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SEMI RECURSIVE ESTIMATION OF CONDITIONAL CUMULATIVE DISTRIBUTION FUNCTION FOR FUNCTIONAL DATA UNDER MIXING CONDITION

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ABSTRACT. This article studies the problem of nonparametric estimation of the conditional model of a scalar response variable Y given a functional random variable X. Our estimate is based on semi recursive approach. The asymptotic properties of the proposed estimators are established Under Mixing Conditions.

1. INTRODUCTION

Functional statistics are dedicated to the study of models involving data of a functional nature, this field of research has experienced an explosion of work under the impetus of the work of Ferraty and Vieu [12] which studies the almost complete convergence of the regression in fractal dimension.

One of the innovations existing in the functional framework since the last decade is the introduction of the conditional allocation function estimator by Laksaci *et al* [12]. Ezzahrioui and Ould said [8] study the asymptotic normality of the conditional distribution function in both cases ($i.i.d$ and α-mixing) using the Laksaci

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estimator *et al* [12]. However, Laksaci and Maref [17] establish the almost complete convergence speed for spatially dependent explanatory variable. Results of uniform convergence of the estimator were established by Laksaci and Maref [17]. For robustness, we refer to Azzedine *and al* [2], and Attouch *and al* [1]. The originality of the work of Tadj *and al* [21] focuses on the study of uniform convergence on the two real and functional arguments of the estimator of several conditional models including the accuracy of the convergence speed. A pioneering work that was highlighted by Laksaci *et al* [12] developing a robust alternative method of conditional quantile estimator. Laksaci and Hachemi [18] study the conditional distribution function by the local linear estimator. The generalization of this latter result to spatial data was published by Rahmani it et al [22]. Thus Demongeot *and al* [22] to model the conditional distribution function for a local linear estimator specify the almost complete convergence speed. Laksaci and Mechab [18] provided a new basis for estimating the conditional chance function.

Historically the recursive kernel estimation method was introduced by Wolverton and Wagner [26], for recent results we refer to kharedani *et al* [18] and Amiri *and al* [4], and Bouadjemi [7]. Recently two main approaches have been developed in the paper of Benziad it and al [22], the first is based on the recursive double-kernels estimate of the conditional distribution function and the second is obtained using the robust approach, This work focuses on functional ergodic data. Notably Ardjoun and Laksaci [5] contribute to this dynamic by their study of conditional mode through the recursive approach.

Our contribution is to introduce a semirecursive approach to the estimation of the conditional distribution function for functional explanatory variable i.e. of random variables with values in a space of infinite dimension.

The outline of this paper is as follows: We present our semi-recursive estimation in Section 2. The asymptotic normality of the proposed estimator is given in Section 3. The proofs of the auxiliary results are relegated to the Appendix.

2. Model and Notations

We beging by recalling the definition of the strong mixing proprety. For this we introduced the following notations let $\Im_i^k(Z)$ denote the σ-algebra generated by $\{Z_i, i \leqslant j \leqslant k\}$.

Definition 2.1. *Let* $\{Z_i, i = 1, 2, \ldots\}$ *be sticly stationary sequence of random variables. Given a positive integer n set*

$$\alpha(n) = Sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \Im_1^k(Z) \ and \ B \in \Im_{k+n}^{\infty}(Z),\ k \in \mathbb{N}^*\}.$$

The sequence is said the α *mixing (strong mixing) if the mixing coefficient*

$$\alpha(n) \to 0 \ \ as \ \ n \to \infty. \tag{2.1}$$

In the remainder of the paper, we suppose that $(X_i, Y_i)_{i=1,2,\ldots}$ is strongly mixing, the process valued in $\Im \times \mathbb{R}$ where $\Im$ is semi metric vector space. $d(.,.)$ denoting the semi metric. Assume that exists a version of the conditional cummultative distribution function of y given $X = x \ / x$ is fixed point in $\Im$:

$$\forall x \in \mathcal{F} \text{ and } \forall y \in \mathbb{R} \qquad F^x(y) = P(Y \le y | X = x).$$

