Advances in Mathematics: Scientific Journal **12** (2023), no.1, 225–278
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.12.1.15

GOODNESS-OF-FIT TESTS FOR THE NEW WEIBULL-G FAMILY OF DISTRIBUTIONS

K.K. Meribout[1], N. Seddik-Ameur, and H. Goual

ABSTRACT. In this paper, we present two New models named New-Weibull-Weibull (NWW) and New-Weibull-Rayleigh (NWR) from The New-Wei- bull-G family recently introduced that can have a variety of hazard rate shapes that allows to describe observations from different fields of study. The unknown parameters of the NWW and NWR models have been estimated under the maximum likelihood estimation method. Moreover, we construct a modified chi-squared goodness-of-fit test based on the *Nikulin– Rao–Robson* (NRR) statistic to verify the applicability of the proposed NWW and NWR models. The modified test shows that the models studied can be used as a good candidate for analyzing a large variety of real phenomena. The NWW and NWR models are applied upon a five different real complete and right-censored data sets in order to evaluate its practicability and flexibility.

1. INTRODUCTION AND MOTIVATION

In reliability studies, Industrials have to analyse feedback experience data with the aim of obtaining reliable results, in survival analysis practitioners are in presence of different observations to study, so all of them have to choose the appropriate model for their analysis.

[1]*corresponding author*

2020 Mathematics Subject Classification. 49K15, 53B20, 53Z30.
Key words and phrases. Extension, fiber parallelizable, Hartogs domain, holomorphically extensifer, smooth algebraic curve.
Submitted: 07.12.2022; *Accepted:* 22.12.2022; *Published:* 26.01.2023.

Among new generalized distributions proposed in the statistical literature, a new generator of statistical distributions introduced by *Tahir et al.* [12] so-called New *Weibull*-G $(NW-G)$ family is an extension of the $T-X$ distributions family *Alzaatreh et al.* [1] which received great attention from researchers. The $NW-G$ family is very interesting in the sense that the obtained rate failures can be constant, increasing, decreasing, bathtub, upside-down bathtub, J, reversed-J, and S which widen its fields of applications. In the recent literature, owing of the significance of the *Weibull* distribution in modelling reliability and survival data, four generators of the T-X family [1] have been derived: bêta Weibull-G [10], Weibull-X [1], exponentiated Weibull-X [2] and Weibull-G [11] with different forms of the distribution generators X.

Depending on the generator distributions,*Tahir et al.* [12] proposed different models such as the New *Weibull*-uniform (NWU), the New *Weibull* exponential (NWE), the New *Weibull*-logistic (Wlo), the New *Weibull*-log-logistic $(NWLL)$, the New *Weibull-Bur XII* $(NWBXII)$ and the New *Weibull*-normal (NWN) distributions. So, they obtained a panoply of new models capable of describing any type of data. Statistical characteristics and properties are derived nevertheless the problem of the validation of these models has not been considered till now, which motivate us to develop some statistic tests to verify how complete and right-censored real data can be fitted by the New *Weibull*-G family. In this work, two new distributions called the *New Weibull-Rayleigh* (NWR) and the *New Weibull-Weibull*(NWW) are studied, we use firstly the classical model selection criteria such as the Akaike information criterion (AIC), the consistent Akaike information criterion $(CAIC)$, the Hannan-Quinn information criterion $(HQIC)$, the Bayesian information criterion (BIC), and the Kolmogorov Smirnov test (KS). Also for testing a composite hypothesis H_0, different EDF statistics are used like the Anderson -Darling statistic, Kolmogorov-Smirnov statistic, Cramer-Von-Mises, and others but the critical values of these statistics are not available for new models in the statistical literature. So we propose in this work, criteria goodness-of-fit tests based on modified Pearson statistic Y^2 called the Nikulin-Rao-Robson (NRR) statistic, [Nikulin [16] [17], Rao and Robson [9]], for the New *Weibull-Weibull* (NWW) and the New *Weibull-Rayleigh* (NWR) models. The proposed statistic which is based on maximum likelihood method for estimating the unknown parameters gives a very powerful test and covers all the information given by the collected data. Its construction depends on the hypothesized distribution. Some Y^2 formulas have been adapted for classical models nevertheless new ones have not been considered yet.

Using the initial data in estimating the unknown parameters, the maximum likelihood estimators and estimated information matrix are also provided in the case of complete and censored data. Theoretical results are confirmed by simulating thousands of samples from different sizes. Also ,examples from real data from different fields are applied to show the practicability of the proposed test statistics and the importance of these new models (NWW, NWR).

2. Characterisation

Generally speaking, there has been a fundamental interest in creating new generators for families of univariate continuous distributions by adding one or more additional shape parameters to the base distribution. the induction of extra parameters turned out that this was helpful in enhancing the family's quality of adjustment. In life testing, engineering, survival theory, and reliability theory, the *Weibull* distribution is one of the most well-liked and frequently applied models for failure time. The hazard rate function of the classical *Weibull* distribution exhibits monotone behavior. whereas the curves of the empirical hazard rate frequently exhibit non-monotonic shapes in the majority of real applications, including bell, bathtub, inverted bathtub forms, and others. On the other hand, *Rayleigh* distribution is widely used in sociological approaches, the physical sciences, engineering (to measure the lifetime of an object) and also in bio-animal analysis. So, the New *Weibull*-G Family was introduced by *Tahir et al.* [12] will enable us to model a broad variety of real phenomena which have the shapes of hazard rate in J, S, increasing and bathtub. We mention briefly two generators that have influenced our New *Weibull*-G family of distributions. By replacing the base distribution of the random variable T of the $T - X$ family [1] by the *Weibull* distribution we obtain the *Weibull*-X family [1], influenced by the G-*Zografos-Balakrishnan* class [10], *Bourguignon et al.* [11] defined the *Weibull*-G family.

We are interested in the New-Weibull-Weibull (NWW) and New-Weibull-Rayleigh (NWR) models of the $NW-G$ family introduced by *Tahir et al.* in 2016 [13] because they can provide a large range of hazard rate shapes : constant, increasing, decreasing, bathtub, J, inverted J, and S.

2.1. **Characterizing the New-Weibull-Weibull model.**

The New-Weibull-Weibull (NWW) distribution is a 4-parameter distribution with the scale parameter represented by α, the shape parameter by β and γ and the location parameter λ. The cumulative distribution function (cdf) F_{NWW}, and its probability density function (pdf) f_{NWW} are given by:

$$F_{NWW}\left(x,\vartheta\right)=\exp\left\{-\alpha\left[-\log\left(1-\exp\left(-\lambda x^{\gamma}\right)\right)\right]^{\beta}\right\},\quad x>0,\quad \vartheta=\left(\alpha,\beta,\lambda,\gamma\right),$$

$$\begin{aligned} f_{NWW}\left(x,\vartheta\right)&=\alpha\beta\frac{\lambda\gamma x^{\gamma-1}\exp(-\lambda x^{\gamma})}{1-\exp\left(-\lambda x^{\gamma}\right)}\left\{-\log\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right\}^{\beta-1}\\ &\quad\times\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right\}^{\beta}\right). \end{aligned}$$

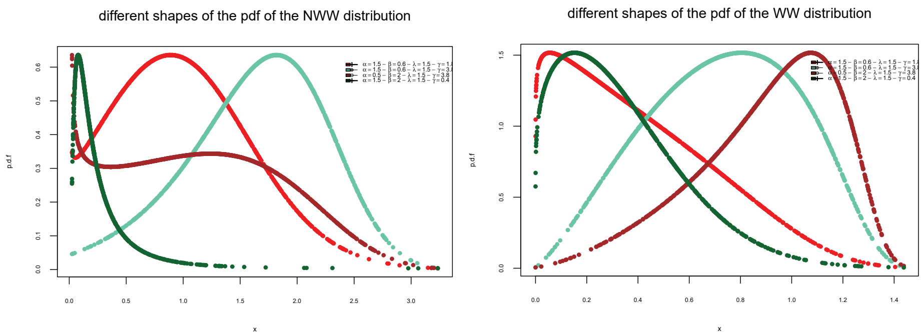
The analytic hazard rate function (hrf) of the NWW distribution is:

$$\begin{aligned} h_{NWW}\left(x,\vartheta\right)&=\frac{\alpha\beta\frac{\lambda\gamma x^{\gamma-1}\exp(-\lambda x^{\gamma})}{1-\exp(-\lambda x^{\gamma})}\left\{-\log\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right\}^{\beta-1}}{1-\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right\}^{\beta}\right)}\\ &\quad\times\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right\}^{\beta}\right), \end{aligned}$$

To show the quality and flexibility of this new distribution (NWW), we plot the probability density function and the hazard rate function by comparing them to the density and hazard rate of the Weibull-Weibull (WW) distribution [5].

