Advances in Mathematics: Scientific Journal **12** (2023), no.1, 193–216
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.12.1.13

MAXIMUM PRINCIPLE FOR SINGULAR CONTROL PROBLEMS OF SYSTEMS DRIVEN BY MARTINGALE MEASURES

Saloua Labed

ABSTRACT. We provide necessary optimality conditions for singular controlled stochastic differential equations driven by an orthogonal continuous martingale measure. The control is allowed to enter both the drift and diffusion coefficient and has two components, the first being relaxed and the second singular, the domain of the first control does not need to be convex, and for the relaxing method, we show by a counter-example that replacing the drift and diffusion coefficients by their relaxed counterparts does not define a true relaxed control problem. The maximum principle for these systems is established by means of spike variation techniques on the relaxed part of the control and a convex perturbation on the singular one. Our result is a generalization of Peng's maximum principle to singular control problems.

1. INTRODUCTION

The purpose of this paper is to study necessary optimality conditions for control problems of systems satisfying the stochastic differential equation (SDE)

$$dx\left(t\right)=b\left(t,x\left(t\right),u\left(t\right)\right)dt+\sigma\left(t,x\left(t\right),u\left(t\right)\right)dB\left(t\right)+c\left(t\right)d\xi\left(t\right),\quad x(0)=x_{0} \tag{1.1}$$

2020 *Mathematics Subject Classification.* 30C80, 34H05, 60H35, 91G80, 93E20.
Key words and phrases. orthogonal continuous martingale measures, maximum principle, singular control, relaxed control, stochastic differential equation.
Submitted: 15.12.2022; *Accepted:* 30.12.2022; *Published:* 23.01.2023.

on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$, where b and σ are deterministic functions, and ξ is an increasing process, continuous on the left with limits on the right with $\xi_0 = 0$. $(B(t), t \geq 0)$ is a Brownian motion, x_0 is the initial state and the control variable has two components, the first being absolutely continuous and the second singular, we denote it by u and ξ respectively. Our control problem consists in minimizing a cost functional of the form

$$J(u,\xi) = E\left[\int_0^T h\left(t, x\left(t\right), u\left(t\right)\right) dt + g(x\left(T\right)) + \int_0^T R\left(t\right) d\xi\left(t\right)\right], \tag{1.2}$$

over the class $\mathcal{U}_1 \times \mathcal{U}_2$ of admissible controls, that is adapted processes, with values in some compact metric space A, called the action space.

Without convexity condition an optimal control dose not necessarily always exist in $\mathcal{U}_1$, this set is not equipped with a compact topology. The thought is then to introduce a larger class of control processes, in which the controller chooses at time t a probability measure denote $q_t(da)$ on the control set $\mathcal{U}_1$, rather than an element $u \in \mathcal{U}_1$. These are called relaxed controls and have a richer topological structure. The problem now is how we define the relaxed systems associated to the relaxed control.

At first look, one is tempted as in [5] to replace simple the drift and diffusion coefficient by the integrals of the drift and diffusion coefficient with respect to the relaxed control, adopting the same method as in deterministic control, but it will be shown by a simple counter example that the suggested "relaxed" state equation is not continuous with respect to the control variable. This implies in particular that the value functions for the original and relaxed problems are not the same. In addition, there is no mean to prove maximum principle for this model.

So that the proposed "relaxed" model in [5] is not a true extension of the original control problem. The abecedarian reason is that one has to relax the quadratic variation, of the stochastic integral part of the state equation, which is a Lebesgue integral, rather than the stochastic integral itself. Roughly speaking, the idea is to relax the generator of the process, which is intimately linked to the weak solutions of the relaxed stochastic equation, rather than the equation itself. As it will be shown, the stochastic equation associated with the relaxed generator will be governed by a continuous orthogonal martingale measure, rather than a Brownian motion. So, we prove the maximum principle without using approximation as in [5]. The methodology that we used to build up our principle result depends on a double perturbation of the optimal control [2]. The first perturbation is a spike variation on the relaxed control as in [10,14] and the second one is convex, on the singular component as in [1, 5, 6]. For the singular part of the control, we apply the Bensoussan's method [3] to derive a first order adjoint process, and a variational inequality which reduces to the computation of a Gâteaux derivative. For the relaxed part, we use a spike variation method directly on the relaxed optimal control as in [10]. As it will be shown, the stochastic equation associated with the relaxed generator will be governed by a continuous orthogonal martingale measure, rather than a Brownian motion. So, we prove our result with means of spike variation techniques as in [10], then by using a suitable predictable representation theorem for martingale measures [13], we derive the variational equation from the state equation to derive the first and second order adjoint process, which are linear backward stochastic differential equations driven by an orthogonal martingale measure as in [10]. Assembling the adjoint processes, and the variational inequalities, we obtain the stochastic maximum principle.

Our outcome might be viewed as a Peng-type general stochastic maximum principle for relaxed controls [14], to singular control problems. This could be viewed as one of the novelties of this paper.

Our paper is composed as follows. In section 2, first we formulate the relaxed control problem, then we give definition and few properties of a class of orthogonal martingale measures, finally we derive the SDE associated to the relaxed control. In section 3, we get a maximum principle of the Pontriagin type for relaxed controls, to singular control problems , extending the well known Peng stochastic maximum principle to the class of measure-valued controls, to singular control problems.

2. Problem formulation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space, and A be some compact metric space, called the action space. We intrigued by optimality necessary conditions for control problems of systems satisfying (1.1) . The cost function over the time $[0, T]$ is given by (1.2) . The strict control problem may fail to have an optimal solution in the absence of convexity conditions, as shown in the following well known example, taken from deterministic control theory (see [11]).

Example 1. *Minimize the cost function*

$$J(u)=\int_0^T x^u(t)^2dt$$

over the set $\mathcal{U}_1$ *of open loop controls, that is, measurable functions* $u:[0,T]\to\{-1,1\}$.

Let $x^u(t)$ *denote the solution of*

$$dx^u(t)=udt,\qquad x(0)=0.$$

We have $\inf_{u\in\mathcal{U}_1}J(u)=0$. *Indeed consider the following sequence of controls*

$$u_n(t)=(-1)^k;\qquad \textit{if}\ \frac{k}{n}\le t\le\frac{k+1}{n}, 0\le k\le n-1.$$

Then clearly $|x^{u_n}(t)|\le 1/n$ *and* $|J(u_n)|\le T/n^2$ *which implies that* $\inf_{u\in\mathcal{U}_1}J(u)=0$. *There is, still, no control* u *such that* $J(u)=0$.

If this had been the case, then for every t, $x^u(t)=0$. *This in turn would imply that* $u(t)=0$, *which is impossible. The problem is that the sequence* (u_n) *has no limit in the space of strict controls. This limit, if it exists, will be the natural candidate for optimality.*

If we identify $u_n(t)$ *with the Dirac measure* $\delta_{u_n(t)}(da)$ *and set* $q_n(dt,da)$ $=\delta_{u_n(t)}(da)dt$, *we get a measure on* $[0,1]\times\mathcal{A}$. *Then* $(q_n(dt,da))_n$ *converges weakly to* $(1/2)\,dt[\delta_{-1}+\delta_1](da)$.