The kernel estimate of the distribution function F^x denoted $\widehat{F}^x$, is defined by

$$\forall x \in \mathcal{F} \text{ and } \forall y \in \mathbb{R} \qquad \widehat{F}^x(y) = \frac{\sum_{i=1}^n K(h_K^{-1} d(x, X_i)) H(h_H^{-1}(y - Y_i))}{\sum_{i=1}^n K(h_K^{-1} d(x, X_i))},$$

where K is a kernel, H is a distribution function and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) is a sequence of positive real numbers. Note that this last estimator has been used by Laksaci *& al.* ([12] and [10])
A semi recursive version of the previous kernel estimator is defined by

$$\forall x \in \mathcal{F} \text{ and } \forall y \in R \qquad \widehat{F}^x(y) = \frac{\sum_{i=1}^n E[K_i]^{-1} K(a_i^{-1} d(x, X_i)) H(b_i^{-1}(y - Y_i))}{\sum_{i=1}^n E[K_i]^{-1} K(a_i^{-1} d(x, X_i))},$$

The function H is a strictly increasing distribution function and b_i (resp.a_i) is a sequence of positive real numbers $lim_{n\to\infty} b_n = 0$.

All along the paper, when no confusion is possible, we will denote by C and C' some strictly positive generic constants. Consider now the following notations, for any $x \in \Im$, and for $i = 1, 2, \ldots$:

$$K_i = K(a_i^{-1} d(x, X_i)), \qquad H_i = H(b_i^{-1}(y - Y_i)),$$

$$\widehat{F}_N^x(y) = \frac{1}{n} \sum_{i=1}^n E[K_i]^{-1} K_i H_i \quad \text{and} \quad \widehat{F}_D(x) = \frac{1}{n} \sum_{i=1}^n E[K_i]^{-1} K_i.$$

3. The Main Result

We introduce now some assumptions that are needed to state our results:

(H1) $\forall r > 0, \mathbb{P}(X \in B(x,r)) =: \phi_x(r) > 0$, where $B(x,r) = \{x' \in \mathcal{F}/d(x,x') < r\}$.

(H2) For all $y \in \mathbb{R}$ $\forall (x_1, x_2) \in N_x^2$,

$$|F^{x_1}(t_1) - F^{x_2}(t_2)| \leq C\left(d(x_1,x_2)^{\beta_1} + |t_1 - t_2|^{\beta_2}\right),$$

with $\quad C > 0, \beta_1 > 0,\ \beta_2 > 0$ and N_x is a fixed neighborhood of x.

(H3) The bandwidths (a_i, b_i) satisfy:

i. $\forall t \in [0,1]\ \ \lim_{n\to\infty} \frac{\phi_x(ta_n)}{\phi_x(a_n)} = \beta_x(t)$,

ii. $n\phi_n(x) \to \infty$ and $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sqrt{\phi_n(x)}\left(a_i^{\beta_1} + b_i^{\beta_2}\right) \to 0$ as $n \to \infty$

iii. $\beta_{n,r} : \dfrac{1}{n}\sum_{i=1}^{n}\left(\dfrac{\phi_x(a_n)}{\phi_x(a_i)}\right)^r = \beta_{ir} < \infty$ as $n \to \infty$

(H4) H has even bounded derivative verifies

$$\int_{\mathbb{R}} |t|^{\beta_2}\, H'(t)dt < \infty.$$

(H5) α-mixing sequence whose coefficients of mixture satisfy:

$$\exists\, a > 0,\ \exists\, c > 0\ :\ \forall\, n \in\ \mathbb{N},\ \alpha(n) \leqslant cn^{-a}.$$

