ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **12** (2023), no.1, 217–224
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.12.1.14

HOLOMORPHIC EXTENSION

Mohamed Nadir Boukhetouta[1], Mostafa Krachni[2], Fares Yazid[3], and Fatima Siham Djeradi[4]

ABSTRACT. In this paper, we prove that if there exists a holomorphic, proper, surjective map defined on a complex manifold X into a smooth algebraic curve with parallelizable fibers, then any holomorphic mappings defined on the Hartogs domain T of $\mathbb{C}^n$ can be extended holomorphically (resp. meromorphically) from $\Delta^n \setminus Z$ into X, where Z is an analytic subset of Δ^n such that codimension of Z at least 2.

1. INTRODUCTION

More than a century ago, Hartogs discovered a phenomenon of forced extension of holomorphic maps. Hartogs proved that any holomorphic map defined on the domain T of $\mathbb{C}^n$ extended holomorphically to the envelope of holomorphy of T denoted $\tilde{T}$ which is equal to the polydisc Δ^n of $\mathbb{C}^n$. T is called the Hartogs domain of $\mathbb{C}^n$.

Note that Bochner gave another proof of Hartogs theorem. Levi proved the case where the map is meromorphic [20].

A complex manifold X is said holomorphically extensifer (resp. meromorphically extensifer), if any holomorphic (resp. meromorphic) mapping defined on

[3]*corresponding author*

2020 *Mathematics Subject Classification.* 49K15, 53B20, 53Z30.
Key words and phrases. Extension, fiber parallelizable, Hartogs domain, holomorphically extensifer, smooth algebraic curve.
Submitted: 03.01.2023; *Accepted:* 18.01.2023; *Published:* 24.01.2023.

the Hartogs domain T into X an be extended holomorphically (resp. meromorphically) extended Δ^n. For example, any compact parallelizable manifold is holomorphically extensifer [24]. Ivashkovich proved that a Kähler manifold is holomorphically extensifer if and only if it does not contain rational curves [12]. We also have that any projective manifold is meromorphically extensifer.

A complex manifold X is said holomorphically (resp. meromorphically) extensifer outside of codimension at least 2, if any holomorphic (resp. meromorphic) mapping defined on the Hartogs domain T into X can be extended holomorphically (resp. meromorphically) from $\Delta^n \setminus Z$ where Z is an analytic subset of Δ^n of codimension a least 2. For example, Krachni proved that any homogeneous compact manifold is holomorphically extensifer to the complement in of an analytic subset of Δ^n of codimension at least 2 [19].

We use properties of projective algebraic curves (compact) and non-compact algebraic curves (Stein manifolds), lemma of Dloussky [7], and theorems of manifolds holomorphically (resp. meromorphically) extensifer, all can be found in [19], to deal with the following problem : if there exists a holomorphic, proper and surjective map φ defined on a complex manifold X with values in a non-singular algebraic curve Δ, with parallelizable fibers, then every holomorphic (resp. meromorphic) mapping from T to X extends holomorphically (resp. meromorphically) to $\Delta^n \setminus Z,$ where Z is an analytic subset of codimension at least 2.

2. Preliminaries

We call the Hartogs domain the open subset T of $\mathbb{C}^n$ defined by:

$$\begin{aligned} T = T_{\rho,\sigma} &= \{z \in \mathbb{C}^n : |z_i| < \rho, i = 1, \cdots, n-1; |z_n| < 1\} \\ &\cup \{z \in \mathbb{C}^n : |z_i| < \rho, i = 1, \cdots, n-1; \tau < |z_n| < 1\}. \end{aligned}$$

Here $0 < \rho < 1$ and $0 < \tau < 1$.

The envelope of holomorphy of T denoted $\tilde{T}$ is equal to the polydisc Δ^n of $\mathbb{C}^n$. An analytic set of pure codimension equal to 1 (which can admit singularities) is called a hypersurface.