This suggests that the set $\mathcal{U}_1$ of strict controls is too narrow and should be embedded into a wider class with a richer topological structure, for which the control problem becomes solvable. The idea of relaxed control is to replace the A-valued process $(u(t))$ with $P(A)$-valued process q, where $P(A)$ is the space of probability measures equipped with the topology of weak convergence. Then q may be identified as a random product measure on $[0,T]\times A$, whose projection on $[0,T]$ coincides with Lebesgue measure.

Let $\mathcal{V}$ be the set of Radon measures on $[0,1]\times A$ whose projections on $[0,1]$ coincide with the Lebesgue measure dt, equipped with the topology of stable convergence of measures. It is clear that every $(q(dt,da))$ in $\mathcal{V}$ may be disintegrated as $q(dt,da)=q_t(da)\,dt$, where $q_t(da)$ is a transition probability. $\mathcal{V}$ is a compact metrizable space. Stable convergence is required for bounded measurable functions $h(t,a)$ such that for each fixed $t \in [0,1]$, $h(t,.)$ is continuous, see [8] for further details.

Now, we give the definition of the relaxed control.

Definition 2.1. *A relaxed control is the term* $q = (\Omega, \mathcal{F}, \mathcal{F}_t, P, B_t, q_t, x(t), a)$ *such that*

(1) $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *is a filtered probability space satisfying the usual conditions;*
(2) (q_t) *is an* $P(A)$*-valued process, progressively measurable with respect to* $(\mathcal{F}_t)$*, and such for that for each* t*,* $1_{(0,1]}.q$ *is* $\mathcal{F}_t$*-measurable.*
(3) $(x(t))$ *is* $\mathbb{R}^d$*-valued,* $\mathcal{F}_t$*-adapted, with continuous paths, such that* $x(0) = x_0$ *and*

$$f(x(t)) - f(x_0) - \int_0^t \int_A Lf(s, x(s), a)\, q_s(w, da) ds \tag{2.1}$$

is a P*-martingale, for each* $f \in C_b^2(\mathbb{R}^d, \mathbb{R})$.

We denote by $\mathcal{R}$ the collection of all relaxed controls, and by a slight abuse of notation, we will often denote a relaxed control by q instead of specifying all the components.

Remark 2.1. *The set* $\mathcal{U}_1$ *of strict controls is embedded into the set* $\mathcal{R}$ *of relaxed controls by the mapping*

$$\Psi : u \in \mathcal{U}_1 \rightarrow \Psi(u)(dt, da) = dt\delta_{u(t)}(da) \in \mathcal{R},$$

where δ_u *is the Dirac measure at a single point* u*.*

Let us come back to the precedent example. We have, if we identify $u_n(t)$ with the Dirac measure $dt\delta_{u_n}(da)$, then it is not difficult to prove that the sequence of product measures $(dt\delta_{u_n}(da))_n$ converges weakly to $(1/2)\, dt\, [\delta_{-1} + \delta_1](da)$, see [11] Lemma 1.1, page 20.

Let us define the relaxed model by

$$dx^q(t) = x(0) + \int_0^t ds \int_{\mathcal{U}_1} u q_t(s, da)$$

and the associated relaxed cost is given by

$$J(q) = \int_0^1 [x^q(t)]^2\, dt.$$

Then it is clear that the strict control problem is generalized by the relaxed problem, by taking measures q of the form

$$q\left(dt,du\right)=dt\delta_{u_t}\left(du\right).$$

Moreover if

$$\hat{q}\left(dt,du\right)=\frac{1}{2}dt\left[\delta_{-1}+\delta_{1}\right]\left(du\right),$$

then we have $J(\hat{q})=0$ and $\hat{q}$ as an optimal relaxed control. Moreover since $\inf\limits_{q\in\mathcal{R}} J(q)=0$, then the value functions of the strict and relaxed control problems are the same.

2.1. SDE associated to the relaxed control.

The question asked here is, what is the natural SDE associated to the relaxed control?

As we have see in the precedent example that in the deterministic case or in the stochastic case where only the drift is controlled, one has just to replace in (1.1) the drift by the same drift integrated against the relaxed control. But the difference here is that both the drift and diffusion coefficient are controlled. Let us try a direct relaxation of the original equation (1.1) as in [5]:

$$\begin{cases} [c]l dx\left(t\right)=\int_A b(t,x(t),a)q_t(da)dt+\int_A \sigma(t,x(t),a)q_t(da)dB\left(t\right)+c\left(t\right)d\xi\left(t\right) \\ x\left(0\right)=x_0 \end{cases}$$

this model does not fulfill the requirements of a true relaxed model, as it will be shown in the next example.

Example 2. *Consider the control problem governed by the following SDE without singular terms*

$$\begin{cases} [c]l dX\left(t\right)=u_t dB\left(t\right) \\ X\left(0\right)=x \end{cases},$$

where the control $u\in\mathcal{U}_1$*, the set of measurable functions* $u:[0,1]\to A=[-1,1]$*. The relaxed model will be governed by the equation*

$$\begin{cases} [c]l dX\left(t\right)=\int_A a q_t\left(da\right)dB\left(t\right) \\ X\left(0\right)=x. \end{cases}$$

Consider the following sequence of controls

$$u_n(t) = (-1)^k; \qquad \textit{if } \frac{k}{n} \leq t \leq \frac{k+1}{n}, 0 \leq k \leq n-1,$$

$dt\delta_{u_n}(da)$ *be the relaxed control associated to* $u_n(t)$*, then the sequence* $(dt\delta_{u_n}(da))$ *converges weakly to* $(1/2)\, dt\, [\delta_{-1} + \delta_1](da)$.

It is clear that $X^n(t) = \int_0^t u_n(s)\, dB(s) = \int_0^t \left[\int_A a\delta_{u_n(s)}(da)\right] dB(s)$ *is a continuous martingale with quadratic variation* $\langle X^n, X^n\rangle_t = \int_0^t u_n^2(s)\, ds = t$*. Therefore* $(X^n(t))$ *is a Brownian motion.*

Since the sequence $(dt\delta_{u_n}(da))$ *converges weakly to* $q^* = (1/2)\, dt\, [\delta_{-1} + \delta_1](da)$*. Let* X^* *be the relaxed state process corresponding to the limit* q^**, then*

$$X^*(t) = \int_0^t \int_A a\,(1/2)\,[\delta_{-1} + \delta_1](da)\, dB(s) = 0.$$

It is obvious that the sequence of state processes $(X^n(t))$ *can not converge in* L^2 *to* $X^*(t)$*. Indeed*

$$E\left[|X^n(t) - X^*(t)|^2\right] = E\left[|X^n(t)|^2\right] = E\left[\left|\int_0^t u_n(s)\, dB(s)\right|^2\right]$$
$$= E\left[\int_0^t u_n^2(s)\, ds\right] = t.$$

This implies in particular that the state process is not continuous in the control variable and as a byproduct, the value functions of the strict and "relaxed" control problems are not equal. Moreover, even if the set $\mathcal{V}$ is compact, there is no mean to prove the existence of an optimal control for this model.

What is the right relaxed state process?

The reason why the proposed model in [5] is not a true extension of the original strict control problem, is that the stochastic integral part does not behave as a Lebesgue integral. In fact, one should relax the drift and the quadratic variation of the martingale part, which is a Lebesgue integral. In the relaxed model, the quadratic variation process must be

$$\int_0^t \int_A \sigma\sigma^*(s, x(s), a)\, q_s(da)\, ds, \tag{2.2}$$

which is more natural than relaxing the stochastic integral itself. Now, one has to look for a square integrable martingale whose quadratic variation is given by

(2.2) which is equivalent to the search of an object which is a martingale whose quadratic variation is $dtq_t\,(da)$. This object is precisely a continuous orthogonal martingale measure, whose covariance measure is $dtq_t\,(da)$. This is equivalent to the relaxation of the infinitesimal generator associated to the state process (2.1).