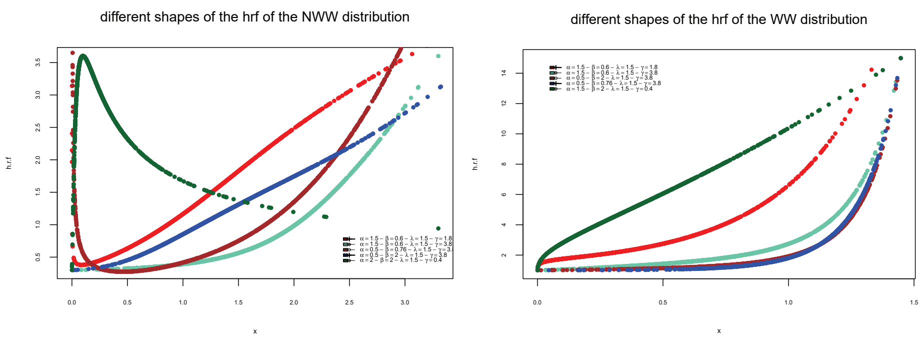
We present in Figure 1 and Figure 2, the pdf and hrf of the NWW distribution and the WW distribution.

According to Figure 1, the NWW distribution produces a variety of forms of pdf, including left-skewed, right-skewed, bathtub, and bell shapes. Also shown in Figure 2, is the family's ability to generate a variety of hazard rate forms, including increasing, bathtub, bell, and J. Indeed, the NWW distribution can be highly helpful for fitting diverse data sets with varying shapes, such as economical and reliability data.



(A) NWW (B) WW

FIGURE 1. pdf of the NWW distribution against the WW distribution.



(A) NWW (B) WW

FIGURE 2. hrf of the NWW distribution against the WW distribution.

2.2. **Characterizing the New-Weibull-Rayleigh model.**

We establish the New-Weibull-Rayleigh (NWR) probability density function (pdf) and cumulative distribution function (cdf) as follows

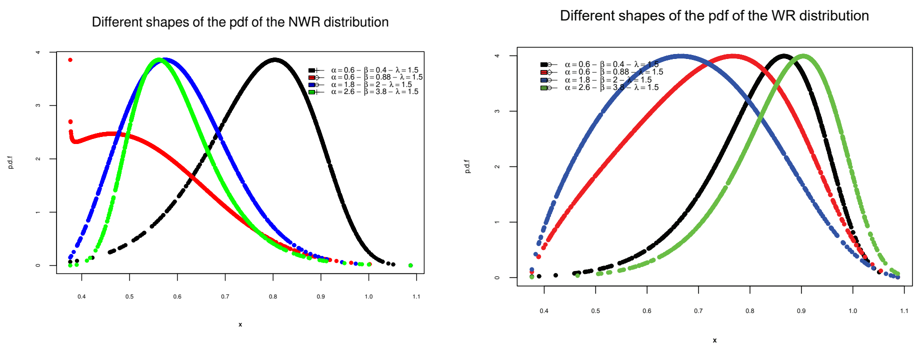
$$F_{NWR}(x,\varsigma) = \exp\left\{-\alpha\left[-\log\left(1-\exp\left(-\lambda x^2\right)\right)\right]^{\beta}\right\}, x>0, \varsigma=(\alpha,\beta,\lambda)>0$$

$$\begin{aligned} f_{NWR}(x,\varsigma) &= \alpha\beta\frac{\lambda\gamma x\exp(-\lambda x^2)}{1-\exp\left(-\lambda x^2\right)}\left\{-\log\left[1-\exp\left(-\lambda x^2\right)\right]\right\}^{\beta-1} \\ &\times\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x^2\right)\right]\right\}^{\beta}\right). \end{aligned}$$

Here $\varsigma = (\alpha, \beta, \lambda)$ is the vector of unknown parameters of the NWR distribution, which β is the shape parameter, α and λ represent the scale and location parameters, respectively. The corresponding hazard rate function (hrf) is

$$h_{NWR}(x,\varsigma) = \frac{\alpha\beta \frac{\lambda\gamma x \exp\left(-\lambda x^2\right)}{1-\exp(-\lambda x^2)} \left\{-\log\left[1-\exp\left(-\lambda x^2\right)\right]\right\}^{\beta-1}}{1-\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x^2\right)\right]\right\}^{\beta}\right)}$$
$$\times \exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x^2\right)\right]\right\}^{\beta}\right),$$

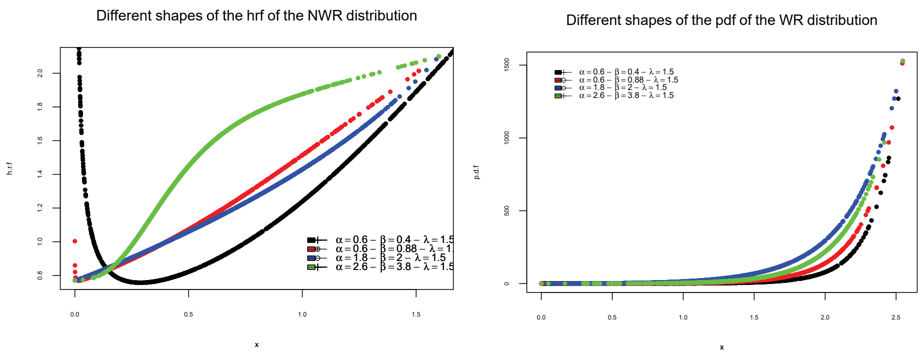
We display In Figure 3 and Figure 4 the pdf and hrf, respectively, of the NWR distribution by comparing them to the pdf and hrf of the Weibull-Rayleigh (WR) [5] distribution in order to demonstrate the flexibility and variability of this novel distribution (NWR).



(A) NWR (B) WR

FIGURE 3. pdf of the NWR distribution against the WR distribution.

For different combination of parameters α, β and λ of the NWR distribution, Figure 3 shows that the NWR distribution generates a range of pdf shapes, including symmetric, left- and right-skewed, decreasing and bell-shaped. and Figure 4 also demonstrates the family's capacity to produce several hazard rate shapes, such as increasing, bathtub, bell, and S form. Indeed, for fitting a variety of data sets with varied forms in different real application the NWR distribution may be quite beneficial.



(A) NWR (B) WR

FIGURE 4. hrf of the NWR distribution against the WR distribution.

3. MLEs of studied models

The maximum likelihood estimators (MLEs) of the unknown parameters are necessary for the construction of the NRR statistic and other selection model criteria, so we provide the score functions for the studied models NWW and NWR, for more details see [18].

3.1. **Completed case.**

3.1.1. *Simulation of the New-Weibull-Weibull model (NWW).* Let us consider a sample $(x_1, \ldots, x_n)$ from the NWW distribution, the likelihood function is given by

$$\begin{aligned} L_{NWW}(x_i, \vartheta) &= \prod_{i=1}^{n} f_{NWW}(x_i, \vartheta) \\ &= \prod_{i=1}^{n} \left[\alpha\beta \frac{\lambda\gamma x_i^{\gamma-1} \exp(-\lambda x_i^{\gamma})}{1-\exp(-\lambda x_i^{\gamma})} \left\{ -\log\left[1-\exp\left(-\lambda x_i^{\gamma}\right)\right]\right\}^{\beta-1} \right. \\ &\quad \left. \times \exp\left(-\alpha \left\{ -\log\left[1-\exp\left(-\lambda x_i^{\gamma}\right)\right]\right\}^{\beta} \right) \right]. \end{aligned}$$

So, the log-likelihood function:

$$l(x_i, \vartheta) = \sum_{i=1}^{n} \log f_{NWW}(x_i, \vartheta)$$

$$= n\log(\alpha\beta\lambda\gamma) + \sum_{i=1}^{n}\log\left(x_i^{\gamma-1}\right) - \sum_{i=1}^{n}(\lambda x_i^{\gamma})$$
$$+ (\beta-1)\sum_{i=1}^{n}\log\{-\log[1-\exp(-\lambda x_i^{\gamma})]\}$$
$$- \alpha\sum_{i=1}^{n}\{-\log[1-\exp(-\lambda x_i^{\gamma})]\}^{\beta} - \sum_{i=1}^{n}\log[1-\exp(-\lambda x_i^{\gamma})].$$

The score functions (Appendix 1) are obtained by deriving the log-likelihood function with respect to the unknown parameters of the NWW distribution. So, the maximum likelihood estimators of the vector ϑ are obtained by equalling to zero the first derivatives above. To solve these equations, iterative methods are required. The elements of the estimated Information Fisher Matrix (IFM) are also derived (see Appendix 1).

We propose a simulation study by using R software, the Newton-Raphson method available in the BB package [25] is used to numerically estimate the unknown parameters $\widehat{\vartheta} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}, \widehat{\gamma}\right)$ of the NWW model with their mean square errors (MSE). We choose different sizes of sample $n = 20 - 50 - 100 - 150 - 250 - 500 - 750$, and the initial values of the parameters ($\alpha = 1, \beta = 1.5, \lambda = 2.5, \gamma = 1.3$). The results of $12,000$ simulations are shown in Table 1.

TABLE 1. The mean square errors of the maximum likelihood estimators $\widehat{\vartheta}$ of the New-Weibull-Weibull distribution.

