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A POISSON ALGEBRA STRUCTURE OVER THE EXTERIOR ALGEBRA OF A QUADRATIC SPACE

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ABSTRACT. We construct a Poisson algebra structure of degree -2 over the exterior algebra of a quadratic space. Here we do not use Clifford algebra as in [4].

1. INTRODUCTION

A graded Lie algebra of degree $-\tau$, where $\tau \geq 0$ is an integer, over a commutative field $\mathbb{K}$, is a graded vector space $\mathcal{G} = \bigoplus\limits_{n\in\mathbb{N}} \mathcal{G}^n$ together with a bilinear map

$$[,] : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}, (x, y) \longmapsto [x, y],$$

called bracket and which satisfies the following conditions:

(1) $[\mathcal{G}^p, \mathcal{G}^q] \subset \mathcal{G}^{p+q-\tau}$;
(2) $[x, y] = -(-1)^{(p-\tau)\cdot(q-\tau)} [y, x], x \in \mathcal{G}^p, y \in \mathcal{G}^q$;
(3) $(-1)^{(p-\tau)(r-\tau)} [x, [y, z]] + (-1)^{(q-\tau)(p-\tau)} [y, [z, x]]$
$+ (-1)^{(r-\tau)(q-\tau)} [z, [x, y]] = 0, x \in \mathcal{G}^p, y \in \mathcal{G}^q, z \in \mathcal{G}^r$.

The identity (3) is equivalent to the following:

$$[x, [y, z]] = [[x, y], z] + (-1)^{(p-\tau)(q-\tau)} [y, [x, z]].$$

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A commutative algebra structure over $\mathcal{G}$ of degree $-\tau$ is the data of a multiplication, denoted by $\cdot$, over $\mathcal{G}$ satisfying

$$x \cdot y = (-1)^{(p-\tau)\cdot(q-\tau)} y \cdot x,$$

with $x \in \mathcal{G}^p, y \in \mathcal{G}^q$.

A Poisson algebra structure of degree $-\tau$ over $\mathcal{G}$ is simultaneously the data of a graded Lie algebra structure of degree $-\tau$ and a graded commutative algebra of degree $-\tau$ over $\mathcal{G}$ satisfying

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{(p-\tau)\cdot q} y \cdot [x, z]\,,$$

with $x \in \mathcal{G}^p, y \in \mathcal{G}^q$.

The goal of the present paper is to show that the exterior algebra of a quadratic space admits a Poisson structure of degree -2.

We organize this paper as follows. In Section 2, we present the notion of extension of the Lie bracket. In Section 3, we recall the definition of a quadratic space. Finally Section 4 deals with Poisson bracket on $\bigwedge(E)$.

2. Extension of the Lie bracket

Let V be a finite-dimensional (complex or real) vector space, and let V^* be its dual vector space.

We consider the exterior algebra of the direct sum of V and V^*

$$\bigwedge \left(V \bigoplus V^*\right) = \bigoplus_{n=-2}^{\infty} \Big(\bigoplus_{p+q=n} \big(\bigwedge^{q+1} V^* \bigoplus \bigwedge^{p+1} V\big)\Big). \tag{2.1}$$

We say that an element of $\bigwedge (V \bigoplus V^*)$ is of bidegree (p, q) and of degree $n = p + q$ if it belongs to $\bigwedge^{q+1} V^* \bigoplus \bigwedge^{p+1} V$. Thus elements of the base field are of bidegree $(-1, -1)$, elements of V (resp. V^*) are of bidegree $(0, -1)$ (resp. $(-1, 0)$), and a linear map $\mu : \bigwedge^2 V \longrightarrow V$ (resp. $\gamma : V \longrightarrow \bigwedge^2 V$) can be considered to be an element of $\bigwedge^2 V^* \bigoplus V$ (resp. $V^* \bigoplus \bigwedge^2 V$) which is of bidegree $(0, 1)$ (resp. $(1, 0)$).

Proposition 2.1. *[3] On the graded vector space $\bigwedge (V \oplus V^*)$ there exists a unique graded Lie bracket, called the big bracket, such that*

(i) *if $x, y \in V$, $[x, y] = 0$,*
(ii) *if $\zeta, \eta \in V^*$, $[\zeta, \eta] = 0$,*

(iii) *if* $x \in V, \eta \in V^*, [x, \eta] =< \eta, x >$,

(iv) *if* $u, v, w \in \bigwedge (V \oplus V^*)$ *are of degree* $|u|, |v|, |w|$ *respectively, then*

$$[u, v \wedge w] = [u, v] \wedge w + (-1)^{|u||v|} v \wedge [u, w] . \tag{2.2}$$

This last formula is called the graded Leibniz rule. The following proposition lists important properties of the big bracket.