2.2. **Martingale measures.**

Let us give the precise definition of a martingale measure introduced by Walsh [15], see also [9, 11, 12] for more details.

Definition 2.2. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space and $(E, \mathcal{E})$ a Lusin space. Then, $\{M_t\,(A)\,, t \geqq 0, A \in \mathcal{E}\}$ is an $\mathcal{F}_t-$martingale measure if and only if:*

1) $M_0 = 0,\ \forall A \in \mathcal{E}$;
2) $\{M_t\,(A)\,, t \geqq 0\}$ *is an $\mathcal{F}_t$-martingale,* $\forall A \in \mathcal{E}$;
3) $\forall t > 0, M_t(.)$ *is a L^2-valued σ-finite measure in the following sense: there exists a non-decreasing sequence $\{E_n\}$ of E with $\cup_n E_n = E$ such that*
   a) *for every $t > 0$, $\sup\limits_{A\in\mathcal{E}_n} E\left[M\,(A,t)^2\right] < \infty$, $\mathcal{E}_n = \mathcal{B}\,(E_n)$;*
   b) *for every $t > 0$, $E\left[M\,(A_j,t)^2\right] \to 0$ for all sequence A_j of $\mathcal{E}_n$ decreasing to $\varnothing$.*

For $A, B \in \mathcal{E}$, there exists a unique predictable process $\langle M(A), M(B)\rangle_t$, such that

$$M(A,t)M(B,t) - \langle M(A), M(B)\rangle_t \text{ is a martingale.}$$

Remark 2.2.

(1) *A martingale measure M is called orthogonal if $M(A,t).M(B,t)$ is a martingale for $A, B \in \mathcal{E}$, $A \cap B = \emptyset$.*
(2) *If M is an orthogonal martingale measure, one can prove the existence of random σ-finite positive measure $\upsilon(ds,dx)$ on $\mathbb{R} \times E$, $\mathcal{F}_t-$predictable, such that for each A of $\mathcal{A}$ the process $(\upsilon\,((0,t] \times A))_t$ is predictable and satisfies*

$$\forall A \in \mathcal{E}, \forall t > 0, \quad \upsilon\,((0,t] \times A) = \langle M(A)\rangle_t \quad P - a.s.$$

υ can be decomposed as follows $\upsilon(dt,da) = q_t(da)dk_t$, where k_t is a random predictable increasing process and $(q_t(da))_{t\geq 0}$ is a predictable family of random $\sigma-$finite measure.

We refer to [9, 12, 15] for more details and a complete construction of the stochastic integral with respect to orthogonal martingale measures.

Predictable representation for orthogonal martingale measures

Let us denote the set of square-integrable martingales over $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ by M^2.

Proposition 2.1. *Let N be in M^2. Then there exist a unique square integrable predictable process n such that*

$$N_t = N_0 + \int_0^t \int_E n(a,s) M(da, ds) + L(t),$$

where L is an L^2-martingale with $\left\langle L, \int_0^{\cdot} \int_E b(a,s) M(da,ds) \right\rangle = 0$ for every predictable b.

Proof. See [13] □

2.3. SDE corresponding to the relaxed martingale problem.

Now, what is the SDE corresponding to the relaxed martingale problem (2.1)? We begin by the following example without singular control.

Example 3. *Let $A = \{a_1, a_2, ..., a_n\}$, then every relaxed control $dtq_t(da)$ will be a convex combination of the Dirac measures $dtq_t(da) = \sum_{i=1}^n \alpha_i(t) \delta_{a_i}(da)\, dt$, where for each $i, \alpha_i(t)$ is a real-valued process such that $0 \leq \alpha_i(t) \leq 1$ and $\sum_{i=1}^n \alpha_i(t) = 1$. It is shown that the solution of the (relaxed) martingale problem (2.1) is the law of the solution of the following SDE (see [9])*

$$dx(t) = \sum_{i=1}^d b(t, x(t), u_i(t)) \alpha_i(t) dt + \sum_{i=1}^d \sigma(t, x(t), u_i(t)) \alpha_i(t)^{1/2} dB^i(t), \tag{2.3}$$

$x(0) = x_0$, where the B^i's are d-dimensional Brownian motions on an extension of the initial probability space. The process M defined by

$$M(A \times [0,t]) = \sum_{i=1}^d \int_0^t \alpha_i(s)^{1/2} \delta_{u_i(s)}(A) dB^i(s),$$

is in fact a strongly orthogonal continuous martingale measure (see [9, 15]) with intensity $q_t(da)dt = \displaystyle\sum_{i=1}^n \alpha_i(t) \delta_{u_i(t)}(da)\, dt$. Thus, the SDE (2.3) can be expressed in *terms of M and q as follows*

$$dx(t) = \int_A b(t, x(t), a) q_t(da) dt + \int_A \sigma(t, x(t), a) M(da, dt). \tag{2.4}$$

The following theorem gives a pathwise representation of the solution of the martingale problem in terms of strongly orthogonal continuous martingale measure whose intensity are our relaxed control.

Theorem 2.1.

(1) *Let P be the solution of the martingale problem* (2.1). *Then P is the law of a d-dimensional adapted and continuous process x defined on an extension of the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t)$ and solution of the following SDE starting at x_0*

$$dx(t) = \int_A b(t, x(t), a) q_t(da) dt + \int_A \sigma(t, x(t), a) M(da, dt) + c\,(t)\, d\xi\,(t)\,, \tag{2.5}$$

where $M = \left(M^k\right)_{k=1}^d$ is a family of d-strongly orthogonal continuous martingale measures with intensity $q_t(da)dt$.

(2) *If the coefficients b and σ are Lipschitz in x, uniformly in t and x_0, the SDE* (2.5) *has a unique pathwise solution.*

Proof.

(1) The proof is based essentially on the same arguments as in [9] Theorem IV-2 and [7] Prop. 1.10.
(2) The coefficients being Lipschitz continuous, following the same steps as in [7] and [9], it is not difficult to prove that (2.5) has a unique solution such that for every $p > 0$ we have $E\left[|X_t|^p\right] < +\infty$.

□

Remark 2.3. *Note that the orthogonal martingale measure corresponding to the relaxed control $q_t(da)dt$ is not unique.*

3. Maximum principle

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}\,, P\right)$ be a probability space equipped with a filtration satisfying the usual conditions, on which a $d-$dimensional orthogonal continuous martingale measures $(M\,(A, t))$ is defined. We assume that $(\mathcal{F}_t)$ is the $P-$augmentation of the natural filtration of $(M(A,t))$. Let T be a strictly positive real number and consider the following sets

- A_1 is non-empty subset of $\mathbb{R}^d$ and $A_2 = ([0,\infty))^m$.
- $\mathcal{R}$ is the class of relaxed control, $q : [0,T] \times \mathcal{A} \to \mathcal{A}_1$.
- $\mathcal{U}_2$ is the class of measurable, adapted processes $\xi : [0,T] \times \mathcal{A} \to A_2$ such that ξ is non decreasing, left-continuous with right limits and ξ_0.