$N = 12,000$	$n = 20$	$n = 50$	$n = 100$	$n = 150$	$n = 250$	$n = 500$	$n = 750$
$\widehat{\alpha}$	0.9894	1.0229	1.0018	1.0013	1.0012	1.0007	1.0005
MSE	8.4764×10^{-4}	6.9313×10^{-4}	5.8781×10^{-4}	5.0308×10^{-5}	4.5366×10^{-5}	4.1171×10^{-5}	3.6648×10^{-5}
$\widehat{\beta}$	1.5084	1.5070	1.5055	1.5034	1.4986	1.4977	1.4953
MSE	3.7397×10^{-4}	2.0336×10^{-4}	1.2016×10^{-4}	8.7691×10^{-5}	6.5800×10^{-5}	5.6558×10^{-5}	4.3126×10^{-5}
$\widehat{\lambda}$	2.5093	2.5100	2.5135	2.5041	2.4987	2.4970	2.4956
MSE	7.9374×10^{-4}	4.1065×10^{-4}	2.5210×10^{-4}	7.7870×10^{-5}	1.3390×10^{-5}	9.4950×10^{-6}	7.6621×10^{-6}
$\widehat{\gamma}$	1.3266	1.3235	1.3201	1.3183	1.3172	1.3154	1.3088
MSE	7.5275×10^{-4}	5.5352×10^{-4}	4.0184×10^{-4}	3.3086×10^{-4}	2.9222×10^{-4}	2.3623×10^{-4}	9.6658×10^{-5}

We want to reinforce the results obtained in Table 1, by plotting in Figure 5 the $\sqrt{n}$-convergence curve compared to the values of the MLEs of the NWW distribution for the different sample sizes used in the simulation study ($n = 20 - 50 - 100 - 150 - 250 - 500 - 750$).

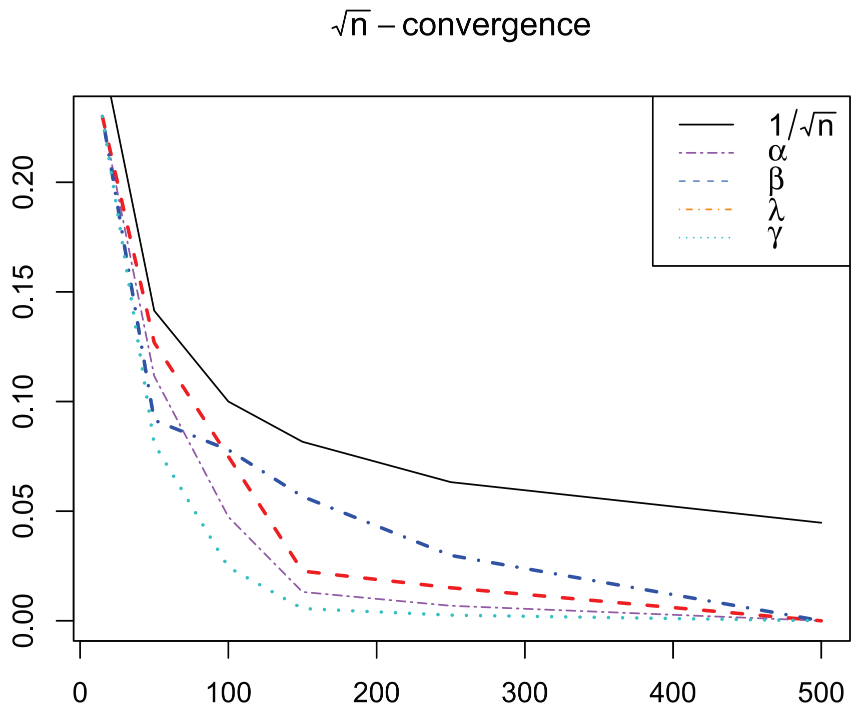


FIGURE 5. The $\sqrt{n}$-convergence of the parameters $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\lambda}$ and $\widehat{\gamma}$ of the New-Weibull-Weibull distribution.

From Figure 5 we can conclude that the MLEs of the parameters $\alpha, \beta, \lambda, \gamma$ of the NWW distribution converge faster than $n^{-1/2}$, this demonstrates that the maximum likelihood estimation method used produces accurate results.

3.1.2. *Simulation of the New-Weibull-Rayleigh model (NWR).* Now, suppose that a sample $(x_1, \ldots, x_n)$ of size n belongs to the NWR distribution. So, the likelihood function is obtained as

$$\begin{aligned}
L_{NWR}(x_i,\varsigma) &= \prod_{i=1}^{n} f_{NWR}(x_i,\varsigma) \\
&= \prod_{i=1}^{n}\left[2\alpha\beta\lambda\frac{x_i\exp(-\lambda x_i^2)}{1-\exp(-\lambda x_i^2)}\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\}^{\beta-1}\right. \\
&\times \left.\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\}^{\beta}\right)\right].
\end{aligned}$$

and the log-likelihood function is

$$\begin{aligned}
l(x_i,\varsigma) &= \sum_{i=1}^{n}\log f_{NWR}(x_i,\varsigma) \\
&= n\log(2\alpha\beta\lambda)+\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\left(\lambda x_i^2\right) \\
&\quad +(\beta-1)\sum_{i=1}^{n}\log\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\} \\
&\quad -\alpha\sum_{i=1}^{n}\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\}^{\beta}-\sum_{i=1}^{n}\log\left[1-\exp\left(-\lambda x_i^2\right)\right].
\end{aligned}$$

The score functions are obtained by deriving the log-likelihood function relative to the unknown parameters ς of the NWR distribution. So, the maximum likelihood estimators of $\widehat{\varsigma}$ are obtained by equalling to zero the first derivatives above. To solve these equations, iterative methods are required. The formula of the score functions and the elements of the estimated Information Fisher Matrix (IFM) are derived and given in Appendix 2.

The same as the previous model, we use the BB package [26] of the R software to estimate the vector of parameters ς of the NWR distribution and their MSE by performing $12,000$ simulations for different sizes of sample: $n = 20-50-100-150-250-500-750$ and the initial values of the parameters $(\alpha = 2.90, \beta = 3.30, \lambda = 1.80)$. The results of simulation are illustrated in Table 2:

To affirm the results of the simulation made above in Table 2, we draw in Figure 6, the $\sqrt{n}$-convergence curve compared to the values of the MLEs of the NWR distribution for the different sample sizes used $n = 20-50-100-150-250-500-750$.

TABLE 2. The mean square errors of the maximum likelihood estimators $\widehat{\varsigma}$ of the New-Weibull-Rayleigh distribution.

$N = 12,000$	$n = 20$	$n = 50$	$n = 100$	$n = 150$	$n = 250$	$n = 500$	$n = 750$
$\widehat{\alpha}$	2.9538	2.9402	2.9333	2.9166	2.9102	2.9055	2.9003
MSE	4.6603×10^{-3}	4.0301×10^{-3}	3.5478×10^{-3}	3.0126×10^{-3}	5.3456×10^{-4}	3.0211×10^{-4}	5.2261×10^{-5}
$\widehat{\beta}$	3.3638	3.3511	3.3409	3.3326	3.3166	3.3018	3.3004
MSE	4.0954×10^{-3}	3.2245×10^{-3}	3.0645×10^{-3}	2.8712×10^{-3}	6.7803×10^{-4}	4.5870×10^{-4}	7.0819×10^{-5}
$\widehat{\lambda}$	1.8538	1.8416	1.8322	1.8266	1.8107	1.8069	1.8001
MSE	7.7508×10^{-3}	6.6421×10^{-3}	5.8941×10^{-3}	4.4587×10^{-3}	2.3133×10^{-3}	7.0213×10^{-4}	1.4503×10^{-4}

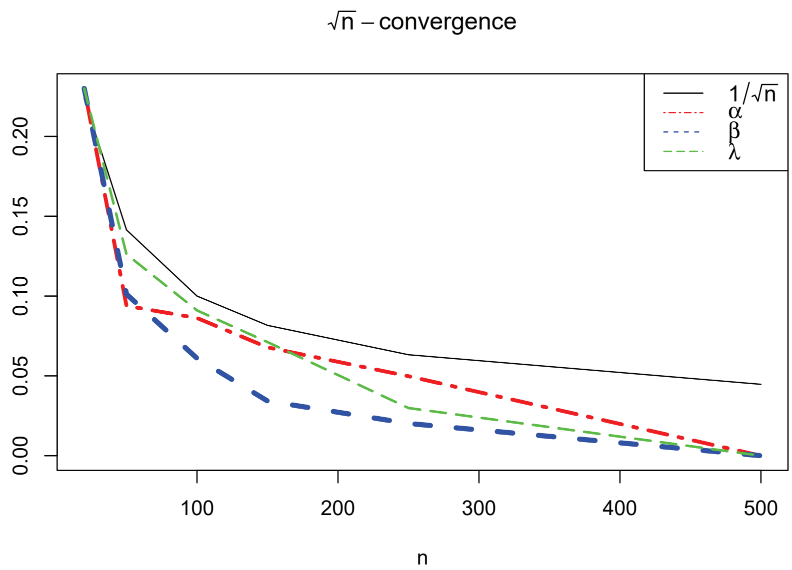


FIGURE 6. The $\sqrt{n}$-convergence of the parameters $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\lambda}$ of the New-Weibull-Rayleigh distribution.

As expected, Figure 6 leads us to conclude that the $MLEs$ of the NWR are consistent for all sample sizes which confirms the property of the maximum likelihood estimation method.

