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AN EXAMPLE OF LOCALLY CONFORMALLY SYMPLECTIC MANIFOLDS

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ABSTRACT. Our aim in this paper is to give an example of locally conformally symplectic manifolds.

1. INTRODUCTION

The notion of locally conformally symplectic manifold was introduced in [6] and has been studied extensively by Vaisman and many others (see e.g. [1, 2, 5, 10, 13]). Locally conformally symplectic manifolds are generalized phase spaces of hamiltonian dynamical systems since the form of the hamiltonian equations is then preserved by homothetic canonical transformations [13]. We recall that a smooth manifold M is a locally conformally symplectic manifold if there exist a d-closed 1-form

$$\alpha : \mathfrak{X}(\mathrm{M}) \longrightarrow C^{\infty}(\mathrm{M}),$$

and a nondegenerate 2-form

$$\Omega : \mathfrak{X}(\mathrm{M}) \times \mathfrak{X}(\mathrm{M}) \longrightarrow C^{\infty}(\mathrm{M}),$$

such that

$$d\Omega = -\alpha \wedge \Omega,$$

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where d is the exterior differentiation operator. The 1-form α is called the Lee form [6, 13]. The triple $(\mathrm{M}, \alpha, \Omega)$ is called a locally conformally symplectic manifold. In particular, if α is an exact 1-form on M, i.e., $\alpha = df$ for some smooth function f on M then Ω is called globally conformally symplectic form on M and it is straightforward to verify that $e^{-f} \cdot \Omega$ is a symplectic form on M. The 1-form α is unique. This implies that α is uniquely determined by Ω on a smooth manifold M of dimension at least 4. The dimension of a locally conformally symplectic manifold has to be even. Since Ω^n is nowhere vanishing, a locally conformally symplectic manifold possesses a canonic orientation [9]. For first properties and examples of locally conformally symplectic manifolds, we refer the reader to [3, 7, 8, 12]. We organize this paper as follows. In Section 2, we study some properties of the Lichnerowicz-de Rham differential. Section 3 deals with the study of example for locally conformally symplectic manifolds.

2. Properties of the cohomology operator d_α

A differential form η of degree p defines a multilinear skew-symmetric function

$$\eta : \underbrace{\mathfrak{X}(\mathrm{M}) \times \cdots \times \mathfrak{X}(\mathrm{M})}_{p \; times} \longrightarrow C^\infty(\mathrm{M}).$$

Its exterior derivative $d\eta$ is defined as follows:

$$d\eta : \underbrace{\mathfrak{X}(\mathrm{M}) \times \cdots \times \mathfrak{X}(\mathrm{M})}_{(p+1) \; times} \longrightarrow C^\infty(\mathrm{M})$$

is the function defined by the formula

$$\begin{aligned}(d\eta)(X_1, \ldots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i-1} X_i \left[\eta(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1})\right] \\ &+ \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{p+1})\end{aligned}$$

for any $X_1, \ldots, X_{p+1} \in \mathfrak{X}(\mathrm{M})$, where the sign $\widehat{}$ indicates the absence of the respective arguments [11].

Proposition 2.1. *When $\Lambda(\mathrm{M})$ is the $C^\infty(\mathrm{M})$-module of differential forms on M and when d is the exterior differentiation operator then for any $\eta \in \Lambda(\mathrm{M})$, we have*

$$d_\alpha\eta = d\eta + \alpha \wedge \eta.$$

Corollary 2.1. *The* 1*-form* α *is* d_α*-closed if, and only if,* α *is* d*-closed.*

Corollary 2.2. *The* 1*-form* α *is* d*-closed if, and only if,* $d_\alpha \circ d_\alpha = 0$.

Proposition 2.2. *We have the following properties:*

(1) $d_\alpha 1 = \alpha$;
(2) $d_\alpha(\xi \wedge \gamma) = (d_\alpha\xi) \wedge \gamma + (-1)^{|\xi|}\xi \wedge (d_\alpha\gamma) - (-1)^{|\xi\wedge\gamma|}\xi \wedge \gamma \wedge d_\alpha 1$;

for any ξ *and* γ *homogeneous.*

Proof. One uses the Proposition 2.1, we have first

$$d_\alpha 1 = d1 + 1 \cdot \alpha = \alpha.$$

And for any ξ and γ homogeneous

$$d_\alpha(\xi \wedge \gamma) = (d\xi) \wedge \gamma + (-1)^{|\xi|}\xi \wedge (d\gamma) + \alpha \wedge \xi \wedge \gamma.$$

That ends the proof. □

The essential difference between d and d_α is that d_α does not satisfy a Stokes' theorem. Let us introduce the linear map

$$\tau : C^\infty(\mathrm{M}) \longrightarrow Ham(\mathrm{M}), f \longmapsto X_f,$$

where $Ham(\mathrm{M})$ is the Lie algebra of hamiltonian vector fields on M, for more details see [4].

Theorem 2.1. *Define* $I_\alpha := \{f \in C^\infty(\mathrm{M}),\ d_\alpha f = 0\}$.

(1) *The set* I_α *is an ideal of the Lie algebra* $(\mathrm{C}^\infty(\mathrm{M}), \{,\})$ *and this ideal is the kernel of the homomorphism* τ.
(2) *The quotient* $C^\infty(\mathrm{M})/I_\alpha$ *is a Lie algebra.*

3. Study of the example of locally conformally symplectic manifolds

We denote $(e_1, e_2, ..., e_{2n})$ the canonical basis of $\mathbb{R}^{2n}$ and $(e_1^*, e_2^*, ..., e_{2n}^*)$ the dual basis. For $i = 1, 2, ..., 2n$, e_i^* is the canonical projection

$$pr_i : \mathbb{R}^{2n} \longrightarrow \mathbb{R}, (t_1, t_2, ..., t_{2n}) \longmapsto t_i.$$

Let $\alpha_0 = de^*_{2n}$ and $\Omega_0 = \sum\limits_{i=1}^{n} d_{\alpha_0} e^*_i \wedge de^*_{n+i}$.