Definition 3.1. *An admissible control is a $\mathcal{F}_t$-adapted process $(q,\xi) \in \mathcal{R} \times \mathcal{U}_2$ such that*

$$E\left[\sup_{t\in[0,T]} |q_t(a)|^2 + |\xi(T)|^2\right] < \infty.$$

For any $(q,\xi) \in \mathcal{R} \times \mathcal{U}_2$, the relaxed control problem is now driven by equation

$$\begin{cases} dx(t) = \int_A b(t,x(t),a) q_t(da)dt + \int_A \sigma(t,x(t),a)M(da,dt) + c(t)\,d\xi(t) \\ x(0) = x_0 \end{cases}, \tag{3.1}$$

where

$$\begin{aligned} &b : [0,T] \times \mathbb{R}^d \times A_1 \to \mathbb{R}^d \\ &\sigma : [0,T] \times \mathbb{R}^d \times A_1 \to \mathbb{R}^d \times \mathbb{R}^k \\ &c : [0,T] \to \mathbb{R}^d \times \mathbb{R}^k. \end{aligned}$$

The expected cost has the form (see [9])

$$J(q,\xi) = E\left[\int_0^T \int_A h(t,x(t),a)\,q_t(da)dt + \int_0^T R(t)\,d\xi(t) + g(x(T))\right], \tag{3.2}$$

where

$$\begin{aligned} &g : \mathbb{R}^d \to \mathbb{R} \\ &h : [0,T] \times \mathbb{R}^d \times A_1 \to \mathbb{R} \\ &R : [0,T] \to ([0,\infty))^m. \end{aligned}$$

The control problem is to minimize the functional $J(.,.)$ over $\mathcal{R} \times \mathcal{U}_2$. If $\left(\hat{q},\hat{\xi}\right) \in \mathcal{R} \times \mathcal{U}_2$ is an optimal solution, that is

$$J\left(\hat{q},\hat{\xi}\right) = \inf_{(q,\xi)\in\mathcal{R}\times\mathcal{U}_2} J(q,\xi).$$

We may ask, how we can characterize it, in other words what conditions must $\left(\hat{q},\hat{\xi}\right)$ necessarily satisfy?

To answer this question, we need the following assumptions throughout this section.

- **(H$_1$)** b, σ, g, h are twice continuously differentiable with respect to x.
- **(H$_2$)** The derivatives $b_x, b_{xx}, \sigma_x, \sigma_{xx}, g_x, g_{xx}, h_x, h_{xx}$ are continuous in (q, ξ) and uniformly bounded. b, σ are bounded by $C(1 + |x| + |q|)$.
- **(H$_3$)** c and R are continuous and c is bounded.

Under the above hypothesis, for every $(q, \xi) \in \mathcal{R} \times \mathcal{U}_2$, equation (3.1) has a unique strong solution given by

$$\begin{aligned} x^{(q,\xi)}(t) = x_0 &+ \int_0^t \int_A b(s, x^{(q,\xi)}(s), a) q_s(da) dt \\ &+ \int_0^t \int_A \sigma(t, x^{(q,\xi)}(s), a) M(da, ds) + \int_0^t c(s)\, d\xi(s), \end{aligned}$$

and the cost functional J is well-defined from $\mathcal{R} \times \mathcal{U}_2$ into $\mathbb{R}$.

The purpose of the stochastic maximum principle is to find necessary condition for optimality satisfied by an optimal control. Suppose that $\left(\hat{q}, \hat{\xi}\right) \in \mathcal{R} \times \mathcal{U}_2$ is an optimal control and $\hat{x}(t)$ denotes the optimal trajectory, that is, the solution of (3.1) corresponding to $\left(\hat{q}, \hat{\xi}\right)$. Let us introduces the following perturbation of the optimal control $\left(\hat{q}, \hat{\xi}\right)$

$$(3.3) \qquad \left(q_t^\theta, \xi^\theta\right) = \begin{cases} [c]c\left(\delta_v, \hat{\xi} + \theta\left(\eta - \hat{\xi}\right)\right) & \text{if } t \in E \\ \left(\hat{q}_t, \hat{\xi} + \theta\left(\eta - \hat{\xi}\right)\right) & \text{if } t \in E^c \end{cases},$$

where $E = \{r \leq t \leq r + \theta\}$, $0 \leq r < T$ is fixed and the E^c is otherwise of E, $\theta > 0$ is sufficiently small, and v is an arbitrary $\mathcal{F}_r$-measurable random variable, η is an increasing process with $\eta(0) = 0$.

Let $x^\theta(t), x^{\left(q^\theta, \hat{\xi}\right)}(t)$ be the trajectories associated respectively with $\left(q^\theta, \xi^\theta\right)$, $\left(q^\theta, \hat{\xi}\right)$, and $q^\theta_.(A)$ is the intensity of the orthogonal continuous martingale measures M^θ, we create it of the form

$$(3.4) \qquad \begin{aligned} M_t^\theta(A) = &\int_0^t \int_A \mathbf{1}_{[r,r+\theta]}(s) \delta_\nu(da) dB(s) \\ &+ \int_0^t \int_A \mathbf{1}_{[r,r+\theta]^C}(s) M(da, ds). \end{aligned}$$

Since $\left(\hat{q},\hat{\xi}\right)$ is optimal, then

$$J\left(q^{\theta},\xi^{\theta}\right)-J\left(\hat{q},\hat{\xi}\right)\geq 0.$$

and we have

$$J\left(q^{\theta},\xi^{\theta}\right)-J\left(\hat{q},\hat{\xi}\right)=J\left(q^{\theta},\xi^{\theta}\right)-J\left(q^{\theta},\hat{\xi}\right)+J\left(q^{\theta},\hat{\xi}\right)-J\left(\hat{q},\hat{\xi}\right).$$

Then we take

$$J_{\xi}=J\left(q^{\theta},\xi^{\theta}\right)-J\left(q^{\theta},\hat{\xi}\right), \tag{3.5}$$

$$J_{q}=J\left(q^{\theta},\hat{\xi}\right)-J\left(\hat{q},\hat{\xi}\right), \tag{3.6}$$

and the variational inequality will be given by

$$\lim_{\theta\to 0}\frac{1}{\theta}J_{\xi}+\lim_{\theta\to 0}\frac{1}{\theta}J_{q}\geq 0. \tag{3.7}$$

For simplicity of notation, we denote

$$f\left(t\right)=\int_{A}f\left(t,\hat{x}\left(t\right),a\right)\hat{q}_{t}\left(da\right),$$

$$f^{\theta}\left(t\right)=\int_{A}f\left(t,x^{\theta}\left(t\right),a\right)q_{t}^{\theta}\left(da\right)$$

where f stands for one of the functions $b, b_x, b_{xx}, \sigma, \sigma_x, \sigma_{xx}, h, h_x, h_{xx}$.

We will proceed by separating the computation of the two limits in (3.7), and obtain a variational inequality from (3.5) and a variational inequality from (3.6). We need the following technical lemma to achieve this goal.