Proposition 2.2. *[3] Let* $[\cdot, \cdot]$ *denote the big bracket. Then*

(i) $\mu : \bigwedge^2 V \longrightarrow V$ *is a Lie bracket if and only if* $[\mu, \mu] = 0$.

(ii) ${}^t\gamma : \bigwedge^2 V^* \longrightarrow V^*$ *is a Lie bracket if and only if* $[\gamma, \gamma] = 0$.

(iii) *Let* $\mathcal{G} = (V, \mu)$ *be a Lie algebra. Then* γ *is a* 1*-cocycle of* $\mathcal{G}$ *with values in* $\bigwedge^2 \mathcal{G}$*, where* $\mathcal{G}$ *acts on* $\bigwedge^2 \mathcal{G}$ *by the adjoint action, if and only if* $[\mu, \gamma] = 0$.

By the graded commutativity of the big bracket,

$$[\mu, \gamma] = [\gamma, \mu] . \tag{2.3}$$

By the bilinearity and graded skew-symmetry of the big bracket, one has

$$[\mu + \gamma, \mu + \gamma] = [\mu, \mu] + 2 [\mu, \gamma] + [\gamma, \gamma] . \tag{2.4}$$

Using the bigrading of $\bigwedge (V \oplus V^*)$, we see that the conditions

$$[\mu + \gamma, \mu + \gamma] = 0 \tag{2.5}$$

and

$$[\mu, \mu] = 0,\ [\mu, \gamma] = 0,\ [\gamma, \gamma] = 0 \tag{2.6}$$

are equivalent.

Lemma 2.1. *Let* $\mathcal{G} = (V, \mu)$ *be a Lie algebra. Then:*

(i) *The map* $d_\mu : a \longmapsto [\mu, a]$ *is a derivation of degree* 1 *and of square* 0 *of the graded Lie algebra* $\bigwedge (V \oplus V^*)$.

(ii) *If* $a \in \bigwedge V$*, then* $d_\mu a = -\delta a$*, where* δ *is the Lie algebra cohomology operator.*

(iii) *For* $a, b \in \bigwedge V$*, let us set*

$$[[a, b]] = [[a, \mu] , b] . \tag{2.7}$$

Then $[[\cdot, \cdot]]$ *is a graded Lie bracket of degree* 1 *on* V *extending the Lie bracket of* $\mathcal{G}$.

3. Quadratic Space

In the following E denotes a vector space over a commutative field $\mathbb{K}$ with a characteristic different from 2 and $\bigwedge(E) = \bigoplus_{n\in\mathbb{N}} \bigwedge^n(E)$ denotes the exterior algebra of E.

Recall that a derivation of $\bigwedge(E)$ of degree r, with $r \in \mathbb{Z}$, is a linear map

$$d : \bigwedge(E) \longrightarrow \bigwedge(E)$$

of degree r satisfying

$$d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^{p\cdot r}\alpha \wedge d(\beta)$$

for all $\alpha \in \bigwedge^p(E)$ and for all $\beta \in \bigwedge(E)$.

It is the same to say that a linear map

$$d : \bigwedge(E) \longrightarrow \bigwedge(E)$$

is a derivation of degree r if and only if d is of degree r and that

$$d(y_1 \wedge \ldots \wedge y_q) = \sum_{j=1}^{q} (-1)^{(j-1)\cdot r}\, y_1 \wedge \ldots \wedge y_{j-1} \wedge d(y_j) \wedge y_{j+1} \wedge \ldots \wedge y_q \tag{3.1}$$

for all $q \in \mathbb{N}$.

Recall that a quadratic form on E is a map $q : E \longrightarrow \mathbb{K}$ such that:

1) $q(\lambda \cdot x) = \lambda^2 \cdot q(x), \lambda \in \mathbb{K}, x \in E$;
2) the map

$$E \times E \longrightarrow \mathbb{K}, (x, y) \longmapsto \frac{1}{2}\left[q(x+y) - q(x) - q(y)\right],$$

is a symmetric bilinear form.

A quadratic space structure on E is given by a symmetric bilinear form f on E.

In this case we say that the pair (E, f) is a quadratic space.