3.2. **Censored case.**

Let X_i be a random variable distributed with the vector of parameters Θ. The data encountered in survival analysis and reliability studies are often censored. A very simple random censoring mechanism that is often realistic is one in which each individual i is assumed to have a life time X_i and a censoring time C_i, where X_i and C_i are independent random variables, for more details see [19]. Suppose that the data consist of n independent observations $x_i = min(X_i, C_i)$, for $i = 1, \ldots, n$. The distribution of C_i does not depend on any of the unknown parameters of X_i. The likelihood function can be given as follow

$$L(\theta) = \Pi_{i=1}^{n} \lambda^{\delta_i} (X_i, \theta) S(X_i, \theta) = 1\{X_i \leq C_i\}, \theta \in \Theta.$$

Then the log-likelihood function

$$l(\theta) = \sum_{i=1}^{n} \delta_i \ln \lambda (X_i, \theta) + \sum_{i=1}^{n} S(X_i, \theta), \quad \theta \in \Theta.$$

3.2.1. *Simulation of the censored MLEs of the NWW model.* We suppose the failure rate $X_i \rightsquigarrow NWW(x, \vartheta)$. We calculate the maximum likelihood estimators of the vector of parameters $\vartheta = (\alpha, \beta, \lambda, \gamma)$. The censored log-likelihood function of the NWW distribution

$$\begin{aligned} l_{NWW}(x, \vartheta) \quad = \quad & r \log(\alpha\beta\lambda\gamma) + (\gamma - 1) \sum_{i \in F} \log(x_i) - \lambda \sum_{i \in F} x_i^{\gamma} \\ & + \sum_{i \in C} \log\left[1 - \exp\left(-\alpha \{-\log[1 - \exp(-\lambda x_i^{\gamma})]\}^{\beta}\right)\right] \\ & - \alpha \sum_{i \in F} \{-\log[1 - \exp(-\lambda x_i^{\gamma})]\}^{\beta} - \sum_{i \in F} \log[1 - \exp(-\lambda x_i^{\gamma})] \\ & + (\beta - 1) \sum_{i \in F} \log\{-\log[1 - \exp(-\lambda x_i^{\gamma})]\}, \end{aligned}$$

Here r is the number of failures and F and C denote the sets of uncensored and censored observations, respectively. The score functions of the unknown vector of parameters ϑ and the IFM of the NWW distribution are calculated and given in Appendix 1.

The right-censored data provided from the NWW distribution was simulated $N = 12,000$ times, with the following initial values of parameters ($\alpha = 9.1$, $\beta = 1.5$, $\lambda = 3.06$, $\gamma = 3.1$). Using the BB package [25] of the R software for right-censored data, we calculate the mean values of the $MLEs$ of the parameters α, β, λ and γ as well as their mean squared errors (MSE) are calculated in Table 3 for different sample sizes ($n = 20 - 50 - 100 - 150 - 250 - 500 - 750$).

TABLE 3. MSE of the censored MLEs $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\lambda}$ and $\widehat{\gamma}$ of the New-Weibull-Weibull distribution.

$N = 12,000$	$n = 20$	$n = 50$	$n = 100$	$n = 150$	$n = 250$	$n = 500$	$n = 750$
$\widehat{\alpha}$	9.1838	9.1716	9.1563	9.1012	9.1004	9.1001	8.9875
MSE	3.1764×10^{-3}	8.2267×10^{-4}	5.9015×10^{-4}	3.1776×10^{-4}	1.5389×10^{-4}	7.6845×10^{-5}	5.2144×10^{-5}
$\widehat{\beta}$	1.5236	1.5133	1.5103	1.5047	1.5005	1.4999	1.4992
MSE	3.9028×10^{-3}	1.5987×10^{-3}	6.6639×10^{-4}	4.7225×10^{-4}	1.0661×10^{-4}	8.4932×10^{-5}	6.2647×10^{-5}
$\widehat{\lambda}$	3.0928	3.0856	3.0831	3.0705	3.0640	3.0611	3.0599
MSE	8.3684×10^{-3}	2.2975×10^{-3}	1.2167×10^{-3}	7.7233×10^{-4}	3.221×10^{-4}	1.0472×10^{-4}	7.5438×10^{-5}
$\widehat{\gamma}$	3.2308	3.1639	3.1344	3.1268	3.1036	3.1002	2.9967
MSE	1.5534×10^{-2}	3.1812×10^{-3}	4.6265×10^{-4}	2.1137×10^{-4}	7.2128×10^{-5}	3.9045×10^{-5}	1.0871×10^{-5}

To study the convergence of the censored $MLEs$ of the parameters α, β, λ and γ of the NWW distribution, we plot, in Figure 7 the calculated mean absolute values of the $MLEs$ compared to the size n of samples chosen above:

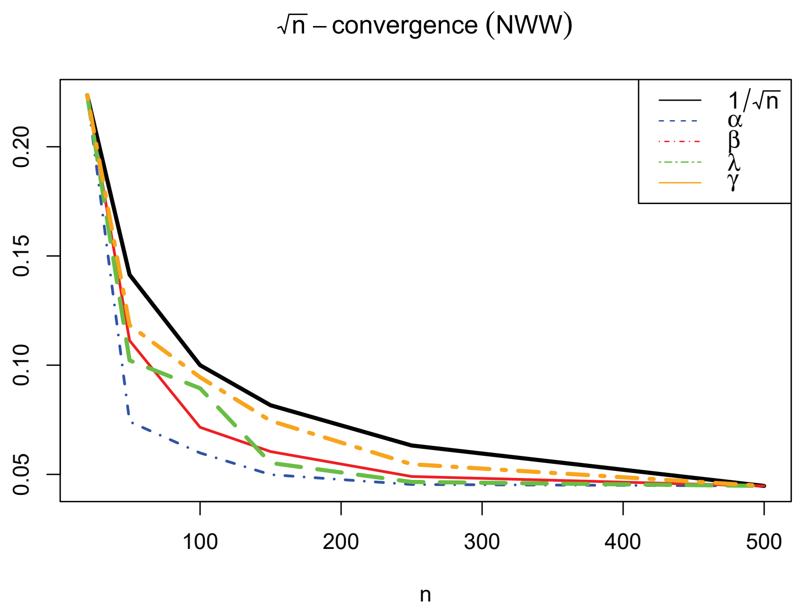


FIGURE 7. The $\sqrt{n}$-convergence of the parameters $\widehat{\alpha}$,$\widehat{\beta}$, $\widehat{\lambda}$ and $\widehat{\gamma}$ of the New-Weibull-Weibull distribution in the right-censored case.

From Figure 7 and the simulation results obtained in Table 3, we notice that all the estimators converge faster than $n^{-0.5}$ by asserting that the maximum likelihood estimators are $\sqrt{n}$-convergent.

3.2.2. *Simulation of the censored MLEs of NWR model.* Here, we consider the failure rate $X_i \rightsquigarrow NWR(x,\varsigma)$. So, the log-likelihood function of the NWR distribution is

$$\begin{aligned}
l(x,\varsigma) &= \sum_{i-1}^{n}\delta_i[\log(2\alpha\beta\lambda)+\log(x_i)-\lambda x_i^2+(\beta-1)\log\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\} \\
&= r\log(2\alpha\beta\lambda)+\sum_{i\in F}\log(x_i)-\lambda\sum_{i\in F}x_i^2 \\
&\quad+\sum_{i\in C}\log\left[1-\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\}^{\beta}\right)\right] \\
&\quad-\alpha\sum_{i\in F}\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\}^{\beta}-\sum_{i\in F}\log\left[1-\exp\left(-\lambda x_i^2\right)\right] \\
&\quad+(\beta-1)\sum_{i\in F}\log\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\}.
\end{aligned}$$

The score functions of the vector of parameters $\widehat{\varsigma}$ and IFM are calculated and given in Appendix 2.

Using the same algorithm (BB) used before, we simulate $N = 12,000$ times data from the NWR distribution, with the following initial parameter values ($\alpha = 9.5$, $\beta = 2.5$, $\lambda = 3$) for the different sample sizes used previously, we calculate the mean values of the $MLEs$ of the estimated parameters α, β and λ and their MSE, the results of the simulation are illustrated in Table 4:

To study the convergence of the MLEs of the parameters of the NWR distribution, we plot in Figure 8 the curve of the mean absolute error of the parameters estimated by the maximum likelihood method compared to the different sample sizes n used above.

From Figure 8 and the simulation results (Table 4), we notice that all the estimators converge faster than $n^{-0.5}$ affirm that the MLEs of the NWR model are $\sqrt{n}-$convergent.

TABLE 4. MSE of the censored MLEs $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\lambda}$ of the New-Weibull-Rayleigh.

