ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **12** (2023), no.1, 153–174
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.12.1.10

ASYMPTOTICALLY STABLE PROCESS AND APPLICATIONS

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ABSTRACT. We remark that some stationary processes do not verify $x_\infty | x_\infty$ is equal to its value. To do this, we propose a new definitions to differentiate it in which a process is asymptotically stable if it verifies this property. We also remark that all processes in all financial models have missed this property. Which leads us to re–examine the models and look the impact and importance of this property.

1. INTRODUCTION

It is not the asymptotic stability of Lyapunov on the dynamic system or these derivatives.

Let x_t be a process, for instance, the price of an action in time t. If we know x in time $T \geq t$, then x_T is logic equal to its value. If $T < \infty$, then all processes must verify this argument, but in the long term (that is for $T = \infty$), there exist a process which verifies or not this argument, see the Example 1. Hence the use of the definitions 2.1 and 2.2.

The asymptotically stable process makes more sense compared to its opposite. However, we have also noticed that the processes used by the authors in their financial models are non–stable, for instance, the volatilities in the Heston model

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2020 *Mathematics Subject Classification.* 62M86, 58K55, 91B25.
Key words and phrases. inference from stochastic process and fuzziness, asymptotic behavior, asset pricing models.
Submitted: 20.12.2022; *Accepted:* 05.01.2023; *Published:* 21.01.2023.

in [6], SV model in [5], WASC model in [3]. These findings therefore lead us to introduce this new process in the domain of finance and to show its importance. We will so modify the SV and WASC models to have the new models whose the specificity is the asymptotic stability. Our task is so to know its law (Laplace transform), stylized facts and the change du to this new specificity.

Our objective is to show the existence, the importance and the contribution of this new process.

2. Asymptotically stable process

Let ($\mathbb{R}^n$, $(\mathfrak{F}_t)_{t\geq 0}$, $\mathbb{P}$) be a filtrate and probability space where

- $\mathfrak{F}_t$ is the natural filter representing the information available in time t;
- $\mathbb{P}$ is the probability density.

Let x_t be a stationary process.

Definition 2.1. *A process x_t is asymptotically strongly $\mathfrak{F}_t$–stable if $x_{t+h}|\mathfrak{F}_t \longmapsto x_\infty$ in law for all $h \geq 0$, as $t \nearrow \infty$ (that is, $x_\infty|\mathfrak{F}_\infty$ exists and has its true value).*

Definition 2.2. *A process x_t is asymptotically weakly $\mathfrak{F}_t$–stable if $x_{t+h}|\mathfrak{F}_t \longmapsto a \in \mathbb{R}$ in law for all $h \geq 0$, as $t \nearrow \infty$ (that is, $x_\infty|\mathfrak{F}_\infty$ has a single value).*

The interest of this process is that a model driven by this process has an information which is stable in long term. In other words, any information on a product x for every moment remains itself and will last a very long time. Moreover, if the stability is a strong type then the past information of a long life remains itself. For the weak type, the information is to be estimated. In this case, we can give a theory on this estimated or another method of estimation so that the estimated value is not open to criticism. But for the moment, we will estimate this value using an available method so as not to inflate this article. In fact, the estimation method only has a criticism if there is any on the asymptotically weakly stable process.

Example 1.

(i) *The OU type process used in the volatility of Heston model (cf. [6]):*

$$dx_t = \alpha x_t dt + \beta dB_t \tag{2.1}$$

is not asymptotically $\mathfrak{F}_t$–stable where $\alpha < 0$, $\beta > 0$ and B_t is a Brownian motion. Indeed, $x_{t+h}|\mathfrak{F}_t \rightsquigarrow N\left(e^{\alpha h}x_t, \frac{-\beta^2(1-e^{2\alpha h})}{2\alpha}\right)$. Thus x_t is stationary where $x_\infty|\mathfrak{F}_t \rightsquigarrow N\left(0, \frac{-\beta^2}{2\alpha}\right)$ and it doesn't depend of the information in time t, for any $t \geq 0$. So, if the limit exists, then we have $x_\infty|\mathfrak{F}_\infty \rightsquigarrow N\left(0, \frac{-\beta^2}{2\alpha}\right)$ which is aleatory.

Thus, the information carried by this process will not last a very long time but it becomes variable in long term.

(ii) *The process x_t defined by:*

$$dx_t = t(\gamma + \alpha x_t)dt + \sqrt{t}e^{-\frac{1}{2}\beta t^2}dB_t \tag{2.2}$$

is asymptotically weakly $\mathfrak{F}_t$–stable where $\alpha < 0$, $\beta > 0$ and $\alpha+\beta \neq 0$. Indeed, let

$$z_t = e^{-\frac{1}{2}\alpha t^2}x_t. \tag{2.3}$$

We have

$$dz_t = \gamma t e^{-\frac{1}{2}\alpha t^2}dt + \sqrt{t}e^{-\frac{1}{2}(\beta+\alpha)t^2}dB_t \tag{2.4}$$

through the Ito's Lemma. Integrating in t to $t+h$, we get

$$z_{t+h} = z_t - \frac{\gamma}{\alpha}(e^{-\frac{1}{2}\alpha(t+h)^2} - e^{-\frac{1}{2}\alpha t^2}) + \int_t^{t+h}\sqrt{u}e^{-\frac{1}{2}(\beta+\alpha)u^2}dB_u.$$

So,

$$\begin{aligned} x_{t+h} &= x_t e^{\alpha th + \frac{1}{2}\alpha h^2} - \frac{\gamma}{\alpha}(1 - e^{\alpha th + \frac{1}{2}\alpha h^2}) \\ &\quad + e^{\frac{1}{2}\alpha(t+h)^2}\int_t^{t+h}\sqrt{u}e^{-\frac{1}{2}(\beta+\alpha)u^2}dB_u \end{aligned}$$

through the equation (2.3). Thus $x_{t+h}|\mathfrak{F}_t$ is a Gaussian random variable with mean and variance:

$$\mathbb{E}(x_{t+h}|\mathfrak{F}_t) = x_t e^{\alpha th + \frac{1}{2}\alpha h^2} - \frac{\gamma}{\alpha}(1 - e^{\alpha th + \frac{1}{2}\alpha h^2}) \tag{2.5}$$

$$Var(x_{t+h}|\mathfrak{F}_t) = \frac{-1}{2(\alpha+\beta)}(e^{-\beta(t+h)^2} - e^{-\beta t^2 + 2\alpha th + \alpha h^2}). \tag{2.6}$$

Since $\alpha < 0$ and $\beta > 0$, then $\mathbb{E}(x_{t+h}|\mathfrak{F}_t) \longrightarrow -\frac{\gamma}{\alpha}$ and $Var(x_{t+h}|\mathfrak{F}_t) \longrightarrow 0$, as $h \nearrow \infty$. So x_t is stationary where $x_{t+h}|\mathfrak{F}_t \longrightarrow -\frac{\gamma}{\alpha}$ in law for any $t \geq 0$, as

$h \nearrow \infty$. Moreover, we have $\mathbb{E}(x_{t+h}|\mathfrak{F}_t) \longrightarrow -\frac{\gamma}{\alpha}$ and $Var(x_{t+h}|\mathfrak{F}_t) \longrightarrow 0$, as $t \nearrow \infty$ for all $h \geq 0$, then $x_{t+h}|\mathfrak{F}_t \longrightarrow -\frac{\gamma}{\alpha}$ in law for all $h \geq 0$, as $t \nearrow \infty$.