Definition 2.1. *A complex manifold X is said holomorphically (resp. meromorphically) extensifer, if any holomorphic (resp. meromorphic) mapping defined on T into X is holomorphically (resp. meromorphically) extended on Δ^n.*

Definition 2.2. *A complex manifold X is said holomorphically (resp. meromorphically) extensifer outside of codimension at least 2, if any holomorphic (resp. meromorphic) mapping on T into X can be extended holomorphically (resp. meromorphically) from $\Delta^n \setminus Z$, where Z is an analytic subset of Δ^n of codimension a least 2.*

Proposition 2.1. *Let X be a complex manifold, if any holomorphic map of T into X is holomorphically extensifer from Δ^n, then every meromorphic map $f : T \to X$ is in fact holomorphic, and extends holomorphically from Δ^n.*

Theorem 2.1. *[19] Let X and Y be two complex manifolds, and let $\varphi : X \to Y$ be a holomorphic map. If:*

- *Y is holomorphically extensifer.*
- *There exists a cover $U = (\mathcal{U}_i)_{i\in I}$ of Y such that $\varphi^{-1}(\mathcal{U}_i)$ is holomorphically extensifer (resp. meromorphically extensifer).*

Then, X is holomorphically extensifer (resp. meromorphically extensifer).

Theorem 2.2. *[19] Let X and Y be two complex manifolds, and let $\varphi : X \to Y$ a holomorphic map. If:*

- *Y is projective.*
- *There exists a cover $U = (\mathcal{U}_i)_{i\in I}$ of Y such that $\varphi^{-1}(\mathcal{U}_i)$ is holomorphically extensifer (resp. meromorphically extensifer).*

Then, any holomorphic mapping from T into X is holomorphically extensifer (resp. meromorphically extensifer) from Δ^n/Z, where Z is an analytic subset of Δ^n, such that $Codim\ Z \geq 2$.

Proposition 2.2. *Let X and Y be two complex manifolds, and let $\varphi : X \to Y$ be an unramified covering, then, X is holomorphically (resp. meromorphically) extensifer, if and only if Y is holomorphically (resp. meromorphically) extensifer.*

Theorem 2.3. *[6] Let H be a hupersurface of T. If there exists a holomorphic map φ of $T \setminus H$, such that H is singularity, then there exists a hypersurface $\tilde{H}$ of $\tilde{T}$ for which $H = \tilde{H} \cap T$, and the holomorphic envelope of $T \setminus H$ is isomorphic to $\tilde{T} \setminus \tilde{H}$.*

Note that Grauert and Remmert proved that if (π, G) is a étale domain over $\mathbb{C}^n$ and $\tilde{H}$ a hypersurface of G then the holomorphic envelope of $G \setminus \lambda^{-1}(\tilde{H})$ is isomorphic to $\tilde{G} \setminus \tilde{H}$.

Theorem 2.4. *[16] Let X and Y two complex manifolds, and let $f : X \to Y$ be a holomorphic and proper map. Let ρ be a coherent analytic sheaf of f on X and let X_y be the fiber of X on the point $y \in Y$ with respect to f and ρ_y the analytical restriction of ρ on X_y. So,*

- *The functions:*

$$d_q(y) = \dim_{\mathbb{C}} H^q(X_y, \rho_y), \qquad q = 0, 1, \cdots$$

are semi-continuous. Moreover, if Y is reduced, then there exists a low-dimensional analytical set N in Y such that all d_q in $Y \setminus N$ are locally constant.

- *If the function d_q for q is constant and Y reduced, then the $q-th$ direct image beam f_q of ρ is locally free.*
- *The Euler-Poincaré's characteristic:*

$$X(Y) = \sum_{q=0}^{\infty} (-1)^q \dim_{\mathbb{C}} H^q(X_y, \rho_y),$$

is locally constant on Y.

3. Discussion and results

Let X be a complex manifold and $\varphi : X \to \Delta$ a holomorphic and surjective map such that Δ is a smooth be algebraic curve.

