlinearities (i.e: $N = N_1 + N_2$):

$$N_1(f) = -\frac{1}{2}\int_0^v f(t, v-u, z) f(t, u, z)\, du \tag{3.12}$$

$$N_2(f) = f(t, v, z)\int_0^{+\infty} f(t, u, z)\, du. \tag{3.13}$$

Seeing that our article focuses on the analytical forms induced by the convergence of series, we just give a remark on the notion of convergence related to our model. General studies of the convergence of ADM is introduced in several works [4, 5, 7, 8], [31]. Concerning the initial value problems, a reliable approach for the convergence of Adomian's method is discussed by [3]. Indeed, the formalism of Cauchy Kovalevskaya gives the convergence analysis sufficient reliability to estimate the truncated maximum absolute error of the Adomian series. Similarly, the work of Turkyilmazoglu [31] based on the disturbance of the initial terms of the iterations allows to better identify the error. Our method converges in the sense that the model is a nonlinear partial differential equation with an integral term so we can apply the same convergence scheme described by [3] and [31].

In [30], R.Singh, J. Saha and J. Kumar were discussed the convergence of the series solution for the fragmentation and aggregation population balance equation and the convergence analysis is reliable enough to estimate the maximum absolute truncated error of the series solution.

4. Applications and Results

In all case studies, the population balance equation are solved by the ADM and all symbolic calculations are done using the MATHEMATICA software.

4.1. **Linear case: Pure breakage.**

In a breakage process, Drops break into two or many fragments. The total number of drops in this process increases while the total mass remains constant. Therefore, Breakage has a significant effect in the number of drops.

In this section, we present three case studies for pure breakage in batch and continuous systems to illustrate how to use this technique. In all these cases, we have a linear breakage frequency $\Gamma(v) = v$ and a uniform daughter droplet distribution $\beta(v, v') = \frac{2}{v'}$.

4.1.1. *PBE in batch system.*

Case study I. PBE with convection term Consider the PBE

$$\frac{\partial f(t,v,z)}{\partial t} + U_d \frac{\partial f(t,v,z)}{\partial z} = -\Gamma(v) f(t,v,z) + \int_v^{+\infty} \beta(v,v')\Gamma(v') f(t,v',z)\, dv' \tag{4.1}$$

Subjected to the boundary condition $f(t, v, 0) = te^{-v}$. And assuming that $U_d = 1$. The equation (4.1) becomes

$$\frac{\partial f(t,v,z)}{\partial t} + \frac{\partial f(t,v,z)}{\partial z} = -v f(t,v,z) + \int_v^{+\infty} 2 f(t,v',z)\, dv'. \tag{4.2}$$

Transforming equation (4.2) to the canonical form and Operating both sides by $L_z^{-1}(.) = \int_0^z (.)\, dz$, we obtain

$$f(t,v,z) - f(t,v,0) = \int_0^z \left(-\frac{\partial f(t,v,z)}{\partial t} - v f(t,v,z) + \int_v^{+\infty} 2 f(t,v',z)\, dv' \right) dz.$$

The solution by ADM is written as follow:

$$f(t,v,z) = \sum_{m=0}^{+\infty} f_m(t,v,z).$$

The solution components $f_m(t, v, z)$ are obtained by the following Adomian recursion scheme:

$$f_0(t, v, z) = te^{-v}$$

$$f_1(t, v, z) = \int_0^z \left(-\frac{\partial f_0(t, v, z)}{\partial t} - v f_0(t, v, z) + \int_v^{+\infty} 2 f_0(t, v', z)\, dv' \right) dz$$
$$= -\frac{(1 + t(-2 + v))z}{e^v}.$$