Proposition 3.1. *If (E, f) is a quadratic space, then the map*

$$q_f : E \longrightarrow \mathbb{K}, x \longmapsto f(x, x),$$

is a quadratic form.

Proof. Simple check. □

4. Poisson bracket on $\bigwedge(E)$

In the following (E, f) is a quadratic space. For $x \in E$ and for $q \geq 1$ an integer, we have:

Proposition 4.1. *The map*

$$\begin{aligned} E^q &\longrightarrow \bigwedge^{q-1}(E), \\ (y_1, \ldots, y_q) &\longmapsto \sum_{j=1}^{q} (-1)^{j-1} f(x, y_j)\, y_1 \wedge \ldots \wedge \widehat{y_j} \wedge \ldots \wedge y_q \end{aligned} \tag{4.1}$$

is alternating multilinear. So there is a unique linear map

$$f_x^q : \bigwedge^{q}(E) \longrightarrow \bigwedge^{q-1}(E) \tag{4.2}$$

such that

$$f_x^q(y_1 \wedge \ldots \wedge y_q) = \sum_{j=1}^{q} (-1)^{j-1} f(x, y_j)\, y_1 \wedge \ldots \wedge \widehat{y_j} \wedge \ldots \wedge y_q. \tag{4.3}$$

Proof. The proof is straightforward. □

For $x = 0$, one has $f_x^q = 0$.

We set

$$f_x = f_x^1 + f_x^2 + \cdots + f_x^q + \cdots .$$

Thus $f_x : \bigwedge(E) \longrightarrow \bigwedge(E)$ is a linear map of degree -1 with $f_x\big|_{\bigwedge^q(E)} = f_x^q$.

Proposition 4.2. *The linear map*

$$f_x : \bigwedge(E) \longrightarrow \bigwedge(E) \tag{4.4}$$

is a derivation of degree -1.

Proof. We have

$$\begin{aligned} f_x(y_1\wedge\ldots\wedge y_q) &= f_x^q(y_1\wedge\ldots\wedge y_q) \\ &= \sum_{j=1}^{q}(-1)^{j-1}\, f\,(x,y_j)\, y_1\wedge\ldots\wedge\widehat{y_j}\wedge\ldots\wedge y_q \\ &= \sum_{j=1}^{q}(-1)^{j-1}\, y_1\wedge\ldots\wedge y_{j-1}\wedge f(x,y_j)\wedge y_{j+1}\wedge\ldots\wedge y_q \\ &= \sum_{j=1}^{q}(-1)^{j-1}\, y_1\wedge\ldots\wedge y_{j-1}\wedge f_x(y_j)\wedge y_{j+1}\wedge\ldots\wedge y_q. \end{aligned}$$

Considering (3.1), we deduce that f_x is a derivation of degree -1. □

For a decomposable element $x_1\wedge\ldots\wedge x_p\in\bigwedge^p(E), p\geq 1$, we have:

Proposition 4.3. *The map*

$$\begin{aligned} E^q \longrightarrow \bigwedge^{q-2}(E),\ &(y_1,\ldots,y_q) \\ &\longmapsto -(-1)^p\sum_{j=1}^{q}(-1)^{j-1}\, f^p_{y_j}\,(x_1\wedge\ldots\wedge x_p)\, y_1\wedge\ldots\wedge\widehat{y_j}\wedge\ldots\wedge y_q \end{aligned} \tag{4.5}$$

being alternating multilinear, then there exists a unique linear map

$$f^q_{x_1\wedge\ldots\wedge x_p}:\bigwedge^{q}(E)\longrightarrow\bigwedge^{q-2}(E) \tag{4.6}$$

such that

$$\begin{aligned} &f^q_{x_1\wedge\ldots\wedge x_p}\,(y_1\wedge\ldots\wedge y_q) \\ &= -(-1)^p\sum_{j=1}^{q}(-1)^{j-1}\, f^p_{y_j}\,(x_1\wedge\ldots\wedge x_p)\, y_1\wedge\ldots\wedge\widehat{y_j}\wedge\ldots\wedge y_q. \end{aligned} \tag{4.7}$$

Moreover, for $p\geq 1$ and $q\geq 1$, we have

$$f^q_{x_1\wedge\ldots\wedge x_p}\,(y_1\wedge\ldots\wedge y_q) = -(-1)^{pq}\cdot f^p_{y_1\wedge\ldots\wedge y_q}\,(x_1\wedge\ldots\wedge x_p)\,. \tag{4.8}$$

Proof. The proof of the existence and uniqueness of the linear map $f^q_{x_1\wedge\ldots\wedge x_p}$ is obvious.