Theorem 3.1. *(Normality asymptotic) Under assumptions (H1)-(H6),then for any* $(x,y) \in \Im \times \mathbb{R}$*, we have*

$$\left(\frac{n\phi_n(x)}{\sigma^2}\right)^{1/2}\left(\widehat{F}^x(y) - F^x(y)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \quad as \quad n \to \infty,$$

where $\sigma^2 = \left(\dfrac{F^x(y)(1-F^x(y))\beta_1\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds\right)}{\left(K(1) - \int_0^1 (K(s))'\beta_x(s)ds\right)^2}\right)$,

$$\mathcal{A}(x,y) = \left\{(x,y) \text{ such that } \frac{F^x(y)(1-F^x(y))\beta_1\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds\right)}{\left(K(1) - \int_0^1 (K(s))'\beta_x(s)ds\right)^2} \neq 0\right\}$$

and $\xrightarrow{\mathcal{D}}$ *means the convergence in distribution.*

Proof. We consider the decomposition

$$\widehat{F}^x(y) - F^x(y) = \frac{1}{\widehat{F}_D^x}\Big(\Big[\Big[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x\Big] - E\Big[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x\Big]\Big]$$

$$+E\Big[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x\Big]\Big) \tag{3.1}$$

The structure of the proof is based on the decomposition (3). Note first that the result to state asymptotic normality will be obtained of first term of the right hand side of numerator suitably normalized is asymptotically normally distributed, the second term is negligible, and the denominator converge in probability. Therefore, Theorem 3.1 is a consequence of the following results. □

Lemma 3.2. *Under the hypotheses of Theorem (3.1), we have*

$$E\left[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x\right] = O\left(\frac{1}{n}\sum_{i=1}^{n}\left(a_i^{\beta_1} + b_i^{\beta_2}\right)\right).$$

Lemma 3.3. *Under the hypotheses of Theorem (3.1), we have*

$$var\left(\left[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x\right]\right) = \frac{F^x(y)(1-F^x(y))\beta_1\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds\right)}{n\phi_x(a_n)\left(K(1) - \int_0^1 (K(s))'\beta_x(s)ds\right)^2} + o\left(\sum_{i=1}^{n}\frac{1}{\phi_x(a_i)}\right).$$

Lemma 3.4. *Under the hypotheses of Theorem (3.1), we have*

$$(n\phi_n(x))^{1/2}\left(\left[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x\right] - E\left[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x\right]\right) \to \mathcal{N}(0,\sigma^2).$$

Lemma 3.5. *Under the hypotheses of Theorem (3.1), we have*

$$\widehat{F}_D^x \to 1 \qquad \textit{in probability}.$$

4. Appendix

Proof. (Proof of Lemma 3.2)

$$E[\tilde{F}_N^x(y)] = \frac{1}{n}\sum_{i=1}^{n} E\left[K_i E[K_i]^{-1} E[H_i|X_i]\right] \text{ with } E\left[H_i|X_i\right] = \int_R H'(t) F^{X_i}(y - b_i t)dt.$$

$$\begin{aligned}
|E[H_i|X_i] - F^x(y)| &\leq \int_R H'(t)\left|F^{X_i}(y-b_it) - F^x(y)\right| \\
|E[H_i|X_i] - F^x(y)| &\leq C\left(a_i^{\beta_1} + b_i^{\beta_2}\right) \\
\left|E\left[\tilde{F}_N^x(y)\right] - F^x(y)\right| &\leq C\frac{1}{n}\sum_{i=1}^{n}\left(a_i^{\beta_1} + b_i^{\beta_2}\right) \\
E\left[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x\right] &= O\left(\sum_{i=1}^{n}\phi_x(a_n)\left(a_i^{\beta_1} + b_i^{\beta_2}\right)\right).
\end{aligned}$$

□

Proof. (Proof of Lemma 3.3)

$$\begin{aligned}
& var\left(\left[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x\right] - E\left[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x\right]\right) \\
&= var[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x] \\
&= var\left[\widehat{F}_N^x(y)\right] + F^x(y)^2 var[\widehat{F}_D^x] \\
&\quad - 2F^x(y)cov\left[\widehat{F}_N^x(y), \widehat{F}_D^x\right]
\end{aligned}$$