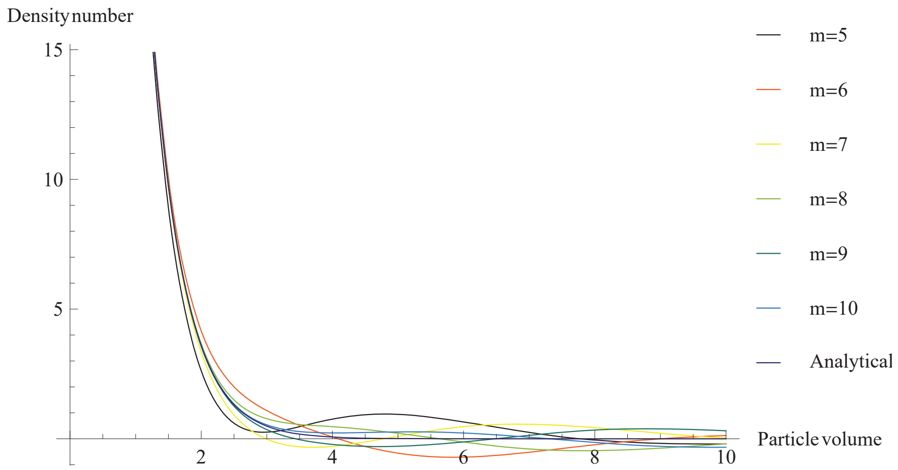

FIGURE 1. The effect of the truncation solution on the density number distribution for droplet breakage in batch system with the boundary condition to be equal to $n(v, t, 0) = te^{-v}$ at $z = 1$ and $t = 50$.

Generally, $f_m(t, v, z)$ is the solution of

$$f_m(t, v, z) = \int_0^z \left(-\frac{\partial f_{m-1}(t, v, z)}{\partial t} - v f_{m-1}(t, v, z) + \int_v^{+\infty} 2 f_{m-1}(t, v', z)\, dv' \right) dz. \tag{4.3}$$

Then we calculate the general term as:

$$f_m(t, v, z) = (-vz)^m \left(\frac{vt\Big(v(v - 2m) + (m - 1)m\Big) + mv\Big(v - 2(m - 1)\Big)}{v^3 e^v m!} \right.$$

$$+\frac{(m-2)(m-1)m}{v^3e^vm!}\Bigg).$$

So,

$$f(t,v,z)=\sum_{k=0}^{+\infty}(-vz)^m\left(\frac{vt\Big(v(v-2m)+(m-1)m\Big)+mv\Big(v-2(m-1)\Big)}{v^3e^vm!}\right.$$
$$\left.+\frac{(m-2)(m-1)m}{v^3e^vm!}\right),$$

which converges to

$$f(t,v,z)=\frac{(z+1)^2(t-z)}{e^{v(z+1)}}.$$

Case study II. PBE with diffusion term Consider the PBE

$$\begin{aligned}\frac{\partial f(t,v,z)}{\partial t}&=D_d\frac{\partial^2(f(t,v,z))}{\partial z^2}-\Gamma(v)f(t,v,z)\\&+\int_v^{+\infty}\beta(v,v')\Gamma(v')f(t,v',z)\,dv'.\end{aligned}\tag{4.4}$$

Subjected to the boundary conditions $f(t,v,0)=te^{-v}$ and $\frac{\partial f(t,v,0)}{\partial z}=e^{-v}$, and assuming that $D_d=1$. The equation (4.4) becomes

$$\frac{\partial^2(f(t,v,z))}{\partial z^2}=\frac{\partial f(t,v,z)}{\partial t}+vf(t,v,z)-\int_v^{+\infty}2f(t,v',z)\,dv'.\tag{4.5}$$

In this case $L_{zz}(.)=\frac{\partial^2}{\partial z^2}(.)$ and its inverse is $L_{zz}^{-1}(.)=\int_0^z\int_0^z(.)\,dz\,dz$. Applying the inverse operator to (4.5) gives

$$\begin{aligned}f(t,v,z)=f(t,v,0)+\frac{\partial f(t,v,0)}{\partial z}z+\int_0^z\int_0^z\Bigg(&\frac{\partial f(t,v,z)}{\partial t}+vf(t,v,z)\\&-\int_v^{+\infty}2f(t,v',z)\,dv'\Bigg)\,dz\,dz.\end{aligned}$$