On the other hand, for the proof of the last assertion, one has

$$
\begin{aligned}
& f^q_{x_1\wedge\ldots\wedge x_p}(y_1\wedge\ldots\wedge y_q) \\
&= (-1)^p \sum_{i=1}^{p}\sum_{j=1}^{q} (-1)^{i+j-1} f(x_i,y_j)\cdot x_1\wedge\ldots\wedge\widehat{x_i}\wedge\ldots\wedge x_p\wedge y_1\wedge\ldots\wedge\widehat{y_j}\wedge\ldots\wedge y_q \\
&= (-1)^p \sum_{i=1}^{p}\sum_{j=1}^{q} (-1)^{i+j-1}\cdot(-1)^{(p-1)(q-1)} f(y_j,x_i)\cdot y_1\wedge\ldots\wedge\widehat{y_j}\wedge\ldots\wedge y_q \\
&\quad \wedge x_1\wedge\ldots\wedge\widehat{x_i}\wedge\ldots\wedge x_p \\
&= -(-1)^{pq}\cdot(-1)^q \sum_{i=1}^{p}\sum_{j=1}^{q} (-1)^{i+j-1} f(y_j,x_i)\cdot y_1\wedge\ldots\wedge\widehat{y_j}\wedge\ldots\wedge y_q \\
&\quad \wedge x_1\wedge\ldots\wedge\widehat{x_i}\wedge\ldots\wedge x_p \\
&= -(-1)^{pq}\cdot f^p_{y_1\wedge\ldots\wedge y_q}(x_1\wedge\ldots\wedge x_p),
\end{aligned}
$$

as desired. □

We set $f_{x_1\wedge\ldots\wedge x_p} = f^1_{x_1\wedge\ldots\wedge x_p} + f^2_{x_1\wedge\ldots\wedge x_p} + \cdots + f^q_{x_1\wedge\ldots\wedge x_p} + \cdots$. From (4.8), we deduce by linearity the following result:

Corollary 4.1. *For $\alpha\in\bigwedge^p(E)$ and $\beta\in\bigwedge^q(E)$, with $p\geq 1$ and $q\geq 1$, we have:*

$$f_\alpha(\beta) = -(-1)^{p\cdot q} f_\beta(\alpha). \tag{4.9}$$

Thus $f_{x_1\wedge\ldots\wedge x_p} : \bigwedge(E)\longrightarrow\bigwedge(E)$ is a linear map of degree $p-2$ with

$$f_{x_1\wedge\ldots\wedge x_p}\big|_{\bigwedge^q(E)} = f^q_{x_1\wedge\ldots\wedge x_p}.$$

Proposition 4.4. *The linear map*

$$f_{x_1\wedge\ldots\wedge x_p} : \bigwedge(E)\longrightarrow\bigwedge(E)$$

is a derivation of degree $p-2$.

Proof. One has

$$
\begin{aligned}
& f_{x_1\wedge\ldots\wedge x_p}\left(y_1\wedge\ldots\wedge y_q\right) \\
= \; & f^q_{x_1\wedge\ldots\wedge x_p}\left(y_1\wedge\ldots\wedge y_q\right) \\
= \; & -\left(-1\right)^p \sum_{j=1}^{q} \left(-1\right)^{j-1} f^p_{y_j}\left(x_1\wedge\ldots\wedge x_p\right)\wedge y_1\wedge\ldots\wedge\widehat{y_j}\wedge\ldots\wedge y_q \\
= \; & -\left(-1\right)^p \sum_{j=1}^{q} \left(-1\right)^{(j-1)\cdot p} y_1\wedge\ldots\wedge y_{j-1}\wedge f^p_{y_j}\left(x_1\wedge\ldots\wedge x_p\right)\wedge y_{j+1}\wedge\ldots\wedge y_q \\
= \; & -\left(-1\right)^p \sum_{j=1}^{q} \left(-1\right)^{(j-1)\cdot p} y_1\wedge\ldots\wedge y_{j-1}\wedge\left[-\left(-1\right)^p f^1_{x_1\wedge\ldots\wedge x_p}(y_j)\right]\wedge y_{j+1} \\
& \wedge\ldots\wedge y_q \\
= \; & \sum_{j=1}^{q} \left(-1\right)^{(j-1)\cdot p} y_1\wedge\ldots\wedge y_{j-1}\wedge f_{x_1\wedge\ldots\wedge x_p}(y_j)\wedge y_{j+1}\wedge\ldots\wedge y_q \\
= \; & \sum_{j=1}^{q} \left(-1\right)^{(j-1)(p-2)} y_1\wedge\ldots\wedge y_{j-1}\wedge f_{x_1\wedge\ldots\wedge x_p}(y_j)\wedge y_{j+1}\wedge\ldots\wedge y_q,
\end{aligned}
$$

as required. □

We denote, $Der_{\mathbb{K}}\left[\bigwedge(E)\right]$, the space of derivations (of all degrees) of $\bigwedge(E)$.