Lemma 3.1. *Let*

$$\begin{aligned}z\left(t\right)=&\int_{0}^{t}\int_{A}b_{x}(s)z\left(s\right)q_{s}^{\theta}(da)dt\\&+\int_{0}^{t}\int_{A}\sigma_{x}(s)z\left(s\right)M^{\theta}(da,ds)+\int_{0}^{t}c\left(s\right)d\left(\eta-\hat{\xi}\right)\left(s\right)\end{aligned}$$

*then, under assumptions (**H_1**-**H_3**), it holds that*

$$\lim_{\theta\to 0}E\left[\left|\frac{x^{\theta}(t)-x^{\left(q^{\theta},\hat{\xi}\right)}(t)}{\theta}-z(t)\right|^{2}\right]=0. \tag{3.8}$$

Proof. The proof is inspired from [2], Lemma 2, page 994. We have

$$x^{\theta}(t)=x_0+\int_0^t\int_A b(s,x^{\theta}(s),a)q_s^{\theta}(da)ds$$
$$+\int_0^t\int_A \sigma(s,x^{\theta}(s),a)M^{\theta}(da,ds)+\int_0^t c(s)\,d\xi^{\theta}(s),$$
$$x^{\left(q^{\theta},\hat{\xi}\right)}(t)=x_0+\int_0^t\int_A b(s,x^{\left(q^{\theta},\hat{\xi}\right)}(s),a)q_s^{\theta}(da)ds$$
$$+\int_0^t\int_A \sigma(s,x^{\left(q^{\theta},\hat{\xi}\right)}(s),a)M^{\theta}(da,ds)+\int_0^t c(s)\,d\hat{\xi}(s),$$

under assumption (**H**$_1$-**H**$_3$), and the definition (3.3) of q^{θ}, and by using Gronwall's and Burkholder-Davis-Gundy's inequality, we get

(3.9)
$$\lim_{\theta\to 0}E\left[\sup_{0\le t\le T}\left|x^{\theta}(t)-x^{\left(q^{\theta},\hat{\xi}\right)}(t)\right|^2\right]=0,$$

(3.10)
$$\lim_{\theta\to 0}E\left[\sup_{0\le t\le T}\left|x^{\left(q^{\theta},\hat{\xi}\right)}(t)-\hat{x}(t)\right|^2\right]=0.$$

(3.11)
$$E\left[|z(t)|^2\right]<\infty.$$

Now we take

$$y^{\theta}(t)=\frac{x^{\theta}(t)-x^{\left(q^{\theta},\hat{\xi}\right)}(t)}{\theta}-z(t)$$

then

$$\begin{aligned}
&dy^{\theta}(t)\\
&=\frac{1}{\theta}\left(\int_A\left[b(t,x^{\theta}(t),a)-b(s,x^{\left(q^{\theta},\hat{\xi}\right)}(t),a)\right]q_t^{\theta}(da)dt\right)\\
&+\frac{1}{\theta}\left(\int_A\left[\sigma(t,x^{\theta}(t),a)-\sigma(t,x^{\left(q^{\theta},\hat{\xi}\right)}(t),a)\right]M^{\theta}(da,dt)\right)\\
&-\theta\left(\int_A b_x(t)z(t)\,q_t^{\theta}(da)dt+\int_A\sigma_x(t)z(t)\,M^{\theta}(da,dt)\right)\\
&=\int_A\int_0^1 b_x(t,x^{\left(q^{\theta},\hat{\xi}\right)}(t)+\lambda\left[x^{\theta}(t)-x^{\left(q^{\theta},\hat{\xi}\right)}(t)\right],a)\left(y^{\theta}(t)+z(t)\right)d\lambda q_t^{\theta}(da)dt
\end{aligned}$$

$$+\int_A \int_0^1 \sigma_x(t, x^{\left(q^\theta,\hat{\xi}\right)}(t) + \lambda\left[x^\theta(t) - x^{\left(q^\theta,\hat{\xi}\right)}(t)\right], a)\left(y^\theta(t) + z(t)\right) d\lambda M^\theta(da, dt)$$

$$-\int_A \int_0^1 b_x(t) z(t)\, d\lambda q_t^\theta(da) dt - \int_A \int_0^1 \sigma_x(t) z(t)\, d\lambda M^\theta(da, dt)$$

it holds that

$$\begin{aligned}
&E\left|y^\theta(t)\right|^2 \\
&\leq 3\int_0^t E\int_A \left(\int_0^1 b_x(s, x^{\left(q^\theta,\hat{\xi}\right)}(s) + \lambda\left[x^\theta(s) - x^{\left(q^\theta,\hat{\xi}\right)}(s)\right], a) y^\theta(s)\, d\lambda q_s^\theta(da) ds\right)^2 \\
&+ 3\int_0^t E\int_A \left(\int_0^1 \sigma_x(s, x^{\left(q^\theta,\hat{\xi}\right)}(s) + \lambda\left[x^\theta(s) - x^{\left(q^\theta,\hat{\xi}\right)}(s)\right], a) y^\theta(s)\, d\lambda\right)^2 q_s^\theta(da) ds \\
&+ 3E\left|\rho^\theta(t)\right|^2
\end{aligned}$$

where ρ^θ is given by

$$\begin{aligned}
\rho^\theta(t) = &\int_0^t \int_A \int_0^1 \left[b_x(s, x^{\left(q^\theta,\hat{\xi}\right)}(s) + \lambda\left[x^\theta(s) - x^{\left(q^\theta,\hat{\xi}\right)}(s)\right], a)\right. \\
&\left. - b_x(s, \hat{x}(s), a)\right] z(s)\, d\lambda q_s^\theta(da) ds \\
&+ \int_0^t \int_A \int_0^1 \left[\sigma_x(s, x^{\left(q^\theta,\hat{\xi}\right)}(s) + \lambda\left[x^\theta(s) - x^{\left(q^\theta,\hat{\xi}\right)}(s)\right], a)\right. \\
&\left. - \sigma_x(s, \hat{x}(s), a)\right] z(s)\, d\lambda M^\theta(da, ds).
\end{aligned}$$

Since b_x, σ_x are bounded, we have

$$E\left|y^\theta(t)\right|^2 \leq 6\int_0^t E\left|y^\theta(t)\right|^2 ds + 3\left|\rho^\theta(t)\right|^2$$

b_x, σ_x being continuous and bounded. Then using (3.9), (3.10), (3.11) and the dominated convergence theorem, we get

$$\lim_{\theta \to 0} E\left|\rho^\theta(t)\right|^2 = 0.$$

We conclude by using Gronwall's lemma. □

Lemma 3.2. *We assume (**H_1-H_3**), then the following estimate holds*

$$E\left[\sup_{0\le t\le T}\left|x^{\left(q^{\theta},\hat{\xi}\right)}(t)-\hat{x}(t)-x_1(t)-x_2(t)\right|^2\right]\le C(\theta)\theta^2, \tag{3.12}$$

where $\lim_{\theta\longrightarrow 0}C(\theta)=0$ *and* $x_1(t)$, $x_2(t)$ *are solutions of the SDEs*

$$\begin{aligned} x_1(t)=&\int_0^t\int_A\left[b(s,x(s),a)q_s^{\theta}(da)-b(s,x(s),a)q_s(da)\right.\\ &\left.+b_x(s,x(s),a)x_1(s)q_s(da)\right]ds\\ &+\int_0^t\int_A\left[\sigma(s,x(s),a)M^{\theta}(da,ds)-\sigma(s,x(s),a)M(da,ds)\right.\\ &\left.+\sigma_x(s,x(s),a)x_1(s)M(da,ds)\right]\end{aligned} \tag{3.13}$$

$$\begin{aligned} &x_2(t)\\ =&\int_0^t\int_A\left[\left(b_x(s,x(s),a)q_s^{\theta}(da)-b_x(s,x(s),a)q_s(da)\right)x_1(s)\right]ds\\ &+\int_0^t\int_A\left[b_x(s,x(s),a)x_2(s)q_s(da)+\frac{1}{2}b_{xx}(s,x(s),a)q_s(da)x_1(s)x_1(s)\right]ds\\ &+\int_0^t\int_A\left[\sigma_x(s,x(s),a)x_1(s)M^{\theta}(da,ds)-\sigma_x(s,x(s),a)x_1(s)M(da,ds)\right]\\ &+\int_0^t\int_A\left[\sigma_x(s,x(s),a)x_2(s)+\frac{1}{2}\sigma_{xx}(s,x(s),a)x_1(s)x_1(s)\right]M(da,ds).\end{aligned} \tag{3.14}$$