The solution by the ADM is determined by the following Adomian recursion scheme:

$$f_0(t,v,z)=te^{-v}+e^{-v}z$$

$$f_{m+1}(t,v,z) = \int_0^z \int_0^z \left(\frac{\partial f_m(t,v,z)}{\partial t} + v f_m(t,v,z) - \int_v^{+\infty} 2 f_m(t,v',z)\, dv' \right) dz\, dz, m \geq 0.$$

From here we calculate the solution components

$$f_1(t,v,z) = z^2 \left(\frac{3+3t(-2+v)+(-2+v)z}{6e^v} \right),$$

$$f_m(t,v,z) = \frac{e^{-v} v^{m-3} z^{2m}}{4^m (1)_m \left(\frac{3}{2}\right)_m} \Bigg(\Big(v(v-2m)+(m-1)m \Big) \Big((2m+1)tv + vz \Big) + m(2m+1) \Big(v(v-2(m-1)) + (m-2)(m-1) \Big) \Bigg).$$

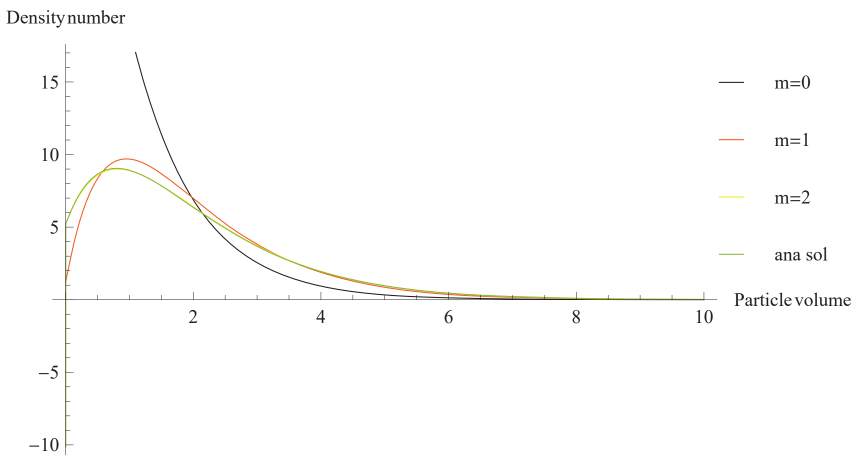

FIGURE 2. The effect of the truncation solution on the density number distribution for droplet breakage in batch system with the boundary condition to be equal to $f(t,v,0) = te^{-v}$, $\frac{\partial f(t,v,0)}{\partial z} = e^{-v}$ at $z = 1$ and $t = 50$.

So,

$$f(t,v,z) = \sum_{m=0}^{+\infty} \frac{e^{-v} v^{m-3} z^{2m}}{4^m (1)_m \left(\frac{3}{2}\right)_m} \Bigg(\Big(v(v-2m)+(m-1)m \Big) \Big((2m+1)tv + vz \Big) + m(2m+1) \Big(v(v-2(m-1)) + (m-2)(m-1) \Big) \Bigg)$$

$$f(t,v,z) = \frac{z^2\,{}_3F_4\left(2,2,2;1,1,1,\frac{5}{2};\frac{vz^2}{4}\right)}{3e^v v^2} + \frac{\cosh\left(\sqrt{v}z\right)}{8e^v v^3}\Bigg(2tv^2\left(4v+z^2\right) - vz\Bigg(4v(z+2)+5z+6\Bigg)+1\Bigg) + \frac{\sinh\left(\sqrt{v}z\right)}{8e^v v^{7/2}z}\Bigg(2vz\Bigg(v\Bigg(2v(-2tz + z+2)+z(-t+z+2)+4\Bigg)+z+3\Bigg)-1\Bigg)$$

4.1.2. *PBE in continuous systems.*

Case study III. PBE with convection term In this case, the boundary condition is $f(t,v,0) = te^{-v}$, the feed distribution is exponential and assuming that $U_d = 1$. We consider the following PBE:

$$\frac{\partial f(t,v,z)}{\partial t} + U_d \frac{\partial f(t,v,z)}{\partial z} = \frac{1}{\theta}\left(f^{feed}(v) - f(t,v,z)\right) - \Gamma(v) f(t,v,z) + \int_v^{+\infty} \beta(v,v')\Gamma(v') f(t,v',z)\,dv'.$$