Proposition 4.5. *The map*

$$E^p \longrightarrow Der_{\mathbb{K}}\left[\bigwedge(E)\right], (x_1,\ldots,x_p)\longmapsto f_{x_1\wedge\ldots\wedge x_p},$$

is alternating multilinear. Thus there exists a unique linear map

$$\widetilde{f}^p : \bigwedge^p(E)\longrightarrow Der_{\mathbb{K}}\left[\bigwedge(E)\right] \tag{4.10}$$

such that

$$\widetilde{f}^p\left(x_1\wedge\ldots\wedge x_p\right) = f_{x_1\wedge\ldots\wedge x_p}. \tag{4.11}$$

Proof. The proof is obvious. □

We set $\widetilde{f} = \widetilde{f}^1 + \widetilde{f}^2 + \cdots + \widetilde{f}^p + \cdots$. So when $\alpha\in\bigwedge^p(E)$, then $\widetilde{f}(\alpha)$ is a derivation of $\bigwedge(E)$ of degree $p-2$.

For $\alpha \in \bigwedge(E)$ and $\beta \in \bigwedge(E)$, we set

$$[\alpha, \beta]_f = [\widetilde{f}(\alpha)](\beta). \tag{4.12}$$

We will, subsequently, show that this bracket defines a Poisson structure of degree -2 on $\bigwedge(E)$.

Note that when $\alpha \in \bigwedge^p(E)$, then

$$\widetilde{f}(\alpha) = f_\alpha. \tag{4.13}$$

By construction, we have:

$$\left[\mathbb{K}, \bigwedge(E)\right]_f = 0. \tag{4.14}$$

Theorem 4.1. *The map*

$$\bigwedge(E) \times \bigwedge(E) \longrightarrow \bigwedge(E), (\alpha, \beta) \longmapsto [\alpha, \beta]_f, \tag{4.15}$$

is bilinear and of degree -2.

Proof. The proof is immediate. □

Theorem 4.2. *For $\alpha \in \bigwedge^p(E)$ and $\beta \in \bigwedge^q(E)$, then*

$$[\alpha, \beta]_f = -(-1)^{p \cdot q} [\beta, \alpha]_f. \tag{4.16}$$

Proof. This follows from (4.13) and Corollary 4.1. □

Theorem 4.3. *For $\alpha \in \bigwedge^p(E), \beta \in \bigwedge^q(E)$ and $\gamma \in \bigwedge(E)$, then*

$$[\alpha, \beta \wedge \gamma]_f = [\alpha, \beta]_f \wedge \gamma + (-1)^{p \cdot q} \beta \wedge [\alpha, \gamma]_f. \tag{4.17}$$

Proof. Since $\widetilde{f}(\alpha)$ is a derivation of degree $p-2$, then we have

$$\begin{aligned}
[\alpha, \beta \wedge \gamma]_f &= [\widetilde{f}(\alpha)](\beta \wedge \gamma) \\
&= [\widetilde{f}(\alpha)](\beta) \wedge \gamma + (-1)^{(p-2) \cdot q} \beta \wedge [\widetilde{f}(\alpha)](\gamma) \\
&= [\alpha, \beta]_f \wedge \gamma + (-1)^{p \cdot q} \beta \wedge [\alpha, \gamma]_f.
\end{aligned}$$

Hence the result. □

Theorem 4.4. *For $\alpha \in \bigwedge^p(E), \beta \in \bigwedge^q(E)$, then*

$$\left[\widetilde{f}(\alpha), \widetilde{f}(\beta)\right] = \widetilde{f}([\alpha, \beta]_f) \tag{4.18}$$

where
$$\left[\widetilde{f}(\alpha),\widetilde{f}(\beta)\right]=\widetilde{f}(\alpha)\circ\widetilde{f}(\beta)-(-1)^{p\cdot q}\widetilde{f}(\beta)\circ\widetilde{f}(\alpha).$$