To calculate

$$\begin{aligned}
var\left[\widehat{F}_N^x(y)\right] &= var\left[\ \tfrac{1}{n}\textstyle\sum_{i=1}^{n} E[K_i]^{-1}K_iH_i\right] \\
&= \tfrac{1}{n^2}\textstyle\sum_{i=1}^{n} var\left[\ E[K_i]^{-1}K_iH_i\right] \\
&\quad + \tfrac{2}{n^2}\textstyle\sum_{i\neq j}^{n} cov\left[E[K_i]^{-1}K_iH_i, E[K_j]^{-1}K_jH_j\right] \\
&= \tfrac{1}{n^2}\textstyle\sum_{i=1}^{n} E[K_i]^{-2} var\left[K_iH_i\right] \\
&\quad + \tfrac{2}{n^2}\textstyle\sum_{i\neq j}^{n} E[K_i]^{-1}E[K_j]^{-1} cov\left[K_iH_i, K_jH_j\right] \\
&= I_1 + I_2 \\
Var\left[K_iH_i\right] &= E\left[K_i^2H_i^2\right] - \left(E\left[K_iH_i\right]\right)^2 \\
Var\left[K_iH_i\right] &= F^x(y)\left(\phi_x(a_i)K^2(1) - \textstyle\int_0^1 (K^2(s))'\phi_x(a_is)ds\right) + o\left(\phi_x(a_i)\right)
\end{aligned}$$

(4.1)
$$E\left[K_i\right] = \phi_x(a_i)K(1) - \int_0^1 (K(s))'\phi_x(a_i s)ds + o(1)$$
$$E\left[K_i^j H_i^j\right] = \left(F^x(y)\int (H^j(t))'dt + O\left(a_i^{\beta_1} + b_i^{\beta_2}\right)\right) E\left[K_i^j\right], \quad j = 1, 2,$$

(4.2)
$$E\left[K_i^j\right] = \phi_x(a_i)K^j(1) - \int_0^1 (K^j(s))'\phi_x(a_i s)ds + o(1), \quad j = 1, 2.$$

For I_1:

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^n E[K_i]^{-2} var\Big[K_i H_i\Big] \\
= &\frac{1}{n^2}\sum_{i=1}^n \frac{F^x(y)\left(\phi_x(a_i)K^2(1) - \int_0^1 (K^2(s))'\phi_x(a_i s)ds\right) + o\left(\phi_x(a_i)\right)}{\left(\phi_x(a_i)K(1) - \int_0^1 (K(s))'\phi_x(a_i s)ds + o(1)\right)^2} \\
= &\frac{1}{n^2\phi_x(a_n)}\sum_{i=1}^n \frac{\phi_x(a_n)}{\phi_x(a_i)} \frac{F^x(y)\left(K^2(1) - \int_0^1 (K^2(s))'\beta_i(s)ds\right) + o\left(\phi_x(a_i)\right)}{\left(K(1) - \int_0^1 (K(s))'\beta_i(s)ds + o(1)\right)^2}
\end{aligned}$$

$$\begin{aligned}
var\Big[\widehat{F}_N^x(y)\Big] &= \frac{F^x(y)\beta_1\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds\right)}{n\phi_x(a_n)\left(K(1) - \int_0^1 (K(s))'\beta_x(s)ds)\right)^2} + o\left(\sum_{i=1}^n \frac{1}{\phi_x(a_i)}\right) \\
var\Big[\widehat{F}_D^x\Big] &= var\Big[\tfrac{1}{n}\sum_{i=1}^n E[K_i]^{-1}K_i\Big] \\
&= \tfrac{1}{n^2}\sum_{i=1}^n var\Big[E[K_i]^{-1}K_i\Big] + \tfrac{2}{n^2}\sum_{i\neq j}^n cov\Big[E[K_i]^{-1}K_i, E[K_j]^{-1}K_j\Big] \\
&= \tfrac{1}{n^2}\sum_{i=1}^n E[K_i]^{-2} var\Big[K_i\Big] + \tfrac{2}{n^2}\sum_{i\neq j}^n E[K_i]^{-1}E[K_j]^{-1} cov\Big[K_i, K_j\Big] \\
Var\left[K_i\right] &= \left(\phi_x(a_i)K^2(1) - \int_0^1 (K^2(s))'\phi_x(a_i s)ds\right) + o\left(\phi_x(a_i)\right).
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^n E[K_i]^{-2} var\Big[K_i\Big] \\
= &\frac{1}{n^2}\sum_{i=1}^n \frac{\left(\phi_x(a_i)K^2(1) - \int_0^1 (K^2(s))'\phi_x(a_i s)ds\right) + o\left(\phi_x(a_i)\right).}{\left(\phi_x(a_i)K(1) - \int_0^1 (K(s))'\phi_x(a_i s)ds + o(1)\right)^2} \\
= &\frac{1}{n^2\phi_x(a_n)}\sum_{i=1}^n \frac{\phi_x(a_n)}{\phi_x(a_i)} \frac{\left(K^2(1) - \int_0^1 (K^2(s))'\beta_i(s)ds\right) + o\left(\phi_x(a_i)\right).}{\left(K(1) - \int_0^1 (K(s))'\beta_i(s)ds + o(1/\phi_x(a_i))\right)^2}
\end{aligned}$$