Rewrite the above equation to the following form

$$\frac{\partial f(t,v,z)}{\partial z} = -\frac{\partial f(t,v,z)}{\partial t} + \frac{1}{\theta} f^{feed}(v) - \left(v + \frac{1}{\theta}\right) f(t,v,z) + \int_v^{+\infty} 2f(t,v',z)\,dv'. \tag{4.6}$$

Integrating the equation 4.6 with respect to z,

$$f(t,v,z) = f(t,v,0) + \frac{1}{\theta} f^{feed}(v) z + \int_0^z \left(\frac{-\partial f(t,v,z)}{\partial t} - \left(v+\frac{1}{\theta}\right) f(t,v,z) + \int_v^{+\infty} 2f(t,v',z)\,dv'\right) dz.$$

Then the Adomian recursion scheme is

$$f_0(t,v,z) = te^{-v} + \frac{ze^{-v}}{\theta},$$

$$f_{m+1}(t,v,z) = \int_0^z \left(\frac{-\partial f_m(t,v,z)}{\partial t} - (v+\frac{1}{\theta}) f_m(t,v,z) + \int_v^{+\infty} 2 f_m(t,v',z)\, dv' \right) dz, \quad m \geq 0.$$

The second term is

$$f_1(t,v,z) = z\left(\frac{2 + (-2+\frac{1}{\theta}+v)(2t+\frac{z}{\theta})}{e^v}\right).$$

The general term is

$$f_m(t,v,z) = \frac{(-1)^m(\frac{1}{\theta}+v)^{m-3} z^m}{(1)_{1+m} e^v}\Bigg((1+m)\Big(m(m-1)(m-2) + \frac{1}{\theta^3} t + \frac{1}{\theta^2}\Big(m + t(-2m+3v)\Big) + v\Big(m(-2(m-1)+v) + t\big(m(m-1) + (-2m+v)v\big)\Big) + \frac{1}{\theta}\Big(2m((1-m)+v) + t\big(m(m-1)+v(-4m+3v)\big)\Big)\Big) + \frac{1}{\theta}\left(\frac{1}{\theta}+v\right)\left(m(m-1) + (\frac{1}{\theta})^2 + \frac{2}{\theta}(-m+v) + (-2m+v)v\right) z\Bigg).$$

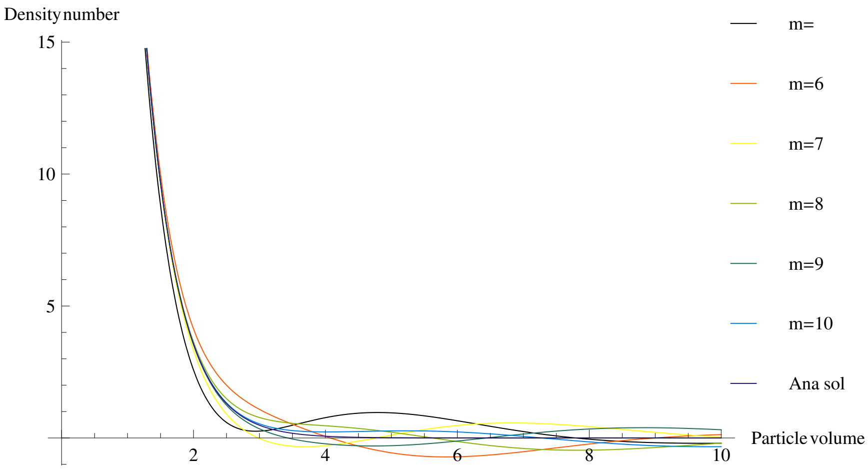

FIGURE 3. The effect of the truncation solution on the density number distribution for droplet breakage in continuous system with the boundary condition to be equal to $f(t,v,0) = te^{-v}$ with exponential feed distribution at $z = 1$ and $t = 50$.