For any point $a \in \Delta$, let $\tau_a$be be a local map from Δ with center a, and $\tau_a(u)$, the value of τ_a at a point u of a neighborhood of a.
Let $z = (z_1, z_2, \cdots, z_n)$ be the local coordinates of a point z on X, if:

$$\sum_{i=1}^{n} \left| \frac{\partial \tau_a}{\partial z_i} (\varphi(z)) \right| > 0,$$

at each point z of $\varphi^{-1}(u)$, we say that $C_u := \varphi^{-1}(u)$ is a regular fiber of X.

In the case where $C_u := \varphi^{-1}(u)$ is a regular fiber of X, we say that φ is a submersion in z.

Lemma 3.1. *Let X be a complex manifold, and Δ a smooth algebraic curve. Let $\pi : X \to \Delta$ be a submersion, such that for all $z \in \Delta, \pi^{-1}(z)$ is a parallelizable curve, then:*

a) X *is holomorphically extensifer, if* Δ *non-compact* $(i.e\Delta \neq P^1(\mathbb{C}))$.

b) Any holomorphic map from T *into* X *extends holomorphically to* $\Delta^n \setminus Z$, *where* Z *is an analytic subset of codimension at least 2, if* Δ *is (i.e*$\Delta = P^1(\mathbb{C})$*).*

Proof. Let Θ be the locally sheaf of rank 1 on X of the vector fields tangent to the fibers of π. For $z \in \Delta$, let Θ_z denote the restriction of Θ to $\pi^{-1}(z)$. Then, π is a submersion, so for any $z \in \Delta, \pi^{-1}(z)$ is a parallelizable curve, which gives us that:

$$\dim_{\mathbb{C}} H^\circ(\pi^{-1}(z), \Theta_z) = 1.$$

By Theorem 2.4, the direct image beam $\pi_*\Theta$ is a locally free beam of rank 1 on Δ. Let $z_0 \in \Delta$, there exists an open disk D_0 centered at z_0 on which $\pi_*\Theta|_{D_0}$ is isomorphic to $\Theta|_{D_0}$, so there exists a vector field $\theta \in \pi_*\Theta(D_0)$ which does not vanish at any point of $\pi^{-1}(D_0)$. As π is a submersion, by restricting D_0 if necessary, we can assume that π admits a section σ above D_0.

Consider the holomorphic map:

$$\begin{aligned} \phi : D_0] \times \mathbb{C} &\longrightarrow \pi^{-1}(D_0) \\ (z, w) &\longrightarrow e^{(w\theta)(\sigma(z))}. \end{aligned}$$

For fixed $z \in D_0, \mathbb{C}$ is the universal covering of $\pi^{-1}(z)$ and $(D_0 \times \mathbb{C}, \phi, \pi^{-1}(D_0))$ is a covering of $\pi^{-1}(D_0)$.

As $D_0 \times \mathbb{C}$ is holomorphically extensifer, and by proposition 2.2, then $\pi^{-1}(D_0)$ is holomorphically extensifer.

In conclusion:

i) If $\Delta \neq P^1(\mathbb{C})$ then, X is holomorphically extensifer, by proposition 2.2 and Theorem 2.1.

ii) If $\Delta = P^1(\mathbb{C})$, then any holomorphic map from T into X extends holomorphically to the complement in Δ^n of an analytic subset Z of Δ^n, such that $Codim\ Z \geq 2$, by Theorem 2.2 and Proposition 2.2.

□

Theorem 3.1. *Let* X *be a complex manifold, and* Δ *be a smooth algebraic curve. Let* $\varphi : X \to \Delta$ *be a holomorphic, proper and surjective map, such that, for all* $z \in \Delta$ *(except a finite number),* $\varphi^{-1}(z)$ *is a parallelizable curve. Then,* X *is holomorphically extensifer to* $\Delta^n \setminus Z$, *where* Z *is an analytic subset of codimension at least 2.*

Before obtaining the final result, we have the following lemma which proves that a hypersurface of Δ^n meets the open set T.