(iii) *The process x_t defined by:*

$$dx_t = \frac{1}{(1+t)^2}(\gamma + \alpha x_t)dt + \frac{\beta}{1+t}dB_t \tag{2.7}$$

is asymptotically strongly $\mathfrak{F}_t$–stable where $\alpha \neq 0$. Indeed, by doing

$$z_t = e^{\frac{-\alpha t}{1+t}} x_t \tag{2.8}$$

and by following the reasoning above, we have $x_{t+h}|\mathfrak{F}_t \rightsquigarrow N\Big(e^{\frac{\alpha h}{(1+t)(1+t+h)}} x_t - \frac{\gamma}{\alpha}\left(1 - e^{\frac{h\alpha}{(t+1)(t+h+1)}}\right), (2\alpha)^{-1}\beta^2\left(e^{\frac{2h\alpha}{(t+1)(t+h+1)}} - 1\right)\Big)$.

So, x_t is stationary where $x_\infty|\mathfrak{F}_t \rightsquigarrow N\Big(e^{\frac{\alpha}{1+t}} x_t - \frac{\gamma}{\alpha}\big(1 - e^{\frac{\alpha}{t+1}}\big), (2\alpha)^{-1}\beta^2\big(e^{\frac{2\alpha}{(t+1)}} - 1\big)\Big)$ for all $t \geq 0$. Moreover, the law of $x_{t+h}|\mathfrak{F}_t$: $N\Big(e^{\frac{\alpha h}{(1+t)(1+t+h)}} x_t - \frac{\gamma}{\alpha}\big(1 - e^{\frac{h\alpha}{(t+1)(t+h+1)}}\big), (2\alpha)^{-1}\beta^2\left(e^{\frac{2h\alpha}{(t+1)(t+h+1)}} - 1\right)\Big) \longrightarrow \delta_{x_\infty}$ (density of Dirac in x_∞), as $t \nearrow \infty$. Thus $x_{t+h}|\mathfrak{F}_t \longmapsto x_\infty$ in law for all $h \geq 0$, as $t \nearrow \infty$.

(iv) *The stability means that to infinite time, $x_{\infty+h} = x_\infty$ for all step time $h \geq 0$. This stability therefore eliminates which say the opposite. For instance, let x_t be a process defined by:*

$$dx_t = (\beta + \alpha x_t)dt + e^{-\beta t}dB_t, \tag{2.9}$$

where $\alpha < 0$ and $\beta > 0$. Let

$$z_t = e^{-\alpha t} x_t \tag{2.10}$$

We have

$$dz_t = \beta e^{-\alpha t}dt + e^{-(\beta+\alpha)t}dB_t \tag{2.11}$$

through the Ito's Lemma. Integrating in t to $t+h$, we get

$$z_{t+h} = z_t - \frac{\beta}{\alpha}(e^{-\alpha(t+h)} - e^{-\alpha t}) + \int_t^{t+h} e^{-(\beta+\alpha)u}dB_u.$$

So,

$$x_{t+h} = x_t e^{\alpha h} - \frac{\beta}{\alpha}(1 - e^{\alpha h}) + e^{\alpha(t+h)} \int_t^{t+h} e^{-(\beta+\alpha)u}dB_u$$

through the equation (2.3). Thus $x_{t+h}|\mathfrak{F}_t$ is a Gaussian random variable with mean and variance:

$$\mathbb{E}(x_{t+h}|\mathfrak{F}_t) = x_t e^{\alpha h} + \frac{\beta}{\alpha}(e^{\alpha h} - 1) \tag{2.12}$$

$$Var(x_{t+h}|\mathfrak{F}_t) = \frac{-1}{2(\alpha+\beta)}(e^{-2\beta(t+h)} - e^{2(-\beta t+\alpha h)}). \tag{2.13}$$

Since $\alpha < 0$ and $\beta > 0$, then $\mathbb{E}(x_{t+h}|\mathfrak{F}_t) \longrightarrow -\frac{\beta}{\alpha}$ and $Var(x_{t+h}|\mathfrak{F}_t) \longrightarrow 0$, as $h \nearrow \infty$. So x_t is stationary where $x_{t+h}|\mathfrak{F}_t \longrightarrow -\frac{\beta}{\alpha}$ in law for any $t \geq 0$, as $h \nearrow \infty$. However, $x_{t+h}|\mathfrak{F}_t \longrightarrow x_\infty e^{\alpha h} + \frac{\beta}{\alpha}(e^{\alpha h} - 1)$ in law where $h \geq 0$, as $t \nearrow \infty$, which means that for a $h > 0$, $x_{\infty+h}|\mathfrak{F}_\infty \neq x_\infty|\mathfrak{F}_\infty$. That is, there exists a difference of infinite. This process is not asymptotically $\mathfrak{F}_t$–stable.

(v) *The OU type process used in the volatility of WASC model (cf. [3]) is not asymptotically $\mathfrak{F}_t$–stable. Indeed, $x_{t+h}|\mathfrak{F}_t \rightsquigarrow N\left(e^{\Phi h}x_t, \int_0^h e^{\Phi u}QQ'e^{\Phi' u}du\right)$. So x_t is stationary where $x_\infty|\mathfrak{F}_t \rightsquigarrow N(0,\Omega_1)$ where $vec(\Omega_1) = -(I_n \otimes \Phi + \Phi \otimes I_n)^{-1} \circ vec(QQ')$ (cf. [2]) and Φ is a negative definite matrix. But, if the limit exists, $x_\infty|\mathfrak{F}_\infty$ is always a random variable $N(0,\Omega_1)$.*

(vi) *The process x_t in $\mathbb{R}^n$ solution of SDE (Stochastic Differential Equation):*

$$dx_t = \frac{1}{(1+t)^2}\Phi x_t dt + \frac{1}{1+t}QdB_t \tag{2.14}$$

is asymptotically strongly $\mathfrak{F}_t$–stable.

Indeed, by following the same reasoning and using the Lemma in [7]:

Lemma 2.1. *Let $A \in \mathcal{M}_p(\mathbb{R})$ and $\mathcal{A} : X \in \mathcal{S}_p \longmapsto \mathcal{A}(X) = AX + XA' \in \mathcal{S}_p$. Then if A^{-1} exists, we have*

$$\mathcal{A}^{-1} = vec^{-1} \circ (I_p \otimes A + A \otimes I_p)^{-1} \circ vec,$$

we have $x_{t+h}|\mathfrak{F}_t \rightsquigarrow N(m_{t,h}, \sigma^2_{t,h})$ where

$$m_{t,h} = e^{\frac{h}{(t+1)(t+h+1)}\Phi}x_t$$
$$vec(\sigma^2_{t,h}) = (I_n \otimes \Phi + \Phi \otimes I_n)^{-1} \circ vec\left(e^{\frac{h}{(t+1)(t+h+1)}\Phi}QQ'e^{\frac{h}{(t+1)(t+h+1)}\Phi'} - QQ'\right),$$

x_t is stationary and $x_\infty|\mathfrak{F}_t \rightsquigarrow N(m_t, \sigma^2_t)$ where

$$m_t = e^{\frac{1}{t+1}\Phi}x_t$$
$$vec(\sigma^2_t) = (I_n \otimes \Phi + \Phi \otimes I_n)^{-1} \circ vec\left(e^{\frac{1}{t+1}\Phi}QQ'e^{\frac{1}{t+1}\Phi'} - QQ'\right).$$