Remark 3.1. *Equation* (3.13) *is called the first-order variational equation. It is the variational equation in the usual sense.* (3.14) *is called the second-order variational equation, without this equation, we cannot derive the variational inequality since* σ *depends explicitly on the control variable.*

Proof. (Proof of Lemma 3.3) We put

$$\tilde{x}(t) = \hat{x}(t) - \int_0^t c(s)\, d\hat{\xi}(s),$$

$$\tilde{x}^{(q^\theta,\hat{\xi})}(t) = x^{(q^\theta,\hat{\xi})}(t) - \int_0^t c(s)\, d\hat{\xi}(s).$$

It is clear that

$$x^{(q^\theta,\hat{\xi})}(t) - \hat{x}(t) - x_1(t) - x_2(t) = \tilde{x}^{(q^\theta,\hat{\xi})}(t) - \tilde{x}(t) - x_1(t) - x_2(t).$$

By using the same argument as in [10], Lemma 2, page 1104, and we have that

$$E\left[\sup_{0\le t\le T} |x_1(t)|^2\right] \le C_k\theta, \tag{3.15}$$

$$E\left[\sup_{0\le t\le T} |x_2(t)|^2\right] \le C_k\theta^2, \tag{3.16}$$

where C_k is a constant. Then we have

$$E\left[\sup_{0\le t\le T} \left|\tilde{x}^{(q^\theta,\hat{\xi})}(t) - \tilde{x}(t) - x_1(t) - x_2(t)\right|^2\right] \le C(\theta)\theta^2,$$

which prove the lemma. □

Lemma 3.3. *Under assumptions of Lemma 3.2, we have*

$$\begin{aligned} \lim_{\theta\to 0}\frac{J_\xi}{\theta} &= E\left[z(t)\, g_x(\hat{x}(T))\right] + E\int_0^T\int_A z(t)\, h_x(t)\, q_t^\theta(da)dt \\ &+ E\int_0^T R(t)\, d\left(\eta - \hat{\xi}\right)(t). \end{aligned} \tag{3.17}$$

Proof. From (3.5), we have

$$\begin{aligned} \frac{J_\xi}{\theta} = E\int_0^T\int_A\int_0^1 \left(\frac{x^\theta(t) - x^{(q^\theta,\hat{\xi})}(t)}{\theta}\right) h_x\Big(t, x^{(q^\theta,\hat{\xi})}(t) \\ +\lambda\left[x^\theta(t) - x^{(q^\theta,\hat{\xi})}(t)\right], a\Big)\, d\lambda \hat{q}_t(da)dt \end{aligned}$$

$$+E\int_0^1 \left(\frac{x^{\theta}(T) - x^{\left(q^{\theta},\hat{\xi}\right)}(T)}{\theta}\right) g_x\left(x^{\left(q^{\theta},\hat{\xi}\right)}(T) + \lambda\left[x^{\theta}(T) - x^{\left(q^{\theta},\hat{\xi}\right)}(T)\right], a\right) d\lambda$$

$$+E\int_0^T R(t)\, d\left(\eta - \hat{\xi}\right)(t).$$

Since b_x and g_x are continuous and bounded, then from (3.3), (3.4), (3.10) and by letting θ going to zero we conclude. □

Lemma 3.4. *We assume (**H**$_1$-**H**$_3$), then the following estimate holds*

$$\begin{aligned} J_q \leq\ & E\left[\int_0^T \int_A \left(h(t,\hat{x}(t),a)\, q_{\theta}(t) - h(t,\hat{x}(t),a)\, q(t)\right) dt\right] \\ & + E\left[g_x(\hat{x}(T))\left(x_1(T) + x_2(T)\right)\right. \\ & \left. + \int_0^T \int_A h_x(t,\hat{x}(t),a)\, q(t)\left(x_1(t) + x_2(t)\right) dt\right] \\ & + \frac{1}{2}E\left[g_{xx}(\hat{x}(T)) x_1(T) x_1(T)\right. \\ & \left. + \int_0^T \int_A h_{xx}(t,\hat{x}(t),a)\, q(t) x_1(t) x_1(t) dt\right] + o(\theta). \end{aligned} \tag{3.18}$$

Proof. Under (3.6), we have

$$\begin{aligned} J_q \leq\ & E\left[\int_0^T \int_A \left(h\left(t, x^{\left(q^{\theta},\hat{\xi}\right)}(t), a\right) q_t^{\theta}(da) - h(t,\hat{x}(t),a)\, \hat{q}(t)\right) dt\right] \\ & + E\left[g(x^{\left(q^{\theta},\hat{\xi}\right)}(T)) - g(\hat{x}(T))\right]. \end{aligned}$$

By using the estimate (3.14), the result follows by mimicking by the same proof as in [10], Lemma 3 page 1108. □

3.1. **The adjoint processes and the variational inequality.**

In this subsection, we will introduce the first and second order adjoint processes involved in the stochastic maximum principle and the associated stochastic Hamiltonian system. These are obtained from the first and second varaitional equations (3.13), (3.14) as well as (3.18).

3.1.1. *The first order terms.* We use the same thing as in [10], we put

$$p_1(t) = \psi_1^*(t)Y_1(t), \tag{3.19}$$

$$Q_1(t) = \int_A \psi_1^*(t)G_1(t,a)q_t(da) - \int_A \sigma_x^*(t,\hat{x}(t),a)q_t(da))p_1(t) \tag{3.20}$$

in which

$$\begin{cases} d\phi_1(t) = \int_A b_x(t,x(t),a)\phi_1(t)q_t(da)dt + \int_A \sigma_x(t,x(t),a)\phi_1(t)M(da,dt) \\ \phi_1(0) = I_d \end{cases}.$$

Then, ϕ_1 is invertible and its inverse ψ_1 satisfies

$$\begin{cases} d\psi_1(t) = \int_A \left[\psi_1(t)\sigma_x(t,x(t),a)\sigma_x(t,x(t),a) - \psi_1(t)b_x(t,x(t),a)\right] q_t(da)dt \\ \qquad\qquad - \int_A \psi_1(t)\sigma_x(t,x(t),a)M(da,dt) \\ \psi_1(0) = I_d. \end{cases},$$

and

$$X_1 = \phi_1(T)g_x\left(\hat{x}(T)\right) + \int_0^T \phi_1(s)\int_A h_x(s,\hat{x}(s),a)\hat{q}_s(da)ds$$

$$Y_1(t) = E\left(X_1/\mathcal{F}_t\right) - \int_0^t \phi_1(s)\int_A h_x(s,\hat{x}(s),a)\hat{q}_s(da)ds.$$

Moreover $p_1(t)$, $Q_1(t)$ satisfies

$$E\left[\sup_{0\leq t\leq T}|p_1(t)|^2 + \sup_{0\leq t\leq T}|Q_1(t)|^2\right] < \infty,$$

the process p_1 is called the first adjoint process.