Lemma 3.2. *Let H be an analytic subset of Δ^n, if H is a hypersurface of Δ^n (i.e. an analytic subset of pure codimension equal to 1), and let the Hartogs be an open set T of $\mathbb{C}^n$.*

Proof. Let H be an analytic subset of pure codimension equal to 1 of Δ^n, hence by definition:

$$H = \{z/f_i(z) = 0,\}$$

such that, f is holomorphic map from T $\left(f_{|H} = 0\right)$. We assume $H \cap T = \emptyset$. As f is holomorphic on Δ^n, then $1/f$ is holomorphic on T. $1/f$ is a holomorphic map on T, so it extends holomorphically to Δ^n, which is impossible, so we conclude that $H \cap T \neq \emptyset$. □

Proof. (Proof of Theorem 3.1) There exists a finite set $\{a_1, a_2, \cdots, a_m\}$ of points $a_i, i = 1, 2, \cdots, m$ on Δ, such that $C_u := \varphi^{-1}(u)$ is a regular fiber for any point $u \in \Delta \setminus \cup_{i=1,\cdots,m} \{a_i\}$.

Let $H_i = \varphi^{-1}(a_i)$, such that $a_i \in \{a_1, a_2, \cdots, a_m\}$, for all $z \in \Delta \setminus \cup H_i, \varphi^{-1}(z)$ is a parallelizable fiber, therefore $\varphi(z)$ is a submersion, and then $X \setminus \varphi^{-1}(\cup H_i)$ is holomorphically extensifer, by Lemma3.1, (because $\Delta \setminus \cup H_i \neq P^1(\mathbb{C})$).

We assume that $X \setminus \cup H_i, i = 1, \cdots, m$ is holomorphically extensifer, that is for any holomorphic map $f : T \to X$, the restriction $f : T \setminus f^{-1}(H_i) \to X \setminus H_i$ extends holomorphically from $T \setminus \widetilde{f^{-1}}(H_l)$ the envelope of holomorphy of $T \setminus f^{-1}(H_i)$.

Let E_i and F_i be an analytic subsets of Δ^n, such that $f^{-1}(H_i) = E_i \cup F_i$ with $Codim\ E_i = 1$ and $Codim\ F_i \geq 2$. We have :

$$T \setminus \widetilde{f^{-1}}(H_l) = T \setminus \widetilde{E}_l \cup F_l = \widetilde{T \setminus E_l} \cup \widetilde{T \setminus F_l} \cong \widetilde{T \setminus E_l}.$$

$\widetilde{T \setminus F_l} = \emptyset$ because $CodimF_i \geq 2$. As $Codim\ E_i = 1$, so E_i is a hypersurface of Δ^n, and we conclude that $\widetilde{T \setminus E_l}$ is isomorphic to $\Delta^n \setminus \widetilde{E}_l$ such that E_i is a hypersurface of Δ^n with $E_i = \widetilde{E}_l \cap T$.

We assume $Z = \cap \widetilde{E}_l$. It suffices to show that Z has codimension at least 2, so that f is holomorphically extensifer from $\Delta^n \setminus Z$. We have:

$$Z \cap T = \cap \widetilde{E}_l \cap T = \cap E_i \subset f^{-1}(H_i) = f^{-1}(\cap H_i) = \emptyset.$$

So, $Z \cap T = \emptyset$, then Z is an analytic subset of Δ^n with $Codim\ Z \geq 2$. $\square$

ACKNOWLEDGMENT

The authors are highly grateful to the anonymous referee for his/her valuable comments and suggestions for the improvement of the paper. This research work is supported by the General Direction of Scientific Research and Technological Development (DGRSDT), Algeria.

REFERENCES

[1] W. BARTH: *A. Van de ven, Compact, complex surfaces*, Springer-Verlag Berlin, New-York, Tokyo, 1984.

[2] H. BENKHE, P. THULLEN: *Theorie der Funktionen mehrerer Komplexen Veranderlishen*, Spriger-Verlag, 1970.

[3] A. BLANCHARD: *Sur les variétés analytiques complexes*, Annales scientifiques de l'École Normale Supérieure, **73**(2) (1956), 157–202.

[4] S. BOCHNER, D. MONTGOMERY: *Groups on analytic manifolds*, Ann. of Math., **48**(2) (1947), 659–669.