Moreover, the law of $x_{t+h}|\mathfrak{F}_t$*:* $N(m_{t,h}, \sigma^2_{t,h}) \longmapsto \delta_{x_\infty}$ *for all* $h \geq 0$*, as* $t \nearrow \infty$*.*

3. Applications in financial market

Lemma 3.1. *For any symmetrical positive definite matrix* Ω *and* $\mu \in \mathbb{R}^n$*, we have*

$$\int_{\mathbb{R}^n} e^{-x'\Omega x + \mu' x} dx = \frac{\sqrt{\pi}^n}{\sqrt{\det \Omega}} e^{\frac{1}{4}\mu'\Omega^{-1}\mu}.$$

Proof. The left integral is equal to

$$\underbrace{\int_{\mathbb{R}^n} e^{-(x-\frac{1}{2}\Omega^{-1}\mu)'\Omega(x-\frac{1}{2}\Omega^{-1}\mu)} dx}_{\text{by doing } y=x-\frac{1}{2}\Omega^{-1}\mu, \text{ which is equal to } \int_{\mathbb{R}^n} e^{-y'\Omega y} dy = \frac{\sqrt{\pi}^n}{\sqrt{\det \Omega}}} \qquad e^{\frac{1}{4}\mu'\Omega^{-1}\mu}.$$

□

Let x_t be a Gaussian process in $\mathbb{R}^n$ such that $x_{t+h}|\mathfrak{F}_t \rightsquigarrow N\big(M(t,h)x_t + R(t,h), \sigma^2_{t,h}\big)$ where $M(t,h)$, $R(t,h)$ and $\sigma^2_{t,h}$ are the deterministic functions which don't depend of x.

We define the volatility of model by

$$\Gamma_t = \sum_{i=1}^{\nu} x_{i,t}(x_{i,t})', \tag{3.1}$$

where ν is a positive integer nonzero.

3.1. Laplace transform of volatility and stationarity.

Let Λ be a $n \times n$ dimensional symmetrical matrix. The conditional Laplace transform of Γ_{t+h} given the information $\mathfrak{F}_t$ is defined by:

$$\Psi_{\Gamma_{t+h},\mathfrak{F}_t}(\Lambda, h) = \mathbb{E}\left\{e^{tr(\Lambda\Gamma_{t+h})}|\mathfrak{F}_t\right\} \text{ where } t, h \geq 0. \tag{3.2}$$

Proposition 3.1. *If* $\nu = n = 1$ *and* $\|2\sigma^2_{t,h}\Lambda\| < 1$*, then we have*

$$\Psi_{\Gamma_{t+h},\mathfrak{F}_t}(\Lambda, h) = \frac{e^{(R(t,h)+M(t,h)\sqrt{\Gamma_t})^2\Lambda[1-2\sigma^2_{t,h}\Lambda]^{-1}}}{[1-2\sigma^2_{t,h}\Lambda]^{\frac{1}{2}}}. \tag{3.3}$$

Proof. Since $\nu = n = 1$, then we have $\Gamma_t = x_{t+h}^2$ and

$$\begin{aligned}\mathbb{E}\{e^{tr(\Lambda\Gamma_{t+h})}|\mathfrak{F}_t\} &= \mathbb{E}\{e^{\Lambda x_{t+h}^2}|\mathfrak{F}_t\}\\ &= e^{(R(t,h)+M(t,h)x_t)^2\Lambda}\mathbb{E}\left[e^{(2R(t,h)+2x_tM(t,h))\sigma_{t,h}\xi\Lambda+\xi^2\sigma_{t,h}^2\Lambda}|\mathfrak{F}_t\right]\\ &\qquad \text{where } \xi \text{ is the random variable of density } \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\varepsilon^2}\\ &= e^{(R(t,h)+M(t,h)x_t)^2\Lambda}\int_{\mathbb{R}} e^{(2R(t,h)+2x_tM(t,h))\Lambda\sigma_{t,h}\varepsilon-\frac{1}{2}(1-2\sigma_{t,h}^2\Lambda)\varepsilon^2}\frac{1}{\sqrt{2\pi}}d\varepsilon\\ &= \frac{e^{(R(t,h)+M(t,h)\sqrt{\Gamma_t})^2\Lambda[1-2\sigma_{t,h}^2\Lambda]^{-1}}}{[1-2\sigma_{t,h}^2\Lambda]^{\frac{1}{2}}} \text{ through the Lemma 3.1.}\end{aligned}$$

□

Proposition 3.2. *If $n \geq 2$, $R(t,h) = 0$ and $\|2\sigma_{t,h}^2\Lambda\| < 1$, then we have*

$$\Psi_{\Gamma_{t+h},\mathfrak{F}_t}(\Lambda,h) = \frac{e^{tr[M(t,h)'\Lambda[I_n-2\sigma_{t,h}^2\Lambda]^{-1}M(t,h)\Gamma_t]}}{[det(I_n-2\sigma_{t,h}^2\Lambda)]^{\frac{\nu}{2}}}. \tag{3.4}$$

Proof. Let firstly $\nu = 1$. We have $\Gamma_t = x_{t+h}x_{t+h}'$ and

$$\mathbb{E}\{e^{tr(\Lambda\Gamma_{t+h})}|\mathfrak{F}_t\} = \mathbb{E}\{e^{x_{t+h}'\Lambda x_{t+h}}|\mathfrak{F}_t\}.$$

Hence

$$\begin{aligned}\mathbb{E}[e^{x_{t+h}'\Lambda x_{t+h}}|\mathfrak{F}_t] &= \mathbb{E}\left[e^{x_t'M(t,h)'\Lambda M(h)x_t+2x_t'M(t,h)'\Lambda\sigma_{t,h}\xi+\xi'\sigma_{t,h}\Lambda\sigma_{t,h}\xi}|\mathfrak{F}_t\right]\\ &\qquad \text{where } \xi \text{ is the random variable of density } \frac{1}{(\sqrt{2\pi})^n}e^{-\frac{1}{2}\varepsilon'\varepsilon}\\ &= e^{x_t'M(t,h)'\Lambda M(t,h)x_t}\mathbb{E}\left[e^{2x_t'M(t,h)'\Lambda\sigma_{t,h}\xi+\xi'\sigma_{t,h}\Lambda\sigma_{t,h}\xi)}|\mathfrak{F}_t\right]\\ &= e^{x_t'M(t,h)'\Lambda M(t,h)x_t}\int_{\mathbb{R}^n} e^{2x_t'M(t,h)'\Lambda\sigma_{t,h}\varepsilon-\varepsilon'\frac{1}{2}(I_n-2\sigma_{t,h}\Lambda\sigma_{t,h})\varepsilon}\frac{1}{(\sqrt{2\pi})^n}d\varepsilon\\ &= \frac{e^{tr[M(t,h)'\Lambda[I_n-2\sigma_{t,h}^2\Lambda]^{-1}M(t,h)\Gamma_t]}}{[det(I_n-2\sigma_{t,h}^2\Lambda)]^{\frac{1}{2}}}\text{through the Lemma 3.1.}\end{aligned}$$

Now, let $\nu \geq 2$. We have

$$\Gamma_t = \sum_{k=1}^{\nu}\Gamma_{k,t}, \tag{3.5}$$

where $\Gamma_{k,t}$ is a matrix process above such that $\nu = 1$ and

$$\Psi_{\Gamma_{k,t+h},\mathfrak{F}_t}(\Lambda, h) = \frac{e^{tr[M(t,h)'\Lambda[I_n - 2\sigma^2_{t,h}\Lambda]^{-1}M(t,h)\Gamma_{k,t}]}}{[det(I_n - 2\sigma^2_{t,h}\Lambda)]^{\frac{1}{2}}}.$$

The result follows using $\Psi_{\Gamma_{t+h},\mathfrak{F}_t}(\Lambda, h) = \prod_{k=1}^{\nu} \Psi_{\Gamma_{k,t+h},\mathfrak{F}_t}(\Lambda, h)$. □

Hence, if $M(t+h) \longrightarrow M(t)$, $R(t+h) \longrightarrow R(t)$ and $\sigma^2_{t+h} \longrightarrow \sigma^2_t$ as $h \nearrow \infty$ then $\Psi_{\Gamma_{t+h},\mathfrak{F}_t}(\Lambda, h)$ converges. Thus, Γ_t is a stationary process.