Let us now define the Hamiltonian

$$H\left(t,x,q,p,Q\right) = \int_A h\left(t,x,a\right)q(da) + p\int_A b\left(t,x,a\right)q(da) + Q\int_A \sigma\left(t,x,a\right)q(da),$$

so, (3.18) can be rewritten

$$\begin{aligned} J_q \leq\ & E\int_0^T \int_A \left[H\left(t,\hat{x}(t),a,p_1(t),Q_1(t)\right) q_t^\theta(da)\right. \\ & \left.-H\left(t,\hat{x}(t),a,p_1(t),Q_1(t)\right) q_t(da)\right] dt \\ & +\frac{1}{2}E\int_0^T \int_A x_1(t) H_{xx}(\hat{x}(t),a,p_1(t),Q_1(t)) x_1^*(t) q_t(da)dt \\ & +\frac{1}{2}E\left[x_1(T) g_{xx}(\hat{x}(T)) x_1^*(T)\right] + o\left(\theta\right). \end{aligned} \tag{3.21}$$

Thus, we can rewrite (3.17) as

$$\lim_{\theta\to 0}\frac{J_\xi}{\theta} = E\int_0^T \left[R\left(t\right)+c\left(t\right)p_1\left(t\right)\right] d\left(\eta-\hat{\xi}\right)\left(t\right). \tag{3.22}$$

For more detail see [10].

3.1.2. *The second order terms.*

The second order estimation concerns the second order derivatives in (3.21). We do the same thing as in [10], then the right third part in (3.21) become

$$\begin{aligned} & E\left[x_1(T) g_{xx}(\hat{x}(T)) x_1^*(T)\right] \\ = & -E\int_0^T \int_A x_1(t) H_{xx}(t,\hat{x}(t),a) x_1^*(t) q_t(da)dt \\ & + E\int_0^T \int_A tr\left[\left(\sigma(t,\hat{x}(t),a) q_t^\theta(da) - \sigma(t,\hat{x}(t),a) q_t(da)\right)^* p_2(t)\right. \\ & \left.\left(\sigma(t,\hat{x}(t),a) q_t^\theta(da) - \sigma(t,\hat{x}(t),a) q_t(da)\right)\right] dt + o\left(\theta\right), \end{aligned} \tag{3.23}$$

where

$$p_2(t) = \psi_2^*(t)\zeta_2(t), \tag{3.24}$$

in which

$$\begin{cases} d\phi_2(t) = \displaystyle\int_A \left[\phi_2(t) b_x^*(t,\hat{x}(t),a) + b_x(t,\hat{x}(t),a)\phi_2(t)\right. \\ \qquad\qquad \left.+\sigma_x(t,\hat{x}(t),a)\phi_2(t)\sigma_x^*(t,\hat{x}(t),a)\right] q_t(da)dt \\ \qquad\qquad + \displaystyle\int_A \left(\phi_2(t)\sigma_x^*(t,\hat{x}(t),a) + \sigma_x(t,\hat{x}(t),a)\phi_2(t)\right) M(da,dt) \\ \phi_2(0) = I_d, \end{cases}$$

where ϕ_2 is invertible and its inverse ψ_2 satisfies

$$\begin{cases} d\psi_2(t) = \displaystyle\int_A \left[(\sigma_x(t,\hat{x}(t),a) + \sigma_x^*(t,\hat{x}(t),a))^2 \psi_2(t) - \psi_2(t) b_x^*(t,\hat{x}(t),a)\right] q_t(da)dt \\ \qquad\qquad - \displaystyle\int_A \left[b_x(t,\hat{x}(t),a)\psi_2(t) + \sigma_x(t,\hat{x}(t),a)\psi_2(t)\sigma_x^*(t,\hat{x}(t),a)\right] q_t(da)dt \\ \qquad\qquad - \left[\psi_2(t)\sigma_x^*(t,\hat{x}(t),a) + \sigma_x(t,\hat{x}(t),a)\psi_2(t)\right] M(da,dt) \\ \psi_2(0) = I_d, \end{cases}$$

and

$$X_2 = \phi_2^*(T) g_{xx}(\hat{x}(T)) + \int_0^T \phi_2^*(s) \int_A H_{xx}(s,\hat{x}(s),a) q_s(da)ds$$
$$\zeta_2(t) = E(X_2/\mathcal{F}_t) - \int_0^t \phi_2^*(s) \int_A H_{xx}(s,\hat{x}(s),a) q_s(da)ds.$$

The process p_2 is called the second adjoint process.

3.1.3. *The adjoint equations.*

By applying Ito's formula to the adjoint processes p_1 in (3.19) and p_2 in (3.24), we obtain the first and second order adjoint equations, which have the forms

$$\begin{cases} -dp_1(t) = \displaystyle\int_A \left[b_x^*(t,\hat{x}(t),a) p_1(t) + \sigma_x^*(t,\hat{x}(t),a) Q_1(t)\right. \\ \qquad\qquad \left. + h_x(t,\hat{x}(t),a)\right] q_t(da)dt - \displaystyle\int_A Q_1(t) M(da,dt) - \psi_1^*(t) dL(t) \\ p_1(T) = g_x(\hat{x}(T)). \end{cases} \tag{3.25}$$

with values in $\mathbb{R}^d$, where L is an L^2-martingale with $\left\langle L, \int_0^\cdot \int_E b(a,s) M(da,ds)\right\rangle = 0$ for every predictable b, Q_1 is given by (3.20) with values in $\mathbb{R}^{d\times k}$. The adjoint equation that $p_1(.)$ satisfied is a linear backward stochastic differential equation**.** This BSDE has a unique adapted solution.

Next, p_2 is a matrix valued and satisfies

$$\begin{aligned} -dp_2(t) = &\int_A \left[b_x^*(t,\hat{x}(t),a) p_2(t) + p_2(t) b_x(t,\hat{x}(t),a)\right. \\ &\left. + \sigma_x^*(t,x(t),a) Q_2(t)\right] q_t(da)dt \\ &+ \int_A \left[\sigma_x^*(t,\hat{x}(t),a) p_2(t) \sigma_x(t,\hat{x}(t),a)\right. \\ &\left. + H_{xx}(\hat{x}(t),a,p_1(t),Q_1(t))\right] q_t(da)dt \end{aligned} \tag{3.26}$$

$$
\begin{aligned}
&+\int_A Q_2(t)\sigma_x\left(t,x(t),a\right)q_t(da)dt\\
&-\int_A Q_2(t)M(da,dt)-\psi_2^*(t)dL'(t)\\
p_2(T)&=g_{xx}\left(\hat{x}(T)\right),
\end{aligned}
$$

where L' is an L^2-martingale with $\left\langle L', \int_0^{\cdot}\int_E b(a,s)M(da,ds)\right\rangle = 0$ for every predictable b, and Q_2 is given by

$$
Q_2(t)=\int_A\left[\psi_2^*(t)G_2(t,a)-p_2(t)\sigma_x\left(t,\hat{x}(t),a\right)+\sigma_x^*(t,\hat{x}(t),a)p_2(t)\right]q_t(da). \tag{3.27}
$$

Note that $p_2(.)$ is also a backward stochastic differential equation with matrix-valued unknowns. This BSDE have a unique adapted solution.