$$\begin{aligned}
var\left[\widehat{F}_D^x\right] &= \frac{\beta_1}{n\phi_x(a_n)} \frac{\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds\right)}{\left(K(1) - \int_0^1 (K(s))'\beta_x(s)ds\right)^2} + o\left(\sum_{i=1}^n \frac{1}{\phi_x(a_i)}\right) \\
cov\left[\widehat{F}_N^x(y), \widehat{F}_D^x\right] &= E[\widehat{F}_N^x(y)\widehat{F}_D^x] - E[\widehat{F}_N^x(y)]E[\widehat{F}_D^x] \\
&= \tfrac{1}{n^2}\sum_{i=1}^n E[K_i]^{-2}E[K_i^2 H_i] - \tfrac{1}{n^2}\sum_{i=1}^n E[K_i]^{-2}E[K_i H_i]E[K_i] \\
&= \tfrac{1}{n^2}\sum_{i=1}^n \frac{\left(F^x(y)\int (H(t))'dt + O\left(a_i^{\beta_1} + b_i^{\beta_2}\right)\right) E\left[K_i^2\right]}{\left(\phi_x(a_i)K(1) - \int_0^1 (K(s))'\phi_x(a_i s)ds + o(1)\right)^2} \\
&- \tfrac{1}{n^2}\sum_{i=1}^n \frac{\left(F^x(y)\int (H(t))'dt + O\left(a_i^{\beta_1} + b_i^{\beta_2}\right)\right) E\left[K_i\right]}{\phi_x(a_i)K(1) - \int_0^1 (K(s))'\phi_x(a_i s)ds + o(1)} \\
&= I - II
\end{aligned}$$

So,

$$\begin{aligned}
II &= O\left(\tfrac{1}{n}\sum_{i=1}^n \left(a_i^{\beta_1} + b_i^{\beta_2}\right)\right) \\
I &= \tfrac{1}{n^2}\sum_{i=1}^n \frac{(F^x(y))\left(\phi_x(a_i)K^2(1) - \int_0^1 (K^2(s))'\phi_x(a_i s)ds\right) + o(\phi_x(a_i))}{\left(\phi_x(a_i)K(1) - \int_0^1 (K(s))'\phi_x(a_i s)ds + o(1)\right)^2} \\
&= \frac{1}{n^2\phi_x(a_n)}\sum_{i=1}^n \frac{\phi_x(a_n)}{\phi_x(a_i)} \frac{F^x(y)\left(K^2(1) - \int_0^1 (K^2(s))'\beta_i(s)ds\right) + o\left(\phi_x(a_i)\right)}{\left(K(1) - \int_0^1 (K(s))'\beta_i(s)ds + o(1)\right)^2}
\end{aligned}$$

$$cov\left[\widehat{F}_N^x(y), \widehat{F}_D^x\right] = \frac{F^x(y)\beta_1}{n\phi_x(a_n)} \frac{\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds\right)}{\left(K(1) - \int_0^1 (K(s))'\beta_x(s)ds\right)^2} + o\left(\sum_{i=1}^n \frac{1}{\phi_x(a_i)}\right)$$