The following proposition is obvious:

Proposition 3.3. *If the function g where $\Gamma_t = g(x_t)$ is continuous, then Γ_t and x_t are the same nature on the stationarity.*

3.2. **Some natures of volatilities.**

- Unidimensional model (n=1): Using the results above,
  - If x_t is the process in the Examples 1 (i), then Γ_t is a stationary process and the conditional Laplace transform of $\Gamma_\infty|\mathfrak{F}_t$ is

(3.6)
$$\Psi_{\Gamma_\infty,\mathfrak{F}_t}(\Lambda) = \frac{1}{\left[1 + \frac{\beta^2}{\alpha}\Lambda\right]^{\frac{1}{2}}}.$$

    So, $\Gamma_\infty|\mathfrak{F}_t$ follows the law of chi–square which doesn't depend of the information in time t, for all $t \geq 0$. Thus, the volatility of Heston model is not asymptotically $\mathfrak{F}_t$–stable.
  - If x_t is the process in the Examples 1 (ii), then Γ_t is a stationary process. Moreover, the Laplace transform of $\Gamma_{t+h}|\mathfrak{F}_t$:

$$\Psi_{\Gamma_{t+h}}(\Lambda, t) = \frac{e^{\left[-\frac{\gamma}{\alpha}\left(1-e^{\alpha th+\frac{1}{2}\alpha h^2}\right)+e^{\alpha th+\frac{1}{2}\alpha h^2}\sqrt{\Gamma_t}\right]^2\Lambda\left[1+\frac{1}{\alpha+\beta}(e^{-\beta(t+h)^2}-e^{-\beta t^2+2\alpha th+\alpha h^2})\Lambda\right]^{-1}}}{\left[1+\frac{1}{\alpha+\beta}(e^{-\beta(t+h)^2}-e^{-\beta t^2+2\alpha th+\alpha h^2})\Lambda\right]^{\frac{1}{2}}} \longmapsto e^{\frac{\gamma^2}{\alpha^2}\Lambda},$$

    as $t \nearrow \infty$. So, $\Gamma_{t+h}|\mathfrak{F}_t \longmapsto \frac{\gamma^2}{\alpha^2}$ in law for all $h \geq 0$, as $t \nearrow \infty$. Then, this volatility is asymptotically weakly $\mathfrak{F}_t$–stable.

- If x_t is the process in the Examples 1 (iii), then Γ_t is a stationary process. Moreover, the Laplace transform of $\Gamma_{t+h}|\mathfrak{F}_t$:

$$\begin{aligned}
&\Psi_{\Gamma_{t+h}}(\Lambda, t) \\
&= \frac{e^{\left[-\frac{\gamma}{\alpha}\left(1-e^{\frac{h\alpha}{(t+1)(t+h+1)}}\right)+e^{\frac{\alpha h}{(1+t)(1+t+h)}}\sqrt{\Gamma_t}\right]^2 \Lambda\left[1-\alpha^{-1}\beta^2\left(e^{\frac{2h\alpha}{(t+1)(t+h+1)}}-1\right)\Lambda\right]^{-1}}}{\left[1-\alpha^{-1}\beta^2\left(e^{\frac{2h\alpha}{(t+1)(t+h+1)}}-1\right)\Lambda\right]^{\frac{1}{2}}} \\
&\longmapsto e^{\Lambda\Gamma_t},
\end{aligned}$$

  as $t \nearrow \infty$. So, $\Gamma_{t+h}|\mathfrak{F}_t \longmapsto \Gamma_\infty$ in law for all $h \geq 0$, as $t \nearrow \infty$. Then, this volatility is asymptotically strongly $\mathfrak{F}_t$–stable.

- Multidimensional model ($n \geq 2$): from the expression (3.4), the volatility of WASC model is not asymptotically $\mathfrak{F}_t$–stable, while the volatility associated to process in Examples 1 (v) is asymptotically strongly $\mathfrak{F}_t$–stable.

The following proposition is obvious:

Proposition 3.4. *If the function g where $\Gamma_t = g(x_t)$ is continuous, then Γ_t and x_t are the same nature on the stability.*

3.3. Application on the unidimensional models.

Let be a model of the form:

$$\left\{\begin{array}{l}
d\log S_t = (r - \frac{1}{2}\Gamma_t)dt + \sqrt{\Gamma_t}dW_t \\
\Gamma_t = g(x_t),\ g \geq 0 \\
dx_t = c(t)\mu(x_t)dt + a(t)\sigma(x_t)dB_t \\
dW_t dB_t = \rho dt
\end{array}\right. \tag{3.7}$$

where

- r and ρ are the reals;
- B_t and W_t are the Brownian motions;
- a and c are the functions of $\mathbb{R}_+$ into $\mathbb{R}$;
- μ and σ are the functions of $\mathbb{R}$ into $\mathbb{R}$.

We have three unidimensional models with theirs specificities that we study below.

If we use for pricing option or hedging, then r is the interest rate associated with risk–neutral probability $\mathbb{P}$.

TABLE 1. Features of some unidimensional models.

Model	Specification	Return to average	clusters of volatility	leverage effect on volatility	stationarity	stability of information in volatility
Heston	$c(t) = 1$; $a(t) = 1$ $\mu(x_t) = \alpha x_t$; $\sigma(x_t) = \beta$ $\beta > 0$, $\alpha < 0$; $g(x) = x^2$	✓	✓	✓	✓	
(ii)	$c(t) = t$; $a(t) = \sqrt{t}e^{-\frac{1}{2}\beta t^2}$; $\mu(x_t) = \gamma + \alpha x_t$; $\sigma(x_t) = 1$ $\beta > 0$; $\alpha < 0$; $g(x) = x^2$	✓	✓	✓	✓	✓ weak
(iii)	$c(t) = \frac{1}{(1+t)^2}$; $a(t) = \sqrt{c(t)}$ $\mu(x_t) = \gamma + \alpha x_t$; $\sigma(x_t) = \beta$ $\alpha \neq 0$; $\beta > 0$; $g(x) = x^2$	✓	✓	✓	✓	✓ strong

3.3.1. *Estimating of parameters and Pricing option.*

To estimate the parameters of model and determine the option price, we use the common technical in the work of [8].

3.3.2. *Empirical result.*

In this article, we propose to estimate the current price of Nasdaq and S&P500. We used the daily data. The time series start the May 5, 2021 and end the June 15, 2021 which are presented by the following Figures 1.