Remark 3.2. *$H_{xx}\left(\hat{x}(t),q_t,p(t),Q(t)\right)$ is the second derivative of the Hamiltonian H at x and it is given by $H_{xx}\left(\hat{x}(t),q_t,p(t),Q(t)\right)=h_{xx}\left(t,\hat{x}(t),q_t\right)+p(t)b_{xx}\left(t,\hat{x}(t),q_t\right)+Q(t)\sigma_{xx}\left(t,\hat{x}(t),q_t\right).$*

3.2. **Main result.**

We are ready now to state the main result of this paper.

Theorem 3.1. (*The stochastic maximum principle*) *Let $\left(\hat{q},\hat{\xi}\right)$ be an optimal control minimizing the cost J over $\mathcal{R}\times\mathcal{U}_2$ and $\hat{x}$ denotes the corresponding optimal trajectory. Then there are two unique couples of adapted processes (p_1,Q_1) and (p_2,Q_2) which are respectively solutions of the backward stochastic differential equations* (3.25) *and* (3.26) *such that*

$$
\begin{aligned}
0\leq\ & H\left(t,\hat{x}(t),\nu,p_1(t),Q_1(t)\right)-H\left(t,\hat{x}(t),q_t,p_1(t),Q_1(t)\right)\\
&+\frac{1}{2}tr\left[\left(\sigma(t,\hat{x}(t),\nu)-\sigma(t,\hat{x}(t),q_t)\right)^*p_2(t)\left(\sigma(t,\hat{x}(t),\nu)-\sigma(t,\hat{x}(t),q_t)\right)\right],
\end{aligned} \tag{3.28}
$$

υ is an arbitrary $\mathcal{F}_r$-measurable random variable with values in $\mathcal{U}_1$, such that

$$
\sup_{w\in\Omega}|\upsilon(w)|<\infty,
$$

$$P\left\{\forall t\in[0,T],\forall i,\left[R^{i}(t)+c_{i}(t)\,p_{1}(t)\right]\geq 0\right\}=1,$$
$$P\left\{\sum_{i=1}^{k}1_{\{R^{i}(t)+c_{i}(t)p_{1}(t)\geq 0\}}d\hat{\xi}(t)=0\right\}=1. \tag{3.29}$$

Proof. From (3.7), (3.21), (3.22) and (3.23), we have for every $\mathcal{F}-$measurable random variable ν, and every increasing process η with $\eta_0=0$,

$$\begin{aligned}J_q\leq{}& E\int_0^T\int_A\left[H\left(t,\hat{x}(t),a,p_1(t),Q_1(t)\right)q_t^{\theta}(da)\right.\\&-H\left(t,\hat{x}(t),a,p_1(t),Q_1(t)\right)q_t(da)]\,dt+\circ(\theta)\\&+\frac{1}{2}E\int_0^T\int_A tr\left[\left(\sigma^{\theta}(t,\hat{x}(t),a)q_t^{\theta}(da)-\sigma(t,\hat{x}(t),a)q_t(da)\right)^{*}p_2(t)\right.\\&\left.\left(\sigma^{\theta}(t,\hat{x}(t),a)q_t^{\theta}(da)-\sigma(t,\hat{x}(t),a)q_t(da)\right)\right]dt.\end{aligned}$$

This equation is the variational inequation of the second order.

We use the definition of q_θ, the last variational inequality becomes

$$\begin{aligned}\frac{1}{\theta}J_q\leq{}&\frac{1}{\theta}E\int_r^{r+\theta}\left[H\left(t,\hat{x}(t),\nu,p_1(t),Q_1(t)\right)-H\left(t,\hat{x}(t),q_t,p_1(t),Q_1(t)\right)\right]dt+o(\theta)\\+\frac{1}{2\theta}E\int_r^{r+\theta}&tr\left[\left(\sigma(t,\hat{x}(t),\nu)-\sigma\left(t,\hat{x}(t),q_t\right)\right)^{*}p_2(t)\left(\sigma(t,\hat{x}(t),\nu)-\sigma(t,\hat{x}(t),q_t)\right)\right]dt.\end{aligned}$$

Then, the desired result follows by letting θ going to zero.

If we put $\eta(t)=\xi(t)$ we obtain (3.28). □

Remark 3.3. *It we suppose that $c=R=0$, then we recover to our work which generalised the Peng's maximum principle see [10].*

References

[1] S. Bahlali, B. Djehiche, B. Mezerdi: *The relaxed stochastic maximum principle in singular optimal control of diffusions*, SIAM Journal on Control and Optimization, **46**(2) (2007), 427-444.

[2] S. Bahlali, B. Mezerd: *A general stochastic maximum principle for singular control problems*, Electronic Journal of Probability, **10** (2005), 988-1004.

[3] A. Bensoussan: *Lectures on Stochastic Control, Nonlinear Filtering and Stochastic Control*, In S.K. Mitter, A. Moro, (eds) Nonlinear Filtering and Stochastic Control. Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, **972**, 1982.

[4] A. Cadenillas, U.G. Haussmann: *The stochastic maximum principle for a singular control problem*, Stochastics: An International Journal of Probability and Stochastic Processes, **49**(3-4) (1994), 211-237.

[5] A. Chala, S. Bahlali: *Stochastic controls of relaxed-singular problems*, Random Operators and Stochastic Equations, **22**(1) (2014), 31-41.

[6] U.G. Haussmann, W. Suo: *Singular optimal stochastic controls I: Existence*, SIAM Journal on Control and Optimization, **33**(3) (1995), 916-936.

[7] B. Jourdain, S. Méléard, W. Woyczynski: *Nonlinear SDEs driven by Lévy processes and related PDEs*, Alea, **4** (2008), 1-29.

[8] N.E. Karoui, N. Du'hŪŪ, J.P. Monique: *Compactification methods in the control of degenerate diffusions: existence of an optimal control*, Stochastics: an international journal of probability and stochastic processes, **20**(3) (1987), 169-219.

[9] N.E. Karoui, S. Méléard: *Martingale measures and stochastic calculus*, Probability theory and related fields, **84**(1) (1990), 83-101.

[10] S. Labed, B. Mezerdi: *The maximum principle in optimal control of systems driven by martingale measures*, Afrika Statistika, **12**(1) (2017), 1095-1116.

[11] L. Mazliak: *An introduction to probabilistic methods In stochastic control*, Laboratory Probabilities, University of Paris, France, 1996.

[12] S. Méléard: *Martingale measure approximation, application to the control of diffusions*, Prépublication du laboratoire de probabilités, université Paris VI, 1992.

[13] L. Overbeck: *On the predictable representation property for super-processes*, Séminaire de probabilités de Strasbourg, **29** (1995), 108-116.

[14] S. Peng: *A general stochastic maximum principle for optimal control problems*, SIAM Journal on control and optimization, **28**(4) (1990), 966-979.

[15] J.B. Walsh: *An introduction to stochastic partial differential equations*, In École d'Été de Probabilités de Saint Flour, Springer, Berlin, Heidelberg, **XIV**(1984) (1986), 265-439.

Department of Mathematics
Laboratory of Applied Mathematic
University of Biskra
Po. Box 145, Biskra (07000)
Algeria.
Email address: `s.labed@univ-biskra.dz`