Then

$$f(t,v,z)=\sum_{m=0}^{+\infty}\frac{(-1)^m(\frac{1}{\theta}+v)^{m-3}z^m}{(1)_{1+m}e^v}\Bigg((1+m)\Big(m(m-1)(m-2)+\frac{1}{\theta^3}t$$
$$+\frac{1}{\theta^2}\Big(m+t(-2m+3v)\Big)+v\Big(m(-2(m-1)+v)+t\big(m(m-1)$$
$$+(-2m+v)v\big)\Big)+\frac{1}{\theta}\Big(2m((1-m)+v)+t\Big(m(m-1)+v(-4m+3v)\Big)\Big)$$
$$+\frac{1}{\theta}\left(\frac{1}{\theta}+v\right)\left(m(m-1)+(\frac{1}{\theta})^2+\frac{2}{\theta}(-m+v)+(-2m+v)v\right)z\Bigg).$$

So,

$$f(t,v,z)=\frac{1}{(\frac{1}{\theta}+v)^3e^ve^{z(\frac{1}{\theta}+v)}}\Bigg(\frac{1}{\theta^3}\Big(e^{z(\frac{1}{\theta}+v)}+(z+1)^2(t-z-1)\Big)+\frac{1}{\theta^2}\Big(2(v+1)$$
$$e^{z(\frac{1}{\theta}+v)}-(z+1)\Big(2-v(z+1)(3t-3z-2)\Big)\Big)+\frac{1}{\theta}\Big(\Big(v(v+2)+2\Big)$$
$$e^{z(\frac{1}{\theta}+v)}-v(z+1)\Big(2-v(z+1)(3t-3z-1)\Big)-2\Big)$$
$$+v^3(z+1)^2(t-z)\Bigg).$$

4.2. **Nonlinear case: Pure Coalescence.** In PBE the machanism of coalescence term poses the greatest numerical difficulty occurred by the non-linearity of this phenomena. we will consider the solution of equation (2.1) in the presence of only two terms which are the convection and coalescence terms in batch flow system in order to show the great accuracy of this technique.

4.2.1. *PBE with convection term.* In this section, we present only one case study for pure coalescence in batch systems .

$$\begin{aligned}\frac{\partial f(t,v,z)}{\partial t}+U_d\frac{\partial(f(t,v,z))}{\partial z}&=\frac{1}{2}\int_0^v\omega(v-v',v')f(t,v-v',z)f(t,v',z)\,dv'\\&\quad-\int_0^{+\infty}\omega(v,v')f(t,v',z)f(t,v,z)\,dv'.\end{aligned}\tag{4.7}$$

Here we take $\omega(v,v')=1$, $U_d=1$ and the boundary condition $f(t,v,0)=te^{-v}$.

By considering $L_t = \frac{\partial}{\partial t}$ and $L_z = \frac{\partial}{\partial z}$, and operating both sides of equation 4.7 by L_z^{-1} (defined as $L_z^{-1}(.) = \int_0^z (.)\, dz$), then we obtain the canonical form

$$f(t,v,z) = f(t,v,0) + \int_0^z \left(-\frac{\partial f(t,v,z)}{\partial t} + \frac{1}{2}\int_0^v f(t,v-v',z)f(t,v',z)\,dv' - \int_0^{+\infty} f(t,v',z)f(t,v,z)\,dv' \right) dz. \tag{4.8}$$

And the nonlinear terms have the Adomian polynomial representation:

$$\begin{aligned} \frac{1}{2}\int_0^v f(t,v-v',z)f(t,v',z)\,dv' &= \frac{1}{2}\int_0^v \sum_{m=0}^{+\infty} A_m(v-v',v',t,x)\,dv' \\ &\quad - \int_0^{+\infty} f(t,v',z)f(t,v,z)\,dv' \\ &= -\int_0^{+\infty} \sum_{m=0}^{+\infty} B_m(v,v',t,x)\,dv'. \end{aligned} \tag{4.9}$$

The polynomials A_m and B_m are obtained by the definitional formula (3.6).