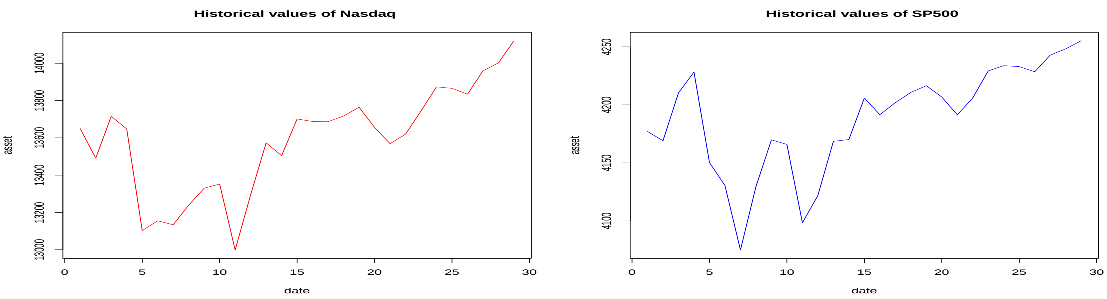

FIGURE 1. Historical Volume of Nasdaq and S&P500 Indexes

pricing number= 3; $r = 0.000045$ (daily interest rate), sample number N= 1000000.

TABLE 2. Estimating parameters by Monte–Carlo method using S&P500 index.

Model	α	β	γ	ρ	objective
Heston	-0.4500398	0.2500076	-	-0.4630357	735.93
(ii)	-5.320534	5.05862848	0.0035096	-0.258254	834.3932
(iii)	-1.23354233	0.06893966	-0.02220208	0.15752659	809.3575

TABLE 3. Pricing option of S&P500 index.

$K = 4200$ (strike); $S_0 = 4255.28$				
Model	T=1 (day)	T=2	T=3	T=4
Heston	87.942	99.08988	109.9039	120.845
(ii)	65.87738	60.68611	59.43442	58.82652
(iii)	66.7912	63.07684	61.43229	60.06095

3.4. **Application on the Multidimensional models.**

We will study the financial market using the WASC model and the two following models of the form:

$$\begin{cases} d\log S_t = \left(r - \frac{1}{2}vec[tr(e_{ii}\Gamma_t)]\right) dt + \sqrt{\Gamma_t}dZ_t \\ \Gamma_t = g(x_t) \text{ is a positive definite matrix} \\ cov(d\log S_t, d\Gamma_t) = k(t,\rho)dt \end{cases} \tag{3.8}$$

where

- r is a vector in $\mathbb{R}^n$;
- x_t is the process in $\mathbb{R}^n$ in the Examples 1 (v);

- If $a_1, \dots, a_n \in \mathbb{R}$, then $vec(a_i) = (a_1, \dots, a_n)'$ which is a vector in $\mathbb{R}^n$;
- e_{ii} is $n \times n$ dimensional matrix defined by $e_{ii} = (\delta_{ijk})_{j,k=1\dots n}$ where $\delta_{ijk} = \begin{cases} 1 \text{ if } (j,k) = (i,i) \\ 0 \text{ otherwise} \end{cases}$;
- Z_t is a n–dimensional vector whose components are the sBm (standard Brownian motion);
- k is a function of $\mathbb{R}_+ \times [-1, 1]^n$ into $\mathbb{R}$;
- $tr(H)$ is the trace of the matrix H.

Let $g(x_t) = \sum_{i=1}^{\nu} x_{i,t}(x_{i,t})'$ where $\nu \geq n$ is integer and the $x_{i,t}$ are the processes in the Examples 1 (v).

There is not the stability in the WASC model, while for the model (3.8), the volatility is stable and we want also the stability in the yield, hence the model of the form:

$$\begin{cases} d\log S_t = \frac{1}{(1+t)^2}\left(r - \frac{1}{2}vec[tr(e_{ii}\Gamma_t)]\right)dt + \frac{1}{1+t}\sqrt{\Gamma_t}dZ_t \\ \Gamma_t = g(x_t) \text{ is a positive definite matrix} \\ cov(d\log S_t, d\Gamma_t) = k(t,\rho)dt \end{cases} \tag{3.9}$$

3.4.1. *Dynamic of volatility.*

Proposition 3.5. *The volatility Γ_t is a positive definite matrix and we obtain*

$$d\Gamma_t = (\nu QQ' + \Phi\Gamma_t + \Gamma_t\Phi')\frac{dt}{(1+t)^2} + \frac{1}{1+t}(\sqrt{\Gamma_t}(d\tilde{B}_t)'Q' + Qd, \tilde{B}_t\sqrt{\Gamma_t}) \tag{3.10}$$

where $\tilde{B}_t$ is a $n \times n$ dimensional stochastic matrix whose components are independent sBm defined by $d\tilde{B}_t = d\check{B}_t X_t(\sqrt{\Gamma_t})^{-1}$, X_t is $n \times \nu$ dimensional process solution of $d(X_t)' = \frac{1}{(1+t)^2}\Phi(X_t)'dt + \frac{1}{1+t}Qd\check{W}_t$, and $\check{B}_t = (B_{1t}, B_{2,t}, \dots, B_{\nu,t})$ is the $n \times \nu$ dimensional matrix where the $B_{i,t}$ are the Brownian motion vectors of $x_{i,t}$, $i = 1, \dots, \nu$.

Proof. The reasoning is meaning that we find in the work of [1]. □

3.4.2. *Study of leverage effect.*

We define the function of covariance between yield and its volatility such that:

$$k(t,\rho) = 2f(t)dt\Gamma_{ii,t}\sum_{l=1}^{n} Q_{li}\rho_l \tag{3.11}$$

where the ρ_i are the components of ρ and $f(t) = \begin{cases} \frac{1}{1+t} \text{ for the model (3.8)} \\ \frac{1}{(1+t)^2} \text{ for the model (3.9)} \end{cases}$.

In this case, the mBs Z_t can define by:

$$dZ_t = \sqrt{1-\rho'\rho}dW_t + d\tilde{B}_t\rho, \tag{3.12}$$

where W_t is a n–dimensional stochastic vector whose components are the sBm and it is independent to $\tilde{B}_t$.

By using the reasoning in the work of [1], the covariance between yield and its correlation is:

$$cov(d\log S_{p,t}, d\zeta_{pq,t}) \quad = \quad f(t)\sqrt{\frac{\Gamma_{pp,t}}{\Gamma_{qq,t}}}\sum_{l=1}^{n} Q_{lq}\rho_l(1-\zeta_{pq,t}^2), \tag{3.13}$$

where $i = 1,\dots,n$; $p, q = 1,\dots,n$ and $p \neq q$, the ρ_i are the components of ρ and $\zeta_{pq,t}$ is the correlation between $\Gamma_{pp,t}$ and $\Gamma_{qq,t}$ defined by:

$$\zeta_{pq,t} = \frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}. \tag{3.14}$$

3.4.3. *Laplace transform of assets returns.*

Let γ be a vector in $\mathbb{R}^n$. The conditional Laplace transform of $\log S_t$ is defined by

$$\Psi_{\log S_T,\mathfrak{F}_t}(\gamma, T) = \mathbb{E}\{e^{\gamma'\log S_T}|\mathfrak{F}_t\} \text{ where } T \geq t \geq 0. \tag{3.15}$$

As the yield $\log S_t$ is affine, we have

$$\Psi_{\log S_T,\mathfrak{F}_t}(\gamma, T) = e^{tr(A(h)\Gamma_t)+B(h)\log S_t+C(h)} \tag{3.16}$$

with $h = T - t$ and $A(h)$, $B(h)$, $C(h)$ are the functions defined by:

Proposition 3.6.