$N = 12,000$	$n = 20$	$n = 50$	$n = 100$	$n = 150$	$n = 250$	$n = 500$	$n = 750$
$\widehat{\alpha}$	9.6616	9.5756	9.5569	9.5206	9.5173	9.5007	9.4988
MSE	8.7025×10^{-3}	9.4710×10^{-4}	5.6230×10^{-4}	1.4288×10^{-4}	6.3754×10^{-5}	4.5073×10^{-5}	2.1344×10^{-5}
$\widehat{\beta}$	2.6211	2.5988	2.5410	2.5259	2.5102	2.5001	2.4987
MSE	7.4461×10^{-3}	2.7906×10^{-3}	8.1866×10^{-4}	5.8379×10^{-4}	1.2121×10^{-4}	6.0230×10^{-5}	3.0871×10^{-5}
$\widehat{\lambda}$	3.1651	3.0822	3.0529	3.0245	3.0017	2.9984	2.9955
MSE	1.0904×10^{-3}	6.0923×10^{-4}	2.2929×10^{-4}	1.0635×10^{-4}	5.4582×10^{-5}	3.7117×10^{-5}	1.1656×10^{-5}

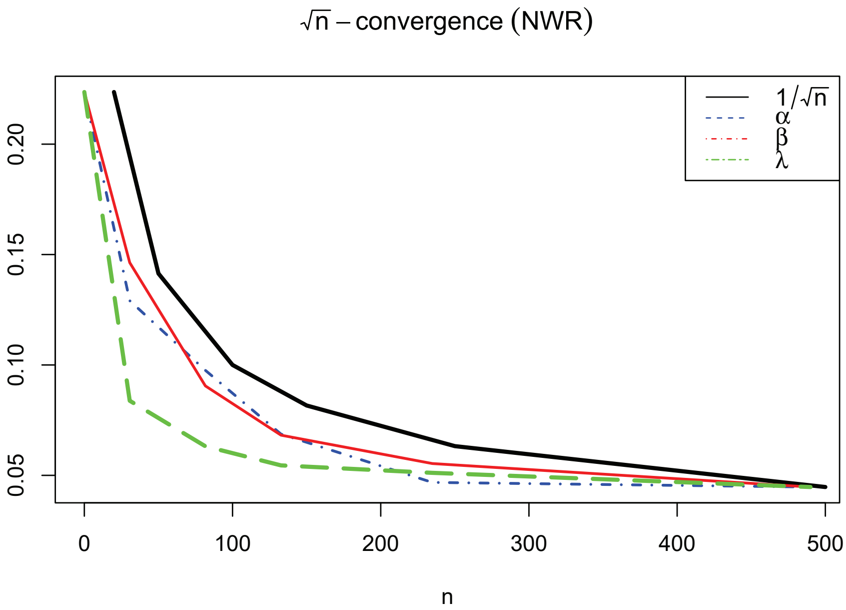


FIGURE 8. The $\sqrt{n}$-convergence of the parameters $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\lambda}$ of the New-Weibull-Rayleigh distribution in the right-censored case.

4. VALIDATION

In this section, numerical illustrations is presented here to exhibit the abilities of the distribution via simulation study of the well known criterion tests: Akaike Information Criteria (AIC), Consistent Akaike Information Criteria ($CAIC$), Bayesian Information Criteria (BIC), Hanan and Quinn Information Criteria ($HQIC$), and Kolmogorov Smirnov (KS).

In 1973, Nikulin ([16], [17]) proposed a modification of Pearson's chi square test for the family of continuous distributions with shift and scale parameters. For their part, Rao and Robson [9] obtained the same result for exponential families, and since 1998 the test is well known as the Nikulin-Rao-Robson test notated NRR test. So, we use the NRR test to fit models in the case of completed data. In 2011, Bagdonavicius and Nikulin ([22], [23]) gave a chi-squared type goodness-of-fit test capable to fit parametric distributions in the case of right-censored data. These tests are based on the maximum likelihood estimation for ungrouped data.

4.1. **Classical tests.**

4.1.1. *The NWW model.* The NWW distribution has been compared with four distributions Weibull (W), Inverse Weibull (IW) [27], Topp-Leone Weibull Weibull ($TLWW$) [3] and Weibull-Weibull (WW) [11]. The R software has been used to compute the analytical measures: AIC, $CAIC$, BIC, $HQIC$ and KS.

We simulated 12,000 times data according to W, IW, $TLWW$, WW and NWW distribution. For different size of samples n, we have calculated the test criteria mentioned above, the results are illustrated in Table 5

TABLE 5. Smaller $AIC/BIC/CAIC/HQIC/KS$ scores in 12,000 simulations from NWW and comparative distributions.

n	Model	AIC	BIC	$CAIC$	$HQIC$	KS
10	W	31.25	32.06	31.22	31.64	31.18
	IW	30.66	30.98	31.57	31.41	31.83
	WW	31.87	31.33	32.14	31.05	30.8
	$TLWW$	31.04	31.07	31.11	31.28	31.22
	NWW	**30.21**	**30.45**	**30.77**	**30.13**	**31.04**
20	W	27.32	27.64	27.44	27.39	24.91
	IW	20.78	21.2	20.99	20.86	24.96
	WW	20.48	21.11	20.90	20.6	24.8
	$TLWW$	20.1	20.74	20.53	20.23	24.79
	NWW	**20.04**	**20.67**	**20.46**	**20.16**	**23.58**

30	W	25.13	26.88	26.91	27.03	25.89
	IW	22.64	25.46	20.08	19.44	24.72
	WW	22.18	23.31	20.31	20.25	25.01
	$TLWW$	21.95	23.11	21.22	19.66	24.64
	NWW	**20.22**	**20.82**	**19.85**	**19.37**	**24.05**
50	W	20.14	20.45	20.16	20.26	25.37
	IW	19.72	20.17	19.76	19.89	25.20
	WW	20.35	21.12	20.44	20.64	25.21
	$TLWW$	20.54	21.15	20.61	20.77	25.31
	NWW	**18.53**	**19.29**	**18.61**	**18.82**	**25.08**
100	W	20.2	20.41	20.21	20.28	25.48
	IW	21.18	21.49	21.31	20.56	25.44
	WW	20.31	20.93	20.33	20.56	25.39
	$TLWW$	21.56	20.93	21.17	21.32	25.44
	NWW	**18.65**	**19.28**	**18.68**	**18.91**	**25.38**

From Table 5 we can analyze that the NWW distribution has the smallest criterion test values used which leads us to deduce that the NWW model will give more reliable results in applications to real data.

4.1.2. *The NWR model.* We use the R software to calculate the classical test criteria mentioned above to validate the model studied by comparing the NWR distribution with Rayleigh (R), Inverse Rayleigh (IR) [24], Rayleigh-Rayleigh ($Ra - Ra$) [19] and Weibull Rayleigh (WR) [11], distributions.

We generate $12,000$ samples with different sizes according to R, IR, $Ra - Ra$, WR and NWR distribution to calculate the test criteria AIC, BIC, $CAIC$, $HQIC$ and KS, the results are illustrated in Table 6:

TABLE 6. Smaller $AIC/BIC/CAIC/HQIC/KS$ scores in $12,000$ simulations from NWR and comparative distributions.

n	Model	AIC	BIC	$CAIC$	$HQIC$	KS
10	R	39.71	39.96	40.13	39.43	0.760
	IR	35.85	36.11	36.27	35.57	0.7624

	$Ra-Ra$	20.70	21.22	22.17	20.13	0.6747
	WR	1.13	0.62	3.22	-0.96	0.7591
	NWR	**0.80**	**0.56**	**3.16**	**-0.03**	**0.7588**
20	R	32.80	32.90	32.82	32.82	0.2521
	IR	31.79	31.89	31.819	31.816	02522
	$Ra-Ra$	28.51	28.75	28.59	28.56	0.2510
	WR	23.09	23.39	23.24	23.15	0.2521
	NWR	**22.96**	**23.26**	**23.11**	**23.02**	**0.2123**
30	R	113.21	114.61	113.35	113.66	0.1179
	IR	100.66	102.06	100.80	101.11	0.1249
	$Ra-Ra$	43.99	46.79	44.43	44.89	0.1186
	WR	-36.48	-32.28	-35.56	-35.14	0.1019
	NWR	**-31.75**	**-27.55**	**-30.83**	**-30.41**	**0.1013**
50	R	187.31	189.22	187.39	188.04	0.0993
	IR	165.71	167.63	165.80	166.44	0.0944
	$Ra-Ra$	70.57	74.39	70.82	72.02	0.0952
	WR	-65.36	-59.62	-64.83	-63.17	0.0789
	NWR	$\mathbf{-56.83}$	$\mathbf{-51.09}$	$\mathbf{-56.31}$	$\mathbf{-54.64}$	**0.0782**
100	R	375.68	378.29	375.72	376.74	0.0709
	IR	331.76	334.37	331.81	332.82	0.0709
	$Ra-Ra$	143.37	148.58	143.50	145.48	0.0739
	WR	-127.86	-120.05	-127.61	-124.70	0.0579
	NWR	$\mathbf{-113.26}$	$\mathbf{-105.44}$	$\mathbf{-113.01}$	$\mathbf{-110.09}$	**0.0572**

Table 6, permits us to determine that the NWR distribution has the smallest test criterion values, which enables us to conclude that the investigated model will provide more accurate findings when applied to actual data.