The solution by the ADM is calculated by the following Adomian recursion scheme:

$$\begin{aligned} f_0(t,v,z) &= te^{-v} \\ f_{m+1}(t,v,z) &= \int_0^z \left(\frac{\partial f_m(t,v,z)}{\partial t} + \frac{1}{2}\int_0^v A_m(t,v-v',v',z)\,dv' \right. \\ &\quad \left. - \int_0^{+\infty} B_m(t,v,v',z)\,dv' \right) dz, m \geq 0. \end{aligned}$$

The first few terms are

$$\begin{aligned} f_1(t,v,z) &= \left(\frac{-2-2t^2+t^2v}{e^v} \right) z, \\ f_2(t,v,z) &= t\left(\frac{-8(-2+v)+t^2(6-6v+v^2)}{8e^v} \right) z^2. \end{aligned}$$

By rearrangement of terms, we can obtain the general term (See the Appendix 6 for more details)

$$f_m^*(t,v,z) = \frac{4v^m z^m (t-z)^{m+1}}{m! e^v (z(t-z)+2)^{m+2}}. \tag{4.10}$$

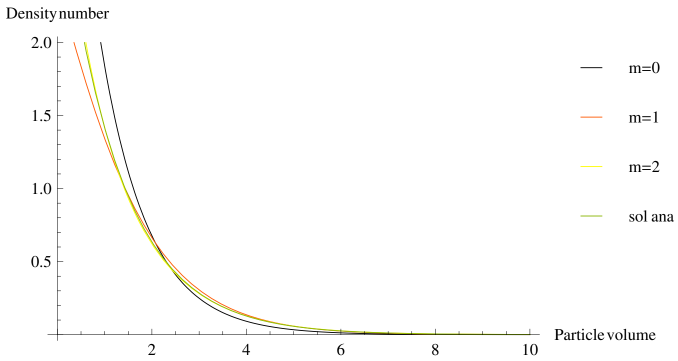

FIGURE 4. The effect of the truncation solution on the density number for droplet Coalescence with the boundary condition to be equal to $f(t, v, 0) = te^{-v}$ at $z = 0.1$ and $t = 5$.

Then,

$$f(t, v, z) = \sum_{m=0}^{+\infty} \frac{4v^m z^m (t-z)^{m+1}}{m! e^v (z(t-z)+2)^{m+2}}. \tag{4.11}$$

So, the exact solution is

$$f(t, v, z) = \frac{4(t-z)e^{\frac{vz(t-z)}{tz-z^2+2}}}{e^v (z(t-z)+2)^2}. \tag{4.12}$$

For nonlinear models, more efficient algorithms and programs in MATHEMATICA for fast generation of Adomian polynomials to high orders have been provided by Duan in [10–12].

The Figure 2 and 4 shows that from the second iteration the m term approximant solutions were nearly identical to the analytical solutions. A rapid convergence was observed in these figures which shows the efficiency of the method.

The Figure 1 and 3 represent the effect of the truncation on the density number for droplet breakage in batch and continuous flow particulate processes, respectively.

5. CONCLUSION

The decomposition method was employed successfully for solving droplet population balance equations in batch and continuous flow systems. It provides a closed form continuous approximation for the solutions unlike other methods as the mesh point techniques which provide the approximation at mesh points only.

Some problems still open until now. For instance, practical convergence of the Adomian decomposition series may be ensured even if the hypotheses of known method are not satisfied. which means that there still exist opportunities for further theoretical studies of convergence for more general situations. In addition, it is not always easy to take into account the boundary conditions for complex domains.

ABBREVIATIONS

ADM Adomian decomposition method.

PBE Population Balance Equation.

NOTATIONS

v	Volume.
t	Time.
z	Spatial location.
$f(t, v, z)$	The particle number density.
$f_m(t, v, z)$	Solution components.
θ	Residence time.
$\Gamma(v)$	The breakage frequency.
$\beta(v, v')\, dv'$	The number of daughter droplets having volume in the range from v to $v + dv$ formed upon breakage of the droplet of volume v'.
$\omega(v, v')$	The coalescence frequency between two droplets of volumes v and v'.
U_d	Droplet velocity.
D_d	Diffusion rate.