- *For the model (3.8):*

$$\begin{aligned} A(h) &= A_{22}(h)^{-1}A_{21}(h), \\ B(h) &= \gamma', \\ C(h) &= tr\left[rh\gamma' - \frac{\nu}{2}\left(\log A_{22}(h) + \Upsilon_2(h)\right)\right], \end{aligned}$$

where $\tilde{1}$ *is a* $n \times n$ *dimensional matrix whose components are equal to* 1 *and*

$$\begin{aligned}
\Upsilon_1(h) &= -\frac{h}{2}\sum_{i=1}^{n}\gamma_i e_{ii} + \frac{h}{2}\gamma\gamma' \\
\Upsilon_2(h) &= \frac{h}{2(t+1)(t+h+1)}(\Phi+\Phi') + \frac{log(1+t+h) - log(1+t)}{2} \\
&\quad (\gamma\rho' Q' + Q\rho\gamma') \\
\Upsilon_3(h) &= \frac{2h}{(t+1)(t+h+1)}QQ' \\
\begin{bmatrix} A_{11}(h) & A_{12}(h) \\ A_{21}(h) & A_{22}(h) \end{bmatrix} &= \exp\left(\begin{bmatrix} \Upsilon_2(h) & -\Upsilon_3(h) \\ \Upsilon_1(h) & -\Upsilon_2(h) \end{bmatrix}\right).
\end{aligned}$$

- *For the model (3.9):*

$$\begin{aligned}
A(h) &= A_{22}(h)^{-1}A_{21}(h), \\
B(h) &= \gamma', \\
C(h) &= tr\left[r\gamma' \frac{h}{(1+t)(1+t+h)} - \frac{\nu}{2}\left(\log A_{22}(h) + \Upsilon_2(h)\right)\right]
\end{aligned}$$

where $\tilde{1}$ *is a* $n \times n$ *dimensional matrix whose components are equal to* 1 *and*

$$\begin{aligned}
\Upsilon_1(h) &= -\frac{h}{2(1+t)(t+h+1)}\sum_{i=1}^{n}\gamma_i e_{ii} + \frac{h}{2(1+t)(1+t+h)}\gamma\gamma' \\
\Upsilon_2(h) &= \frac{h}{2(t+1)(t+h+1)}(\Phi+\Phi') + \frac{h}{2(t+1)(t+h+1)}(\gamma\rho' Q' + Q\rho\gamma') \\
\Upsilon_3(h) &= \frac{2h}{(t+1)(t+h+1)}QQ' \\
\begin{bmatrix} A_{11}(h) & A_{12}(h) \\ A_{21}(h) & A_{22}(h) \end{bmatrix} &= \exp\left(\begin{bmatrix} \Upsilon_2(h) & -\Upsilon_3(h) \\ \Upsilon_1(h) & -\Upsilon_2(h) \end{bmatrix}\right).
\end{aligned}$$

Proof.

- For the model (3.8): using the Feynmann–Kac argument to the model, we have

(3.17)
$$\frac{\partial \Psi_{\log S_T, \mathfrak{F}_t}(\gamma, T)}{\partial h} = \mathcal{L}_{\log S, \Gamma}\Psi_{\log S_T, \mathfrak{F}_t}(\gamma, T)$$

where $T = t + h$ and $\mathcal{L}_{\log S,\Gamma}$ is the infinitesimal generator of the joint $(\log S_t, \Gamma_t)$ defined by:

(3.18)
$$\mathcal{L}_{\log S,\Gamma} = tr\left[\frac{1}{(1+T-h)^2}\left(\nu QQ' + \Phi\Gamma_t + \Gamma_t\Phi\right)D + \frac{2}{(1+T-h)^2}\Gamma_t DQQ'D\right] + \nabla_Y\left(r - \frac{1}{2}vec[tr(e_{ii}\Gamma_t)]\right) + \frac{1}{2}\nabla_Y\Gamma_t\nabla_Y' + \frac{1}{1+T-h}$$

(3.19)
$$tr\left(DQ\rho\nabla_Y\Gamma_t + \Gamma_t\nabla_Y'\rho' Q' D\right)$$

with

- $D = (D_{ij})_{1\leq i,j\leq n}$ where $D_{ij} = \frac{\partial}{\partial\Gamma_{ij,t}}$ and $\Gamma_{ij,t}$, $1 \leq i,j \leq n$ are the components of the volatility matrix Γ_t;
- $\nabla_Y = \left(\frac{\partial}{\partial Y_1}, \cdots, \frac{\partial}{\partial Y_n}\right)'$ where $Y_i = \log S_{i,t}$ is the yield of the i-th underlying in the basket, $i = 1, \ldots, n$.

We have

$$\frac{\partial\Psi_{\log S_T,\mathfrak{F}_t}(\gamma,T)}{\partial h} = \left[tr\left(\frac{\partial A(h)}{\partial h}\Gamma_{t^-}\right) + \frac{\partial B(h)}{\partial h}\log S_{t^-} + \frac{\partial C(h)}{\partial h}\right]\Psi_{\log S_T,\mathfrak{F}_t}(\gamma,T).$$

Then, from the expression (3.17), we deduce

$$tr\left[\frac{\partial A(h)}{\partial h}\Gamma_t\right] + \frac{\partial B(h)}{\partial h}\log S_t + \frac{\partial C(h)}{\partial h}$$
$$= B(h)(r - \frac{1}{2}vec[tr(e_{ii}\Gamma_t)]) + \frac{1}{2}B(h)\Gamma_t B(h)' + \frac{1}{(1+T-h)^2}$$
$$tr\left[\left(\nu QQ' + \Phi\Gamma_t + \Gamma_t\Phi\right)A(h) + 2\Gamma_t A(h)QQ'A(h)\right] + \frac{1}{1+T-h}$$

(3.20)
$$tr\left[A(h)Q\rho B(h)\Gamma_{t^-} + \Gamma_{t^-}B(h)'\rho' Q' A(h)\right]$$

with the initial conditions $A(0) = 0$, $B(0) = \gamma'$ and $C(0) = 0$.

By identifying the coefficient of $\log S_t$, we have $\frac{\partial B(h)}{\partial h} = 0$ which follows that $B(h) = B(0) = \gamma'$ for all $h \geq 0$.

Identifying the coefficient of Γ_t, we have

$$\frac{\partial A(h)}{\partial h} = -\frac{1}{2}\sum_{i=1}^{n}\gamma_i e_{ii} + \frac{1}{2}\gamma\gamma' + \frac{1}{(1+T-h)^2}\left(\Phi A(h) + A(h)\Phi'\right)$$

$$+\frac{2}{(1+T-h)^2}A(h)QQ'A(h)+\frac{1}{1+T-h}(A(h)Q\rho\gamma'+\gamma\rho'Q'A(h)). \tag{3.21}$$

Suppose

$$\begin{aligned}
\Xi_1(h) &= -\frac{1}{2}\sum_{i=1}^{n}\gamma_i e_{ii}+\frac{1}{2}\gamma\gamma' \\
\Xi_2(h) &= \frac{1}{2(1+T-h)^2}(\Phi+\Phi')+\frac{1}{2(1+T-h)}(\gamma\rho'Q'+Q\rho\gamma') \\
\Xi_3(h) &= \frac{2}{(1+T-h)^2}QQ',
\end{aligned}$$

and

$$A(h)=F(h)^{-1}G(h) \text{ with } F(h)\in GL_n(\mathbb{R}) \text{ and } G(h)\in \mathcal{M}_n(\mathbb{R}). \tag{3.22}$$