So,

$$\begin{aligned}
var[\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x] = \frac{(1 - F^x(y))\beta_1}{n\phi_x(a_n)} \frac{\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds\right)}{\left(K(1) - \int_0^1 (K(s))'\beta_x(s)ds\right)^2} \\
+ o\left(\sum_{i=1}^n \frac{1}{\phi_x(a_i)}\right).
\end{aligned}$$

□

Proof. (Proof of Lemma 3.4) We use the technique of masry [19] for proof of lemma. For beguin we need further notations:

$$\begin{aligned} Q_n &:= \tfrac{1}{n^2}\textstyle\sum_{i=1}^n E[K_i]^{-1}K_iH_i - \tfrac{F^x(y)}{n}\sum_{i=1}^n E[K_i]^{-1}E[K_iH_i] \\ &- E\left[\tfrac{1}{n}\textstyle\sum_{i=1}^n E[K_i]^{-1}K_iH_i - \tfrac{F^x(y)}{n}\sum_{i=1}^n E[K_i]^{-1}E[K_iH_i]\right] \\ &= \tfrac{1}{n}\textstyle\sum_{i=1}^n [E[K_i]^{-1}K_i[H_i - F^x(y)] - E[E[K_i]^{-1}K_i[H_i - F^x(y)]]] \\ &= \tfrac{1}{n}\textstyle\sum_{i=1}^n Z_i(x,y). \end{aligned}$$

To establish the asymptotic normality of Q_n. We normalize $Z_i(x,y)$ by: $\tilde{Z}_i = \sqrt{\phi(a_n)}Z_i$, $S_n = \sum_{i=1}^n \tilde{Z}_i$. So, that $Var(\tilde{Z}_i) = \phi(a_n)varZ_i \quad \longrightarrow \sigma^2$ as $n \to \infty$. Now we can write:

$$\begin{aligned} \sqrt{n\phi(a_n)}Q_n &= \tfrac{1}{n}\sqrt{n\phi(a_n)}\textstyle\sum_{i=1}^n Z_i(x,y) \\ &= \tfrac{1}{\sqrt{n}}\textstyle\sum_{i=1}^n \sqrt{\phi(a_n)}Z_i(x,y) \\ &= \tfrac{1}{\sqrt{n}}\textstyle\sum_{i=1}^n \tilde{Z}_i \\ &= \tfrac{1}{\sqrt{n}}S_n. \end{aligned}$$

We need show that:

$$\frac{1}{\sqrt{n}}S_n \xrightarrow{\mathcal{D}} N(0,\sigma^2).$$

The proof of asymptotic normality for S_n we employ bernstein's "big block-small block" procedure use in Masry [19]. partition the set $\{1,\ldots,n\}$ into $2k+1$ subsets with large blocks of size $u = u_n$ and small blocks of size $v = v_n$ and set $k = k_n = \left[\dfrac{n}{v_n + u_n}\right]$.

Let $\{v_n\}$ be a sequence of positive integers satisfying $v_n \to \infty$ such that $v_n = o(\sqrt{n\phi(a_n)})$ *and* $\sqrt{\dfrac{n}{\phi(a_n)}}\alpha(v_n) \to 0$ as $n \to \infty$ implies that there exists a sequence of positive integers $\{q_n\}$, $q_n \to \infty$ such that:

$$q_nv_n = o(\sqrt{n\phi(a_n)}) \;\; q_n\sqrt{\frac{n}{\phi(a_n)}}\alpha(v_n) \to 0 \;\; as \;\; n \to \infty. \tag{4.3}$$

Now define the large block size as $u_n = \left[\dfrac{\sqrt{n\phi(a_n)}}{q_n}\right]$. When $n \to \infty$ we have:

$$\frac{v_n}{u_n} \to 0,\; \frac{u_n}{n} \to 0,\; \frac{n}{u_n}\alpha(v_n) \to 0. \tag{4.4}$$

Let δ_j Ψ_j ζ_j be define as follows:

(4.5) $$\delta_j := \sum_{i=j(u+v)+1}^{j(u+v)+u} \tilde{Z}_i \; 0 \leqslant j \leqslant k-1,$$