6. APPENDIX

(1) Here we summarize only all different cases symbols and functions defined as

- **Pochhammer Symbol:**
  Given $a \in \mathbb{R} \setminus \mathbb{Z}_{-}$ and $n \in \mathbb{N}$ the Pochhammer symbol is defined by
  $$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)},$$
  where Γ is the Gamma function which is given by
  $$\Gamma(a) = \int_0^{+\infty} e^{-t} t^{a-1}\, dt.$$
- **The generalized Hypergeometric function:** The generalized hypergeometric function, denoted ${}_pF_q(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; z)$ and defined by
  $${}_pF_q(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_k}{\prod_{j=1}^{q} (\gamma_j)_k} \frac{z^k}{k!}.$$

In the last example, by rearrangement of the terms according to the powers $t^i v^j z^m$, the series solution could be written as

$$\begin{aligned}
f(t, v, z) &= \sum_{k=0}^{+\infty} f_k(t, v, z) \\
&= \sum_{k=0}^{+\infty} f_k^*(t, v, z) \\
&= \sum_{j=0}^{+\infty} \sum_{i=0}^{+\infty} \sum_{m=0}^{+\infty} \left(a_{i,j,m} t^i v^j z^m\right) \\
&= \sum_{j=0}^{+\infty} v^j \left(\sum_{i=0}^{+\infty} t^i \left(\sum_{m=0}^{+\infty} \left(a_{i,j,m} z^m\right) \right) \right) \\
&= \sum_{j=0}^{+\infty} v^j \left(\sum_{i=0}^{+\infty} t^i b_{i,j}(z) \right)
\end{aligned}$$

Examples of $a_{i,j,m}$

- for $i = 0$

- for $j = 0$

$$a_{0,0,0} = 0$$
$$a_{0,0,1} = -1$$
$$a_{0,0,2} = 0$$
$$\vdots$$

- for $j = 1$

$$a_{0,1,0} = 0$$
$$a_{0,1,1} = 0$$
$$a_{0,1,2} = 0$$
$$a_{0,1,3} = \frac{1}{2}$$
$$\vdots$$

- for $i = 1$
  - for $j = 0$

$$a_{1,0,0} = 1$$
$$a_{1,0,1} = 0$$
$$a_{1,0,2} = 2$$
$$\vdots$$

  - for $j = 1$

$$a_{1,1,0} = 0$$
$$a_{1,1,1} = 0$$
$$a_{1,1,2} = -1$$
$$a_{1,1,3} = 0$$
$$a_{1,1,4} = \frac{-9}{4}$$
$$\vdots$$

Examples of $b_{i,j}(z)$

- for $i = 0$

$$b_{0,0}(z) = \frac{-4z}{e^v(-2+z^2)^2}$$
$$b_{0,1}(z) = \frac{-4vz^3}{e^v(-2+z^2)^3}$$
$$b_{0,2}(z) = \frac{-2v^2z^5}{e^v(-2+z^2)^4}$$
$$\vdots$$

- for $i = 1$

$$b_{1,0}(z) = \frac{-4t(2+z^2)}{e^v(-2+z^2)^3}$$
$$b_{1,1}(z) = \frac{-4vtz^2(4+z^2)}{e^v(-2+z^2)^4}$$
$$b_{1,2}(z) = \frac{-2v^2tz^4(6+z^2)}{e^v(-2+z^2)^5}$$
$$\vdots$$

Therefore

$$f(t,v,z) = \sum_{j=0}^{+\infty} v^j \left(\sum_{i=0}^{+\infty} t^i b_{i,j}(z) \right) = \sum_{j=0}^{+\infty} v^j c_j(t,z)$$
$$= \sum_{j=0}^{+\infty} v^j \frac{4z^j(t-z)^{j+1}}{j!e^v(z(t-z)+2)^{j+2}} = \frac{4(t-z)e^{\frac{vz(t-z)}{tz-z^2+2}}}{e^v(z(t-z)+2)^2}$$

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LABORATORY OF APPLIED MATHEMATICS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BISKRA, PB 7000, BISKRA, ALGERIA.

Email address: imane.achour@univ-biskra.dz

LABORATORY OF APPLIED MATHEMATICS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BISKRA, PB 7000, BISKRA, ALGERIA.

Email address: abdelghani.bellagoun@univ-biskra.dz