4.2. **NRR test for completed case.**

For testing the null hypothesis H_0 that $X_i = (x_1, \ldots, x_n)$, follow any parametric model $F(x, \theta)$, where θ represents the parameters vector. The authors proposed a new statistic called the NRR statistic Y^2 based on the maximum likelihood estimators on non-grouped data, this statistic is a modified chi-square test statistic which can be written as the sum of the famous Pearson statistic X_n^2 and a quadratic form Q, for more details see [9, 16, 17]:

$$Y^2 = X_n^2 + Q.$$

The observations are grouped into r intervals I_j, $I_j =]a_{j-1}, a_j]$, with $j = 1, \ldots, r$, where $\nu = (\nu_1, \ldots, \nu_r)^T$ represents the corresponding empirical frequencies

$$\nu_j = card\{i : x_i \in I_j,\}\, i = 1, 2, \ldots, n.$$

For this criteria the classes are assumed to be equiprobables, so the corresponding probabilities are given by:

$$p_j\left(\widehat{\theta}_n\right) = \int_{a_{j-1}}^{a_j} dF\left(t, \widehat{\theta}_n\right) = \frac{1}{r}, \qquad j = \overline{1, r-1}.$$

Therefore the limits a_j are obtained from these equations:

$$a_j = F^{-1}\left(\frac{j}{r}\right), \qquad j = \overline{1, r-1}.$$

The components of this statistic are

$$\begin{aligned}
X_n^2 &= \sum_{j=1}^{r} \frac{(\upsilon_j - np_j)^2}{np_j}, \quad Q = \frac{1}{n} L^T\left(\widehat{\theta}\right) G^{-1} L\left(\widehat{\theta}\right), \quad \theta = (\theta_1, \theta_2, \ldots, \theta_s) \\
L\left(\widehat{\theta}\right) &= \left[L_1\left(\widehat{\theta}\right), \ldots, L_s\left(\widehat{\theta}\right)\right]^T, \quad L_k(\theta) = \sum_{j=1}^{r} \frac{\upsilon_j}{p_j} \frac{\partial p_j(\theta)}{\partial \theta_k}, \\
&\qquad k = 1, \ldots, s, \; j = 1, \ldots, r, \\
\widehat{G} &= [\widehat{g}_{ll'}]_{r\times r}, \quad g_{ll'} = i_{ll'} - \sum_{j=1}^{r} \frac{1}{p_j}\left(\frac{\partial p_j(\theta)}{\partial \theta_l} \frac{\partial p_j(\theta)}{\partial \theta_{l'}}\right), \quad \widehat{i}_{ll'} = \frac{\partial^2 l\left(x_i, \widehat{\theta}\right)}{\partial \widehat{\theta}_l \partial \widehat{\theta}_{l'}}.
\end{aligned}$$

Note that this approach was used for fitting the competing risk model [20], the generalized Rayleigh distribution [6], the extension Weibull distribution [27], the Burr XII inverse Rayleigh model [8], and others.

4.2.1. *NWW model.* Suppose that a sample x_i, $(i = 1, \ldots, n)$ is distributed by a NWW distribution, the problem is to test the null hypothesis H_0 such

$$H_0 : P(x_i \leq X) = F_{NWW}(x, \vartheta), x > 0, \vartheta = (\alpha, \beta, \lambda, \gamma)^T.$$

The observations are grouped into r intervals I_j, ($I_j =]a_{j-1}, a_j]$. After computing the MLEs of the unknown parameters vector ϑ, we calculated the estimated limits $\widehat{a}_j$ such that the grouped intervals are equiprobable, we obtain

$$\widehat{a}_j = \left[-\frac{\log\left(1 - \exp\left\{ - \left[-\frac{\log\left(\frac{j}{r}\right)}{\widehat{\alpha}} \right]^{\frac{1}{\widehat{\beta}}} \right\} \right)}{\widehat{\lambda}} \right]^{\frac{1}{\widehat{\gamma}}}.$$

We propose a simulation study using R software to show the performances of the constructed test statistic Y^2. For that, we compute $N = 12,000$ samples from the NWW distribution with different sizes n. The mean of the simulated levels of significance calculated Y^2 are compared to their corresponding theoretical ones $\epsilon = 1\% - 5\% - 10\%$, the results of the simulation are given in Table **??**

TABLE 7. Simulated levels of significance of Y^2 of the New-Weibull-Weibull distribution.

$N = 12,000$

ϵ \ n	20	50	100	150	250	500	750
1%	0.0094	0.0136	0.0098	0.0088	0.0122	0.0108	0.0112
5%	0.0342	0.0414	0.0426	0.0442	0.0497	0.0505	0.0509
10%	0.0578	0.0772	0.0806	0.0878	0.0922	0.1006	0.1056

As can be seen, the empirical values of the level of significance are very close to the corresponding theoretical ones which shows that the proposed modified goodness-of-fit statistics (NRR) are able to verify the fit of datasets to these model.

4.2.2. *NWR model.* Let us consider the null hypothesis H_0 that a random sample $X = (x_1, \ldots, x_n)$ follow the New-Weibull-Rayleigh distribution. If $\nu = (\nu_1, \ldots, \nu_r)^T$ represents the observed numbers of observations to fall into the grouped intervals

I_j, as it's shown above, the estimated limit intervals a_j are obtained by

$$\widehat{a}_j = \left[-\frac{\log\left(1-\exp\left\{-\left[-\frac{\log\left(\frac{j}{r}\right)}{\widehat{\alpha}}\right]^{\frac{1}{\beta}}\right\}\right)}{\widehat{\lambda}} \right]^{\frac{1}{2}}, \qquad j = 1,\ldots,r.$$

By simulating $12,000$ samples from the NWR distribution, Table **??** shows a calculated risk of rejection of the NRR statistic test Y^2 for different sample sizes compared to different theoretical risks of error $\epsilon = 1\% - 5\% - 10\%$.

TABLE 8. simulated levels of significance of the Y^2 for the New-Weibull-Rayleigh distribution.

$N = 12,000$

ϵ \ n	20	50	100	150	250	500	750
1%	0.0069	0.007	0.009	0.012	0.017	0.011	0.015
5%	0.0340	0.0441	0.0476	0.0491	0.0503	0.0506	0.0510
10%	0.0918	0.0921	0.1019	0.0955	0.1007	0.0901	0.0984

As can be observed, the suggested modified goodness-of-fit statistics (NRR) are able to confirm the fit of datasets to these models because the empirical values of the level of significance are extremely similar to the corresponding theoretical ones.

4.3. **NRR test for censored case.**

We use the statistic test of Bagdonavicius and Nikulin [22], [23] Y_n^2 to fit right-censored data based on the maximum likelihood estimation for ungrouped data and random intervals are considered as data function. For that, we suppose the null hypothesis H_0:

$$H_0 : F(x) \in F_0 = \{F_0(x,\theta)\, ; x \in \mathbb{R}^s ; \theta \in \Theta \subset \mathbb{R}^s\},$$

where $\theta = (\theta_1,\ldots,\theta_s)^T \in \Theta \subset \mathbb{R}^s$ is unknown s-dimensional vector parameter and F_0 is a known cumulative distribution function.

Let us consider a finite time interval only say $[0, \tau]$ and divide it into $k > s$ smaller intervals $Ij = (a_j - 1, a_j]$, where

$$0 < a_1 < a_2 < \ldots < a_k = \tau.$$

So,

$$\widehat{a}_j = \Lambda^{-1}\left\{\frac{E_j - \sum_{l=1}^{i-1}\Lambda\left(X_{(l)}, \widehat{\theta}\right)}{n-i+1}, \widehat{\theta}\right\}, \widehat{a}_k = \max\left(X_{(n)}, \tau\right).$$

Here $\widehat{\theta}$ is the maximum likelihood estimator of the parameter θ, Λ^{-1} is the inverse of the cumulative hazard function, $X_{(i)}$ represent the ith element in the ordered statistics $\left(X_{(1)}, \ldots, X_{(n)}\right)$ and $E_j = (n-i+1)\,\Lambda\left(\widehat{a}_j, \widehat{\theta}\right) + \sum_{l=1}^{i-1}\Lambda\left(X_{(l)}, \widehat{\theta}\right)$. a_j are random data functions such as the k intervals chosen have equal expected numbers of failures e_j. The chi-square test [*Bagdonavicius and Nikulin* (2011)] is based on the statistic:

$$Y_n^2 = Z^T \widehat{\Sigma}^- Z,$$

where $\widehat{\Sigma}^- = \widehat{A}^{-1} + \widehat{A}^{-1}\widehat{C}^T\widehat{G} - \widehat{C}\widehat{A}^{-1}$. The vector Z is given by

$$Z = (Z_1, \ldots, Z_k)^T; \; Z_j = \frac{1}{\sqrt{n}}(U_j - E_j); \; j = \overline{1,k}.$$

Here, U_j represent the numbers of observed failures in the intervals I_j. Under the hypothesis H_0, the limit distribution of the test statistic

$$Y_n^2 = \sum_{i=1}^{n}\frac{(U_j - e_j)^2}{U_j} + Q,$$

where,

$$Q = W^T\widehat{G}^{-1}W, \quad W = \widehat{C}\widehat{A}^{-1}Z = (W_1, \ldots, W_s)^T,$$

$$\widehat{G} = [\widehat{g}_{ll'}]_{s\times s}, \;\; \widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{k}\widehat{C}_{lj}\widehat{C}_{l'j}\widehat{A}_j^{-1}, \;\; W_l = \sum_{j=1}^{k}\widehat{C}_{lj}\widehat{A}_j^{-1}Z_j,$$

$$\widehat{i}_{ll'} = \frac{1}{n}\sum_{j=1}^{n}\delta_i\frac{\partial\ln\lambda\left(X_i, \widehat{\theta}\right)}{\partial\theta_l}\frac{\partial\ln\lambda\left(X_i, \widehat{\theta}\right)}{\partial\theta_{l'}}, \;\; \widehat{\theta} = (\theta_1, \ldots, \theta_s),$$

$$\widehat{C}_{lj} = \frac{1}{n}\sum_{i:X_i\in I_j}\delta_i\frac{\partial}{\partial\theta_l}\ln\lambda\left(X_i, \widehat{\theta}\right), \;\; j = 1, \ldots k, \;\; l, l' = 1, \ldots, s.$$

We reject the null hypothesis H_0 if $Y_n^2 > \chi_\epsilon^2(r)$ with approximate significance level α. This test statistic was used to fit different distributions, for that see [14].