Proof. Taking into account (3.1), for all $z\in E$, we check that
$$\left[\widetilde{f}(\alpha),\widetilde{f}(\beta)\right](z)=\widetilde{f}([\alpha,\beta]_f)(z).$$
The result follows. □

Theorem 4.5. *The pair $(\bigwedge(E),[,]_f)$ is a Poisson algebra of degree -2.*

Proof. Theorems 4.1, 4.2 and 4.4 mean that the pair $(\bigwedge(E),[,]_f)$ is a graded Lie algebra of degree -2. Theorem 4.3 means that the triple $(\bigwedge(E),[,]_f,\wedge)$ is a Poisson algebra of degree -2. □

As $(\bigwedge(E),[,]_f)$ is a graded Lie algebra, we denote $\widetilde{f}$ by ad_f. Thus we have $[ad_f(\alpha)](\beta)=[\alpha,\beta]_f$ and for $\alpha\in\bigwedge^p(E)$, the linear map

$$ad_f(\alpha):\bigwedge(E)\longrightarrow\bigwedge(E) \tag{4.19}$$

is simultaneously a derivation (of degree $p-2$) of graded Lie algebra and graded commutative Lie algebra.

An element $M\in\bigwedge^3(E)$ is said to be a proto-Lie bialgebra of the quadratic space (E,f) when $[M,M]_f=0$. In this case, we say that the quadruple $(\bigwedge(E),[,]_f,\wedge,M)$ is a proto-Lie bialgebra (for further details, we refer to [3] and references therein).

Proposition 4.6. *When the quadruple $(\bigwedge(E),[,]_f,\wedge,M)$ is a proto-Lie bialgebra, then the map*

$$ad_f(M):\bigwedge(E)\longrightarrow\bigwedge(E),P\longmapsto[M,P]_f, \tag{4.20}$$

is a coboundary operator.

Proof. The map $ad_f(M)$ is obviously of degree $+1$. Since $ad_f(M)$ is a derivation of graded Lie algebra, then for $P\in\bigwedge(E)$, we have

$$\begin{aligned}
[ad_f(M)]^2(P) &= \left[M,[M,P]_f\right]_f\\
&= \left[[M,M]_f,P\right]_f+(-1)^{3\times3}\left[M,[M,P]_f\right]_f\\
&= -\left[M,[M,P]_f\right]_f\\
&= -[ad_f(M)]^2(P).
\end{aligned}$$

We deduce that $[ad_f(M)]^2(P) = 0$. Since P is arbitrary, it follows that $[ad_f(M)]^2 = 0$. This means that $ad_f(M)$ is a coboundary operator. $\square$

For $p \in \mathbb{N}$, we denote

$$H_f^p(M) = Ker([ad_f(M)]|_{\bigwedge^p(E)})/Im([ad_f(M)]|_{\bigwedge^{p-1}(E)}) \tag{4.21}$$

the cohomology space of degree p.

Proposition 4.7. *We have:*

(1) $H_f^0(M) = \mathbb{K}$;
(2) $H_f^1(M) = Ker([ad_f(M)]|_{\bigwedge^1(E)})$.

Proof. Simple check. $\square$

When V is a vector space over $\mathbb{K}$ and when V^* is the dual of V, then for $E = V + V^*$, the map

$$E \times E \longrightarrow \mathbb{K}, (v + \phi, w + \psi) \longmapsto \phi(w) + \psi(v), \tag{4.22}$$

is a symmetric bilinear form.

The Poisson bracket over $\bigwedge(E)$ defined by (4.22) is called "Big bracket" [3].

REFERENCES

[1] N. BOURBAKI: *Eléments de Mathématique : Algèbre, chapitres* 1 *à* 3, Springer-Verlag, 2007.

[2] M. HENNAUX: *Hamiltonian form of the path integral for theories with a gauge freedom*, Phys. Rep., **126**(1) (1985), 1–66.

[3] Y. KOSMANN-SCHWARZBACH: *Jacobian quasi-bialgebras and quasi Poisson Lie groups*, Contemp. Math., Amer. Math. Soc., **132** (1992), 459–489.

[4] B. KOSTANT, S. STERNBERG: *Symplectic reduction, BRS cohomology, and infinite dimensional Clifford algebras*, Ann. Physics., **176**(01) (1987), 49–113.

[5] E. OKASSA: *Generalized Lie-Rinehart algebras and applications in differential geometry*, English Edition, 2022.

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