(4.6) $$\Psi_j := \sum_{i=j(u+v)+u+1}^{(j+1)(u+v)} \tilde{Z}_i \quad 0 \leqslant j \leqslant k-1$$

and

(4.7) $$\zeta_k := \sum_{i=k(u+v)+1}^{n} \tilde{Z}_i.$$

Then S_n can be writtes as:

$$S_n = \sum_{j=0}^{k-1} \delta_j + \sum_{j=0}^{k-1} \Psi_j + \zeta_k = S^{(1)} + S^{(2)} + S^{(3)}.$$

The technique consists to show that:

(4.8) $$\frac{1}{n}\left[E[S^{(2)}]^2\right] \to 0, \; \frac{1}{n}\left[E[S^{(3)}]^2\right] \to 0,$$

(4.9) $$|E[exp(itn^{-1/2}S^{(1)}] - \prod_{j=0}^{k-1} E[exp(itn^{-1/2}\delta_j]| \to 0,$$

(4.10) $$\frac{1}{n}\left[S^{(1)}\right] \to \sigma^2,$$

(4.11) $$\frac{1}{n}\sum_{j=0}^{k-1} E[\delta_j^2 I_{|\delta_j| > \epsilon\sqrt{n\sigma^2}}] \to 0.$$

For $\epsilon > 0$ equation (4.8) show that $S^{(2)}$ and $S^{(3)}$ are asymptotically negligible, (4.9) implies the summands $\{\delta_j\}$ in $S^{(1)}$ are asymptotically independent, and (4.10)-(4.11) are the standar Linderberg-Feller conditions for asymptotic normality of $S^{(1)}$ under independence. Since $E[Z_i] = 0$, we have $E[\delta_j] = E[\Psi_j] = E[\zeta_k] = 0$.

We begin by verified (4.8).

$$\begin{aligned} \frac{1}{n}\left[E[S^{(2)}]^2\right] &= \frac{1}{n}\left[E[\textstyle\sum_{j=0}^{k-1}\Psi_j]^2\right] \\ &= \frac{1}{n}\left[var[\textstyle\sum_{j=0}^{k-1}\Psi_j]\right] \\ &= \frac{1}{n}\textstyle\sum_{j=0}^{k-1}\left[var[\Psi_j]\right] + \frac{2}{n}\sum_{|i-j|>0}\left[cov[\Psi_i,\Psi_j]\right] \\ &= A_1 + A_2 \end{aligned} \tag{4.12}$$

Now, from the Davydov's inequality we get:

$$\begin{aligned} &\frac{1}{n}\textstyle\sum_{j=0}^{k-1}\left[var[\Psi_j]\right] \\ &= \frac{1}{n}\left[var[\textstyle\sum_{i=j(u+v)+u+1}^{(j+1)(u+v)}\tilde{Z}_i]\right] \\ &= \frac{1}{n}\textstyle\sum_{i=0}^{v}\left[var[\tilde{Z}_i]\right] + \frac{1}{n}\sum_{i=0}^{v}\sum_{j=0}^{v}\left[cov[\tilde{Z}_i,\tilde{Z}_j]\right] \\ &\leqslant \frac{1}{n}\textstyle\sum_{i=0}^{v}\left[var[\tilde{Z}_i]\right] + \frac{4}{n}\sum_{i=0}^{v}\sum_{j=0}^{v}\alpha(i-j)\left[[E[\tilde{Z}_i]^2]^{1/2}[E[\tilde{Z}_j]^2]^{1/2}\right]. \end{aligned} \tag{4.13}$$

Here

$$\frac{1}{n}\sum_{i=0}^{v}\left[var[\tilde{Z}_i]\right] = \frac{\phi(a_n)}{n}\sum_{i=0}^{v}\left[var[Z_i]\right],$$

so that a constant C_x exists independent of i, such that

$$\frac{1}{n}\sum_{i=0}^{v}\left[var[\tilde{Z}_i]\right] \leqslant \frac{C_x}{n} \quad \text{for all } i = 1, 2, \ldots.$$