4.3.1. *NWW model.* The choice of $\widehat{a}_j$ when the baseline distribution is a New-Weibull-Weibull, is obtained as follow

$$\widehat{a}_j = \left\{ -\frac{1}{\widehat{\lambda}} \log \left[1 - \exp \left(- \left\{ -\frac{1}{\widehat{\alpha}} \log \left[1 - \exp \left(\frac{\sum_{l=1}^{i-1} \Lambda \left(X_{(l)}, \widehat{\vartheta} \right) - E_j}{N - i + 1} \right) \right] \right\}^{\frac{1}{\widehat{\beta}}} \right) \right] \right\}^{\frac{1}{\widehat{\gamma}}},$$

$\widehat{a}_k = T_{(n)}$,, The elements of the Y_n^2 statistic test are calculated and expressed in Appendix 1. We demonstrate through numerical simulation that the statistic test Y_n^2 of the distribution NWW is a chi-square with k degree of freedom. To do this, we use the R software to calculate the statistic's average number of rejections by comparing it to the corresponding theoretical risk of error $\epsilon = 1\% - 5\% - 10\%$. The results of simulation are shown in Table 9.

TABLE 9. Rejected cases of the censored Y_n^2 of the New-Weibull-Weibull distribution.

$N = 12,000$

ϵ \ n	20	50	100	150	250	500	750
1%	0.0087	0.0090	0.0096	0.0102	0.0108	0.0112	0.0115
5%	0.0326	0.0413	0.0442	0.0446	0.0486	0.0503	0.0509
10%	0.0853	0.0892	0.0964	0.1031	0.1046	0.1083	0.0110

Table 9 reveals that the rejected Y_n^2 values are quite near to the corresponding theoretical values ϵ, allowing us to conclude that the statistic test Y_n^2 is a chi-square with k degrees of freedom.

4.3.2. *NWR model.* The selection of the limits of intervals $\widehat{a}_j$ of the New-Weibull-Rayleigh distribution is made as follows

$$\widehat{a}_j = \left\{ -\frac{1}{\widehat{\lambda}} \log \left[1 - \exp \left(- \left\{ -\frac{1}{\widehat{\alpha}} \log \left[1 - \exp \left(\frac{\sum_{l=1}^{i-1} \Lambda \left(X_{(l)}, \widehat{\theta} \right) - E_j}{N - i + 1} \right) \right] \right\}^{\frac{1}{\widehat{\beta}}} \right) \right] \right\}^{\frac{1}{2}},$$

$\widehat{a}_k = x_{(n)}$. The elements of the statistic test Y_n^2 of the NWR distribution are calculated and obtained in Appendix 2.

Using R software, we estimate the average number of rejections of the estimated statistic Y_n^2 by comparing it to the corresponding theoretical risk of error, $\epsilon = 1\% - 5\% - 10\%$. The results of the simulation are displayed in Table 10:

TABLE 10. Rejected cases of the censored Y_n^2 of the NWR distribution.

$N = 12,000$

ϵ \ n	20	50	100	150	250	500	750
$\alpha = 1\%$	0.0076	0.0082	0.0089	0.0091	0.0097	0.0106	0.0124
$\alpha = 5\%$	0.0376	0.0428	0.0487	0.0501	0.0506	0.0512	0.0517
$\alpha = 10\%$	0.0665	0.0722	0.0890	0.0900	0.0935	0.1031	0.1180

We notice from Table 10 that the values of the rejected Y_n^2 are very close to the corresponding theoretical risk of error ϵ, then we can say that the statistic test of the modified NRR test of the NWR model is a chi-square with degree of freedom k.

5. APPLICABILITY OF STUDIED MODEL

To demonstrate the applicability and usefulness of these new distributions, we will apply the studied NWW and NWR models to a several complete and censored real data in a different of application fields.

5.1. **Antibiotic dataset.**

We used the 'antibio' dataset from the isdals packages of the R software [5], these data represent the amount of organic material in $n = 34$ heifer dung was measured after eight weeks of decomposition. We suppose H_0 that these observations are modelled by the New *Weibull-Rayleigh* distribution. Data are given in Table 11.

TABLE 11. The amount organic in 34 heifer dung.

3.03,2.81,3.06,3.11,2.94,3.06,3.00,3.02,2.87,2.96,2.77,2.75, 2.74,2.88, 2.42,2.73,2.83,2.66,2.80,2.85,2.84,2.93,2.74,2.88, 2.85,3.02,2.85,2.66,2.43,2.63,2.56,2.76,2.70,2.54.

- Graphical analysis: We made the QQ-plot and PP-plot graphs, which QQ-plot allows us to compare the theoretical quantiles calculated from the NWR distribution and the empirical distribution of the real data used. While at the PP plot gives us a visual comparison of the theoretical probabilities of the NWR distribution and the empirical distribution of the actual data used in this example. We plot in Figure 9 the estimated pdf, QQ-plot, PP-plot and the estimated cdf:

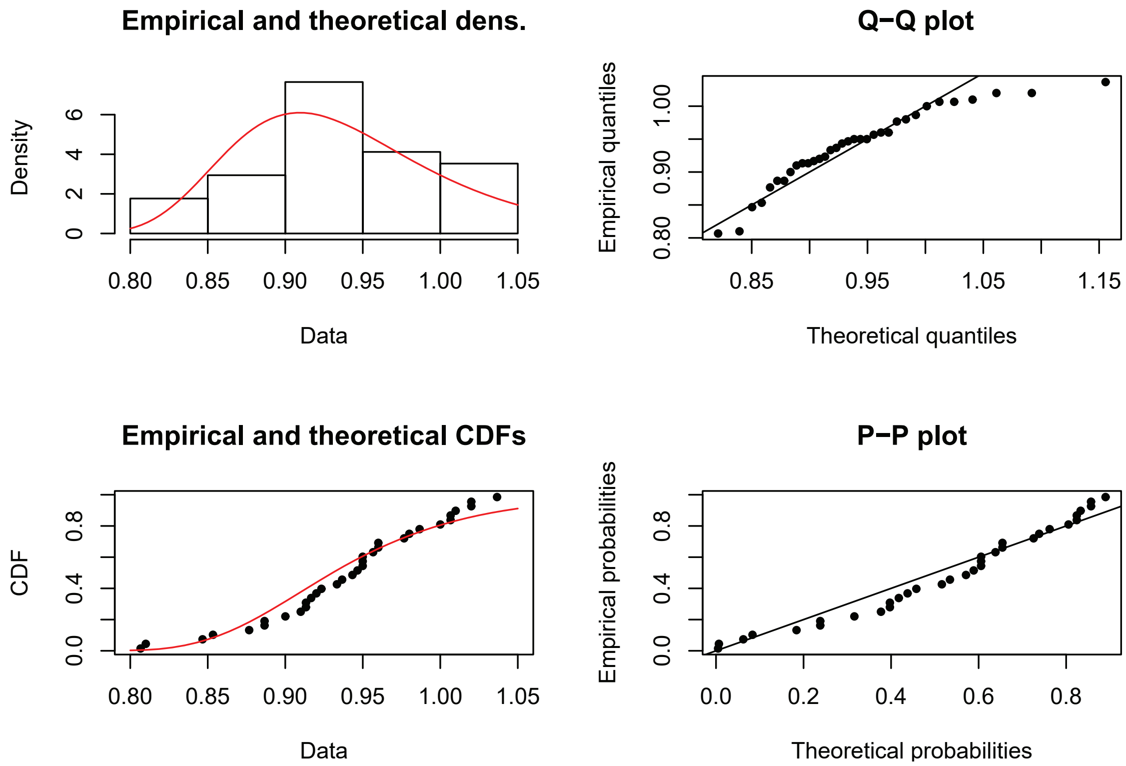


FIGURE 9. Estimated pdf, Q-Q plot, estimated cdf and P-P plot of the amount organic in 34 heifer dung.

As we can see in Figure 9 the values of real data used has the same shape as the New-Weibull-Rayleigh distribution we have supposed.

- Classical tests: We calculate the statistic of the well-known Information Criteria AIC, BIC, $CAIC$, $HQIC$ and KS using the data used in this example for the WR, R, IR, $Ra-Ra$ and NWR distributions, the results are illustrated in Table 12.

TABLE 12. The AIC, BIC, $CAIC$, $HQIC$ and KS of the NWR model based on data set.

Model	AIC	BIC	$CAIC$	$HQIC$	KS
NWR	**-9.04**	**-4.46**	**-8.24**	**-7.48**	**0.1117**
R	93.60	95.13	93.73	94.12	0.5212
IR	93.63	95.15	93.75	94.14	0.5548
$Ra-Ra$	-12.67	9.62	-12.28	-11.63	0.1323
WR	-18.09	-13.51	-17.29	-16.52	0.1166

Table 12, shows that NWR model has the lower Information Criteria score, so we can decide that model NWR is the best model to fit our study.