We have $0 = A(0) = F(0)^{-1}G(0)$. In this case, we take $G(0) = 0$ and $F(0) = I_n$. Since $\frac{\partial[F(h)A(h)]}{\partial h} = \frac{\partial F(h)}{\partial h}A(h) + F(h)\frac{\partial A(h)}{\partial h}$, then we have

$$\begin{aligned}
\frac{\partial G(h)}{\partial h}-\frac{\partial F(h)}{\partial h}A(h) &= F(h)\frac{\partial A(h)}{\partial h} \\
&= F(h)\Xi_1(h)+G(h)\Xi_2(h)+F(h)\Xi_2(h)A(h) \\
&+ G(h)\Xi_3(h)A(h).
\end{aligned}$$

So,

$$\begin{cases}
\frac{\partial G(h)}{\partial h} = G(h)\Xi_2(h)+F(h)\Xi_1(h), \\
\frac{\partial F(h)}{\partial h} = -G(h)\Xi_3(h)-F(h)\Xi_2(h).
\end{cases}$$

Thus,

$$\frac{\partial}{\partial h}\begin{bmatrix} G(h) \\ F(h)\end{bmatrix} = \begin{bmatrix}\Xi_2(h) & -\Xi_3(h) \\ \Xi_1(h) & -\Xi_2(h)\end{bmatrix}\begin{bmatrix} G(h) \\ F(h)\end{bmatrix}.$$

Finally, by identification

$$\frac{\partial C(h)}{\partial h} = tr\left[r\gamma'+\frac{\nu}{(T-h+1)^2}QQ'A(h)\right],$$

where $C(0) = 0$.

Since

$$tr\left[\frac{\nu}{(T-h+1)^2}QQ'A(h)\right] = tr\left(\frac{\nu}{2}F(h)^{-1}G(h)\Xi_3(h)\right) = tr\left(\frac{-\nu}{2}F(u)^{-1}\frac{\partial F(u)}{\partial u} - \frac{\nu}{2}\Xi_2(h)\right),$$

then, we have

$$C(h) = tr\left[rh\gamma' - \frac{\nu}{2}\left(\log A_{22}(h) + \Upsilon_2(h)\right)\right].$$

- Similary reasoning for the model (3.9).

□

3.4.4. *Some natures of assets returns.*

For the model (3.9), its Laplace transform

(3.23) $$\Psi_{\log S_{t+h},\mathfrak{F}_t}(\gamma, t+h) \longrightarrow e^{tr(A(\infty)\Gamma_t)+\gamma' logS_t + C(\infty)}, \text{ as } h \nearrow \infty,$$

where

$$A(\infty) = A_{22}(\infty)^{-1}A_{21}(\infty),$$

$$C(\infty) = tr\left[r\gamma'\frac{1}{1+t} - \frac{\nu}{2}\left(\log A_{22}(\infty) + \Upsilon_2(\infty)\right)\right],$$

$$\begin{bmatrix} A_{11}(\infty) & A_{12}(\infty) \\ A_{21}(\infty) & A_{22}(\infty) \end{bmatrix} = \exp\left(\begin{bmatrix} \Upsilon_2(\infty) & -\Upsilon_3(\infty) \\ \Upsilon_1(\infty) & -\Upsilon_2(\infty) \end{bmatrix}\right),$$

$$\Upsilon_1(\infty) = -\frac{1}{2(1+t)}\sum_{i=1}^{n}\gamma_i e_{ii} + \frac{1}{2(1+t)}\gamma\gamma',$$

$$\Upsilon_2(\infty) = \frac{1}{2(t+1)}(\Phi + \Phi') + \frac{1}{2(t+1)}(\gamma\rho' Q' + Q\rho\gamma'),$$

$$\Upsilon_3(\infty) = \frac{2}{t+1}QQ'.$$

Thus, $\log S_t$ is a stationary process. Moreover, $\Psi_{\log S_{t+h},\mathfrak{F}_t}(\gamma, t+h) \longrightarrow e^{\gamma' logS_\infty}$, as $t \nearrow \infty$. So, $\log S_{t+h}|\mathfrak{F}_t \longmapsto \log S_\infty$ in law for all $h \geq 0$, as $t \nearrow \infty$. Then, $logS_t$ is asymptotically strongly $\mathfrak{F}_t$–stable. Hence, the model (3.9) is asymptotically strongly $\mathfrak{F}_t$–stable.

We have three multidimensional models with theirs specificities that we study below in this paper.

TABLE 4. Features of some multidimensional models

Model	Return to average	clusters of volatility	leverage effects on correlation	leverage effects on correlation	stationarity	stability of information volatility	stability of information on the asset
WASC	✓	✓	✓	✓	✓		
Model (3.8)	✓	✓	✓	✓	✓	✓ strong	
Model (3.9)	✓	✓	✓	✓	✓	✓ strong	✓ strong

3.4.5. *The market without arbitrage.*

Let $n = 2$. Let ι be the interest rate. For the WASC, the models (3.8) and (3.9), if $r = \iota\check{1}$, then the market is without arbitrage. Indeed, in the market without arbitrage, the hoped value of $j = 1; 2$ asset under the risk–neutral probability $\mathbb{P}$ is:

$$\begin{aligned} S_{j,0}e^{\iota t} &= \mathbb{E}[S_{j,t}] \\ &= \mathbb{E}[exp(\log S_{j,t})] \\ &= \Psi_{\log S_0}(\gamma_j, t) \\ &= e^{tr[A(t)\Gamma_0]+\log S_{j,0}+C(t)} \\ &= S_{j,0}e^{tr[A(t)\Gamma_0]+C(t)} \end{aligned} \tag{3.24}$$

where $\gamma_j = \begin{cases} (1,0)' \text{ if } j = 1 \\ (0,1)' \text{ if } j = 2 \end{cases}$ and $\Psi_{\log S_0}(\gamma_j, t)$ is the Laplace transform above by taking $h = t$; $t = 0$ and $\gamma = \gamma_j$.

We have

$$A(t) = A_{22}(t)^{-1}A_{21}(t), \tag{3.25}$$

where $\begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}$ is of the form $\exp\left(\begin{bmatrix} M_{11} & M_{12} \\ 0 & -M_{11} \end{bmatrix}\right)$ and M_{ij}, 2×2–dimensional matrix. This latter is equal to $\begin{bmatrix} e^{M_{11}} & -\frac{1}{2}M_{11}^{-1}(e^{M_{11}} - e^{-M_{11}})M_{12} \\ 0 & e^{-M_{11}} \end{bmatrix}$. Indeed, let be $T = \begin{bmatrix} M_{11} & M_{12} \\ 0 & -M_{11} \end{bmatrix}$, $T^s = (T_{ij}^{(s)})_{ij}$, $s \in \mathbb{N}$. In the trace operator, we have