Proposition 3.1. *For any vector field X on $\mathbb{R}^{2n}$, we have*

$$\begin{aligned} i_X\Omega_0 = &-\sum_{i=1}^{n} X\left(e^*_{n+i}\right)\cdot de^*_i \\ &+\sum_{i=1}^{n}\left(X\left(e^*_i\right)+e^*_i\cdot X\left(e^*_{2n}\right)-\delta_{ni}\cdot\left[\sum_{j=1}^{n} e^*_j\cdot X\left(e^*_{n+j}\right)\right]\right)\cdot de^*_{n+i}. \end{aligned}$$

Proof. Since

$$i_X\Omega_0 = \sum_{i=1}^{n}\Omega_0\left(X,\frac{\partial}{\partial e^*_i}\right)\cdot de^*_i + \sum_{i=1}^{n}\Omega_0\left(X,\frac{\partial}{\partial e^*_{n+i}}\right)\cdot de^*_{n+i},$$

we have

$$\Omega_0\left(X,\frac{\partial}{\partial e^*_i}\right) = \left(\sum_{j=1}^{n} d_{\alpha_0} e^*_j\wedge de^*_{n+j}\right)\left(X,\frac{\partial}{\partial e^*_i}\right) = -X\left(e^*_{n+i}\right)$$

and

$$\begin{aligned} \Omega_0\left(X,\frac{\partial}{\partial e^*_{n+i}}\right) &= \left(\sum_{j=1}^{n} d_{\alpha_0} e^*_j\wedge de^*_{n+j}\right)\left(X,\frac{\partial}{\partial e^*_{n+i}}\right) \\ &= \sum_{j=1}^{n}\left(de^*_j + e^*_j\cdot de^*_{2n}\right)(X)\cdot\delta_{ij} \\ &\quad -\sum_{j=1}^{n}\left(de^*_j + e^*_j\cdot de^*_{2n}\right)\left(\frac{\partial}{\partial e^*_{n+i}}\right)\cdot X\left(e^*_{n+j}\right) \\ &= X\left(e^*_i\right)+e^*_i\cdot X\left(e^*_{2n}\right)-\delta_{ni}\cdot\sum_{j=1}^{n} e^*_j\cdot X\left(e^*_{n+j}\right). \end{aligned}$$

The result follows. □

Proposition 3.2. *The 2-form Ω_0 is nondegenerate.*

Proof. The map

$$\mathfrak{X}\left(\mathbb{R}^{2n}\right)\longrightarrow\Lambda^1\left(\mathbb{R}^{2n}\right), X\longmapsto i_X\Omega_0$$

is injective. Indeed $i_X\Omega_0 = 0$ implies $X\left(e_{n+i}^*\right) = 0$ for any $i = 1, 2, ..., n$ and $X\left(e_i^*\right) + e_i^* \cdot X\left(e_{2n}^*\right) - \delta_{ni} \cdot \left[\sum_{j=1}^{n} e_j^* \cdot X\left(e_{n+j}^*\right)\right] = 0$ for any $i = 1, 2, ..., n$.

Since $X\left(e_{n+i}^*\right) = 0, i = 1, 2, ..., n$ then $X\left(e_{2n}^*\right) = 0$ and $X\left(e_{n+j}^*\right) = 0$ for all $j = 1, 2, ..., n$. We deduce that $X\left(e_i^*\right) = 0$ for $i = 1, 2, ..., n$, so $X = 0$.

The map

$$\mathfrak{X}\left(\mathbb{R}^{2n}\right) \longrightarrow \Lambda^1\left(\mathbb{R}^{2n}\right), X \longmapsto i_X\Omega_0$$

is surjective.

For $\vartheta \in \Lambda^1\left(\mathbb{R}^{2n}\right)$, we verify that if

$$Y = \sum_{i=1}^{n}\left[\vartheta\left(e_{n+i}^*\right) + e_i^* \cdot \vartheta\left(e_n^*\right) - \delta_{ni} \cdot \left(\sum_{j=1}^{n} e_j^* \cdot \vartheta\left(e_j^*\right)\right)\right] \cdot \frac{\partial}{\partial e_i^*} - \sum_{i=1}^{n} \vartheta\left(e_i^*\right) \cdot \frac{\partial}{\partial e_{n+i}^*}$$

we obtain

$$i_Y\Omega_0 = \vartheta.$$

The proof is complete. □

Proposition 3.3. *We get*

$$d_{\alpha_0}\left(\Omega_0\right) = 0.$$

Proof. Since

$$\begin{aligned} d_{\alpha_0}\left(\Omega_0\right) &= d_{\alpha_0}\left(\sum_{i=1}^{n} d_{\alpha_0}e_i^* \wedge de_{n+i}^*\right) \\ &= -\sum_{i=1}^{n}\left[d_{\alpha_0}e_i^* \wedge d_{\alpha_0}\left(de_{n+i}^*\right) + \alpha_0 \wedge d_{\alpha_0}e_i^* \wedge de_{n+i}^*\right] \\ &= 0, \end{aligned}$$

as desired. □

Theorem 3.1. *The triple* $\left(\mathbb{R}^{2n}, \alpha_0, \Omega_0\right)$ *is a locally conformally symplectic manifold.*

Proof. Indeed

$$d\alpha_0 = d\left(de_{2n}^*\right) = d^2\left(e_{2n}^*\right) = 0.$$

This completes the proof. □

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