Hence, by (4.13),

$$\begin{aligned} \frac{1}{n}\left[var[\Psi_j]\right] &\leqslant \frac{v}{n}C_x + 4\textstyle\sum_{i=0}^{v}\sum_{j=0}^{v}\alpha(i-j) \\ &\leqslant vC_x + 4\textstyle\sum_{i=0}^{v}(v-i)\alpha(i) \\ &\leqslant C''\frac{v}{n}. \end{aligned}$$

Thus,

$$A_1 \leqslant C''\frac{kv}{n}.$$

Next,

$$A_2 \leqslant \frac{2}{n}\sum_{i=0}^{k-1}\sum_{i=0}^{k-1}\sum_{i=j(u+v)+u+1}^{(j+1)(u+v)}\sum_{i=j(u+v)+u+1}^{(j+1)(u+v)} cov[\tilde{Z}_i,\tilde{Z}_j] \leqslant C_3\sum_{i=0}^{v}\alpha(i).$$

Therefore

$$\frac{1}{n}\left[E[S^{(2)}]^2\right] \leqslant C_4\left[\frac{kv}{n}+\sum_{i=0}^{v}\alpha(i)\right] \to 0 \text{ as } n\to\infty.$$

By similar arguments for [4.12], we find

$$\frac{1}{n}\left[E[S^{(3)}]^2\right] \leqslant C_5\frac{u}{n}\left[1+\frac{v}{u}\right] \to 0 \;\; as \;\; n\to\infty.$$

This yields the proof of (4.8).

Property (4.9) is an sequence of the following lemma of Volkonsky and Rozanov (see the lemma (5) in Masry [19]). Using this lemma, we have

$$|E[exp(itn^{-1/2}S^{(1)}] - \prod_{j=0}^{k-1} E[exp(itn^{-1/2}\delta_j]| \leqslant 16K_n(\alpha(v_n)) \backsim 16\frac{n}{u_n}(\alpha(v_n)),$$

which tends to zero by (4.4).

Now we establish 4.10:

$$\begin{aligned} & \frac{1}{n}\sum_{j=0}^{k-1} E[\delta_j]^2 \\ = & \frac{1}{n}\sum_{j=0}^{k-1} var[\delta_j] \\ = & \frac{1}{n}\sum_{j=0}^{k-1}\sum_{i=j(u+v)+1}^{j(u+v)+u} var[\tilde{Z}_i] + \frac{1}{n}\sum_{j=0}^{k-1}{}_{i\neq j}\sum_{i=j(u+v)+1}^{j(u+v)+u} cov[\tilde{Z}_i,\tilde{Z}_j] \\ = & \frac{u_n+v_n}{n}\sigma^2 + o(u_n). \end{aligned}$$

The definition of u_n and v_n, we get the result which completes the proof of 4.10. By the hypothesis on $H(.)$ and $F(. \mid x)$, we have $\mid H(y) - F(y\mid x)\mid \leqslant 2$, it follows that

$$\frac{1}{n}\sum_{i=1}^{n} E\left[(\Lambda_i^2)\mathbb{I}_{\{|\Lambda_i|>\epsilon\sigma_{S_n}\}}\right] \leq C\sum_{i=1}^{n} P\left[|\Lambda_i| > \epsilon\sigma_{S_n}\right].$$

On the other hand, we have

$$\frac{|\Lambda_i|}{\sigma_{S_n}} \leq \frac{C}{(n\varphi_n(x))} \to 0.$$

So, for all ϵ, and if n us biggish, then $P\left[|\Lambda_i| > \epsilon\sigma_{S_n}\right]$=0 which implies that $\frac{1}{n}\sum_{i=1}^{n} E\Big[\Lambda_i^2\mathbb{I}_{\{|\Lambda_i|>\epsilon\sigma_{S_n}\}}\Big] = 0$ for n biggish. We finish the proof. □

Proof. (Proof of Lemma 3.5) So, by using the same arguments as those used by Bouadjemi [7] we get

$$E\left[(\widehat{F}_D^x - 1)\right] \to 0$$
$$Var\left[\widehat{F}_D^x\right] \to 0$$
$$\widehat{F}_D^x - 1 \to 0 \quad \text{in probability.}$$

□

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