$T_{ij}^{(0)} = I_n$ if $i = j$ and 0 otherwise; $T_{11}^{(1)} = M_{11}$; $T_{12}^{(1)} = M_{12}$; $T_{21}^{(1)} = 0$; $T_{22}^{(1)} = -M_{11}$; $T_{11}^{(2)} = (M_{11})^2$; $T_{12}^{(2)} = M_{11}M_{12} - M_{21}M_{11} = 0$; $T_{21}^{(2)} = 0$; $T_{22}^{(2)} = (M_{11})^2$. Now, let us consider $p \geq 1$, in the trace operator, reasoning by recurrence, we have $T_{11}^{2(p+1)} = T_{11}^{(2)}T_{11}^{(2p)} + T_{12}^{(2)}T_{21}^{(2p)} = (M_{11})^{2(p+1)}$; $T_{12}^{2(p+1)} = T_{11}^{(2)}T_{12}^{(2p)} + T_{12}^{(2)}T_{22}^{(2p)} = 0$; $T_{21}^{2(p+1)} = T_{21}^{(2)}T_{11}^{(2p)} + T_{22}^{(2)}T_{21}^{(2p)} = 0$; $T_{22}^{2(p+1)} = T_{21}^{(2)}T_{12}^{(2p)} + T_{22}^{(2)}T_{22}^{(2p)} = (M_{11})^{2(p+1)}$. Then using the values $T_{ij}^{(1)}$ and $T_{ij}^{(2p)}$ above, we have, for all $p \geq 1$ $T_{11}^{2p+1} = T_{11}^{(1)}T_{11}^{(2p)} + T_{12}^{(1)}T_{21}^{(2p)} = (M_{11})^{2p+1}$; $T_{12}^{2p+1} = T_{11}^{(1)}T_{12}^{(2p)} + T_{12}^{(1)}T_{22}^{(2p)} = (M_{11})^{2p}M_{12}$; $T_{21}^{2p+1} = T_{21}^{(1)}T_{11}^{(2p)} + T_{22}^{(1)}T_{21}^{(2p)} = 0$; $T_{22}^{2p+1} = T_{21}^{(1)}T_{12}^{(2p)} + T_{22}^{(1)}T_{22}^{(2p)} = -(M_{11})^{2p+1}$. Well, we have

$$\begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} = e^T = \sum_{s=0}^{+\infty} \frac{T^s}{s!} = \begin{bmatrix} e^{M_{11}} & -\frac{1}{2}M_{11}^{-1}(e^{M_{11}} - e^{-M_{11}})M_{12} \\ 0 & e^{-M_{11}} \end{bmatrix}.$$

Thus, $A(t) = 0$ and $C(t) = r_j t$ through the expression (3.25) and the $A_{ij}(t)$ above where r_j is the j-th component of vector r.

Hence $(3.24) = S_{j,0}e^{r_j t}$ and the result follows by using identification method.

3.4.6. *Method of Monte Carlo.*

We estimate the models by the C.GMM method in the work of [1].

After compilation, the graphs of CGMM estimation criterion of WASC, the models (3.8) and (3.9) are identical represented by figure 2.

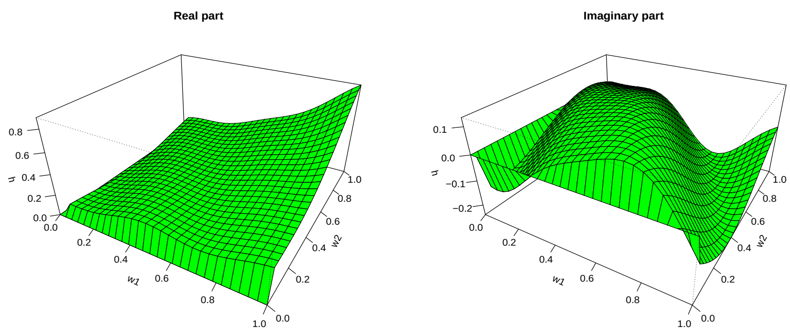

FIGURE 2. C-GMM estimation criterion

These figures show us the values taken by real and imaginary part of the empirical moment of continuum of C-GMM method. Thus, we can minimize this function.

TABLE 5. C-GMM estimator $\hat{\theta}_1$

parameter	WASC	Model (3.8)	Model (3.9)
ρ_1	0.01678155	0.7	0.7
ρ_2	-0.07435433	-0.7	-0.696520888
Q_{11}	0.1	0.1	0.1
Q_{12}	0.00993433	0.0009982766	0.001459835
Q_{21}	0	0	0
Q_{22}	0.1	0.1	0.1
Φ_{11}	-5.32958516	-6.508131	-6.072791349
Φ_{12}	-0.09259241	-0.1	-0.1
Φ_{21}	-0.07924856	-0.1	-0.1
Φ_{22}	-23.54376148	-23.99445	-23.730125497
ν	2	2	2
objective	3.765762×10^{-6}	2.847425×10^{-6}	2.990823×10^{-6}

The Table 7 presents the estimated parameters of model with its standard deviations errors.

Table 6: C-GMM estimator $\hat{\theta}$

parameter	estimator of WASC	standard deviation error	estimator of Model (3.8)	standard deviation error
ρ_1	0.3802324	9.164767×10^{-13}	0.68406	1.603321×10^{-7}
ρ_2	0.1899477	9.72752×10^{-13}	0.6916546	1.324813×10^{-7}
Q_{11}	$0,1$	3.211584×10^{-13}	0.1	3.394635×10^{-8}
Q_{12}	0.001274546	2.654682×10^{-13}	0	2.386973×10^{-8}
Q_{21}	9.328739×10^{-6}	4.962719×10^{-13}	6.229411×10^{-3}	2.678919×10^{-8}
Q_{22}	$0,1$	7.093314×10^{-13}	0.1	2.175998×10^{-8}
Φ_{11}	-20.40753	2.189006×10^{-13}	-27.37276	2.495404×10^{-8}
Φ_{12}	$-0,1$	1.321677×10^{-13}	-9.999269×10^{-2}	1.290915×10^{-8}
Φ_{21}	$-0,1$	1.683202×10^{-13}	-9.999135×10^{-2}	1.403338×10^{-8}
Φ_{22}	-24.25246	2.108821×10^{-13}	-25.97838	1.824978×10^{-8}
ν	2	10.03475	2	0.7836279
objective	5.183242×10^{-6}		8.543687×10^{-6}	

estimator of Model (3.9)	standard deviation error
0.7	7.227369×10^{-12}
0.496825366	7.000169×10^{-12}
0.1	8.68137×10^{-13}
0.004483379	8.350945×10^{-13}
0	1.191963×10^{-12}
0.1	7.956168×10^{-13}
-17.602156567	7.723051×10^{-13}
-0.1	3.585049×10^{-13}
-0.1	8.757168×10^{-13}
-23.479428046	8.776381×10^{-13}
2	1.050625
5.411212×10^{-6}	

From the expression of correlations, the asset and its volatility (resp correlation) are positively correlated.

3.4.7. *European call option of the basket Nasdaq and S&P500.*

Let be a European call of the basket of indexes (Nasdaq, S&P500) and note by K the strike of index quoted by points. We use the spread option in the work of [4].

TABLE 8. Spread option of Nasdaq and S&P500 indexes.

$\epsilon_1 = -3$; $\epsilon_2 = 1$ and $\bar{u} = 39, 28664$. $S_0^1 = 14120.81$ and $S_0^2 = 4255.28$ and $K = 9853.265$.

Model	T=1 (day)	T=2	T=3	T=4
WASC	29.23415	29.25375	29.26164	29.26439
Model (3.8)	35.80677	39.89873	42.83697	45.10411
Model (3.9)	33.47105	34.99683	35.7779	36.25174

4. DISCUSSION AND CONCLUSION

In this article, we have proposed a property of a process: asymptotic stability.

We have shown the logic, the existence, the importance and the contribution of this property. For the first, we explained mathematically as being a stationary process which verifies or not a condition. If it does verify, it is said to be stable and otherwise it is not stable. The stability is marked by its strong or weak type. We use them in the field of finance where we have shown its importance and its impact namely: the realism and the change in the value of an option price.

This suggested process may be useful for a wider class of models beyond financial engineering including e.g. the management of the smart grid and oil production.

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