[5] A. BOREL, R. REMMERT: *Über kompakte homogene kählersche Mannigfaltigkeiten*, Mathematische Annalen, **145**(5) (1962), 429–439.

[6] G. DLOUSSKY: *Enveloppes d'holomorphie et prolongements d'hypersurfaces*, Séminaire Pierre Lelong (Analyse) Année 1975/76: Journées sur les Fonctions Analytiques, Toulouse, 1976, (2006), 217–235.

[7] G. DLOUSSKY: *Prolongement d'applications holomorphes*, Serninaire Pierre Lelong-Henri Skoda (Analyse) Année 1976/77, Springer, (1978), 42–95.

[8] F. DOCQUIER, H. GRAUERT: *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, Mathematische Annalen, **140**(2) (1960), 94–123.

[9] A, HUCKELBERRY, E. OELJEKLAUSS: *Classification theorems for Almost Homogeneous spaces*, Institut Elie Carton, 1984.

[10] S.M. IVASHKOVICH: *Extension of locally biholomorphic mappings into product of complex manifolds*, Mathematics of the USSR-Izvestiya, **27**(1) (1986), 193–199.

[11] S.M. IVASHKOVICH: *Extension of locally biholomorphic mappings of domains into complex projective space*, Mathematics of the USSR-Izvestiya, **22**(1) (1984), 181–189.

[12] S.M. IVASHKOVICH: *The Hartogs phenomen for holomorphically convex Kahler manifolds*, Math. USSR Izvestiya. **29**(1) (1987), 225–232.

[13] K. KODAIRA: *On compact analytic surfaces: II*, Annals of Mathematics, **77**(3) (1963), 563–624.

[14] B. MALGRANGE: *Lectures on the theory of functions of several complex variables, Tata Inst*, Fund. Res. Bombay, 1958.

[15] J. POTTERS: *On almost homogeneous compact complex surfaces*, Inventiones mathematicae, **8**(3) (1969), 244–266.

[16] O. RIEMENSCHNEIDER: *Über die Anwendung algebraischer Methoden in der Deformationstheorie komplexer Räume*, Mathematische Annalen, **187**(1) (1970), 40–55.

[17] R. REMMERT: *Holomorphe und meromorphe Abbildungen komplexer Räume*, Mathematische Annalen, **133**(4) (1957), 328–370.

[18] R. REMMERT, T. VAN DE VEN: *Über holomorphe Abbildungen projektiv-algebraischer Mannigfaltigkeiten auf komplexe Räume*, Mathematische Annalen, **142**(5) (1961), 453–486.

[19] B. SHIFFMAN: *Extension of holomorphic maps into Hermitian manifolds*, Mathematische Annalen, **194**(4) (1971), 249–258.

[20] Y.T. SIU: *Techniques of extension of analytic objects, Mercel Dekker*, Inc. New York, 1974.

[21] Y.T. SIU: *Some recent developments in complex differential geometry*, Proceedings of the International Congress of Mathematicians, **1** (1983), 287–297.

[22] Y.T. SIU: *Every Stein subvariety admits a Stein neighborhood*, Inventiones mathematicae, **38**(1) (1976), 89–100.

[23] K. STEIN: *Überlagerungen holomorph-vollständiger komplexer Räume*, Archiv der Mathematik, **7**(5) (1956), 354–361.

[24] H.C. WANG: *Complex parallisable manifolds*, Proceedings of the American Mathematical Society, **5**(5) (1954), 771–776.

[1,2]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FERHAT ABBAS SETIF 1, CAMPUS EL BEZ, SETIF, ALGERIA.

Email address: mohamednadir19@hotmail.fr or nadir.boukhetouta@univ-constantine2.dz

Email address: mkrachni@univ-setif.dz

[3,4]DEPARTMENT OF MATHEMATICS, AMAR TELIDJI UNIVERSITY OF LAGHOUAT, LABORATORY OF PURE AND APPLIED MATHEMATICS, 37G GHARDAIA ROAD, LAGHOUAT, ALGERIA.

Email address: f.yazid@lagh-univ.dz

Email address: fs.djeradi@lagh-univ.dz