- Modified NRR test: We use the NRR statistic Y^2 to validate the null hypothesis H_0.We obtain the $MLEs$ vector $\widehat{\varsigma}$ of parameters of the NWR model:

$$\widehat{\varsigma} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}\right)^T = (3.20, 2.45, 3.60)^T .$$

The data are grouped into $r = 5$ classes, so we calculate the estimated classes limits $\widehat{a_j}$ and the corresponding v_j and p_j, the results are shown in Table 13:

TABLE 13. Values of $\widehat{a}_j$, v_j and p_j.

$\widehat{a}_j$	0.4017	0.4379	0.4747	0.5254	103.1100
v_j	2	11	13	5	3
p_j	0.2	0.2	0.2	0.2	0.2

Therefore, we obtain the NRR statistic value $Y^2 = 9,60$, then we compare it to the $(r-1)$ chi-square critical values χ^2 for different level of significance $\epsilon = 1\%$;

5%:

$$Y^2 < \chi^2_{1\%}(5-1) = 15,086; \quad Y^2 < \chi^2_{5\%}(5-1) = 11,075.$$

These obtained results indicate the NWR distribution can be used to accurately fit the amount of organic matter in heifer dung that was measured after eight weeks of decomposition.

5.2. **Economic dataset.**

We suppose these $n = 50$ economic observations available in https://data.world/datasets/data, which represent the Annual Inflation in Algeria between 1961 and 2011 are modelled by the New Weibull-Weibull distribution.

TABLE 14. Annual Inflation in Algeria ($1961 - 2011$).

$3.471720042, 2.351279502, 0.5493313109, 1.695183177,$
$1.501331249, 1.817814694, 1.312040991, 3.142056013,$
$1.921084494, 4.940445977, 17.15196386, 4.606461074,$
$9.627611595, 48.89659052, 5.914022113, 10.8405927, 11.92709947,$
$10.08512101, 13.98783797, 25.86203875, 14.35399954, 1.93979417,$
$6.804795897, 8.433505552, 4.972526406, 2.405343265,$
$8.842020401, 9.060963496, 16.01137351, 30.25959848,$
$53.78860423, 21.92611451, 13.62442466, 29.07764734,$
$28.5770375, 24.02190406, 7.001963063, 3.131088695, 10.85640762,$
$24.59809885, 0.7112095185, 1.906328849, 8.323802693,$
$10.62932921, 16.45925846, 11.28281156, 7.331055187,$
$14.60217944, 11.2666112, 16.24561679, 11.43116826.$

- Graphical analysis: We plot in Figure 10 the estimated cdf and the estimated pdf, also, the QQ-plot and PP-plot to compare visually the theoretical quantiles, theoretical probabilities of the NWW distribution and the empirical distribution of the inflation rate in Algeria between 1961 and 2011, respectively.

Figure 10 allows us to see at-a-glance annual inflation in Algeria between 1961 and 2011 can be adjusted by the NWW distribution is plausible.

- Classical tests: In Table 15, we compare the NWW distribution with distributions generally used to model economic data. So, we calculate the statistic

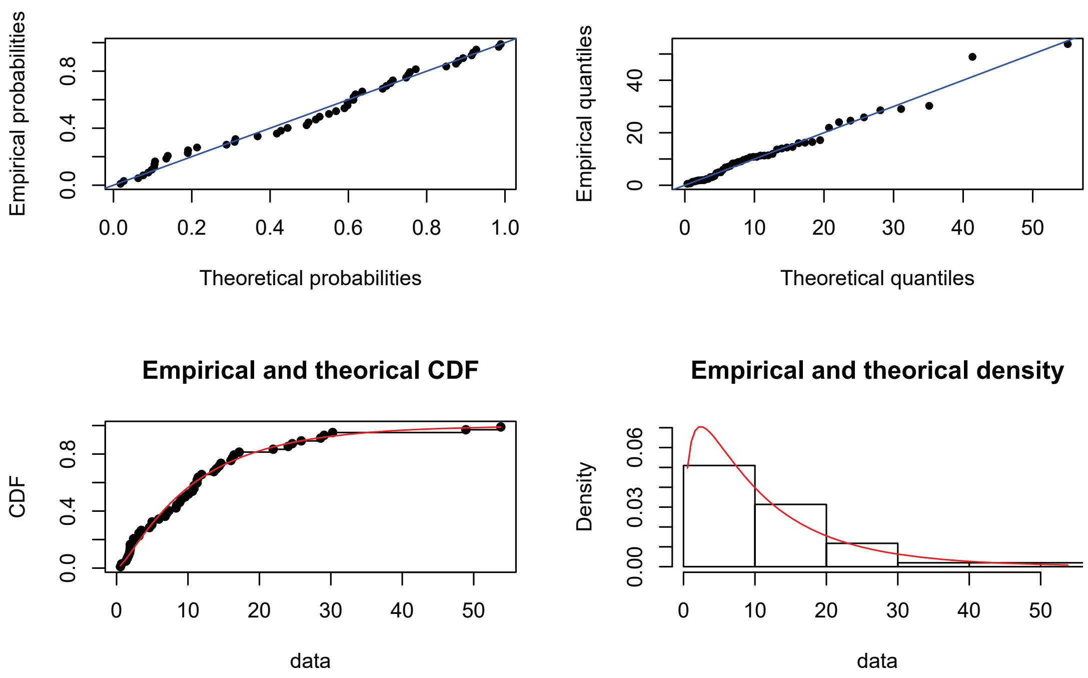


FIGURE 10. Estimated pdf, Q-Q plot, estimated cdf and P-P plot of annual inflation in Algeria $2000 - 2011$.

of the well-known Information Criteria mentioned above for the data used in this example for the NWW, WW, R, *Gumbel*-exponential (GE) and *Pareto* (Pa) distributions:

TABLE 15. The AIC, BIC, $CAIC$, $HQIC$ and KS of the NWW model based on economic data.

Model	AIC	BIC	$CAIC$	$HQIC$	KS
NWW	**361.63**	**369.36**	**362.50**	**364.58**	**0.0577**
WW	363.09	370.81	363.96	366.04	0.0733
$Rayleigh$	398.17	400.10	398.25	398.91	0.2633
$Pareto$	363.27	371.44	364.12	365.87	0.1157
GE	365.03	368.89	365.28	366.50	0.4548

Table 15 demonstrates that the NWW model has the lower Information Criteria value, allowing us to determine that it is the most appropriate model to fit our economic data.

- Modified NRR test: We assume H_0, such that the inflation data are adjusted by the NWW model. Using the maximum likelihood method to estimate the vector of parameters ϑ of the NWW distribution

$$\widehat{\vartheta} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}, \widehat{\gamma}\right)^T = (0.7633, 2.7023, 0.0954, 0.8217)^T .$$

We decide that the data are classified into five classes ($r = 5$), thus we estimate the classes bounds $\widehat{a}_j$ and the corresponding v_j and p_j, the results are shown in Table 16

TABLE 16. Values of $\widehat{a}_j$, v_j and p_j.

$\widehat{a}_j$	6.5450	9.7173	13.8092	20.9757	153.7886
v_j	18	8	9	7	9
p_j	0.2	0.2	0.2	0.2	0.2

Therefore, we obtain the statistic value $Y^2 = 7.6422$, then we compare it to the $(r-1)$ chi-square critical values χ^2 for different significance levels $\epsilon = 1\%$; 5%; 10%:

$$Y^2 < \chi^2_{1\%}(4) = 15.0862; \quad Y^2 < \chi^2_{5\%}(4) = 11.0705; \quad Y^2 < \chi^2_{10\%}(4) = 9.2363.44.$$

From the results obtained, we can conclude that H_0 is not rejected, which leads us to say that the inflation rates in Algeria between 1961 and 2011 can be adjusted by the NWW model at different risk of error ϵ.

So we can deduce that this new generalization of Weibull can indeed model economic data.

5.3. **Drugs mortality dataset.**

The data set taken from Mathers et al. [4] representing the crude mortality rate (CMR) among people who inject drugs. The data set consist of the following observations given in Table 17.

- Graphical analysis: To compare the theoretical quantiles, theoretical probabilities of the NWR distribution to the empirical distribution of the crude mortality rate among drug users who inject drugs, we present in Figure 11 the estimated pdf and cdf, the QQ-plot and PP-plot:

TABLE 17. CMR among people who inject drugs

$2.01, 6.32, 3.52, 2.15, 5.42, 2.04, 2.77, 2.26, 1.95, 1.00, 2.45, 0.74,$
$0.98, 1.27, 2.77, 3.68, 1.18, 1.09, 1.60, 0.57, 3.33, 0.91, 7.14,$
$2.08, 3.85, 1.99, 7.76, 2.52, 1.57, 4.67, 4.22, 1.92, 1.59, 4.08, 2.02,$
$0.84, 6.85, 2.18, 2.04, 1.05, 2.91, 1.37, 2.43, 2.28, 3.74, 1.30, 1.59,$
$1.83, 3.85, 6.30, 4.83, 0.50, 3.40, 2.33, 4.25, 3.49, 2.12, 0.83, 0.54,$
$3.23, 4.50, 0.71, 0.48, 2.30, 7.73$

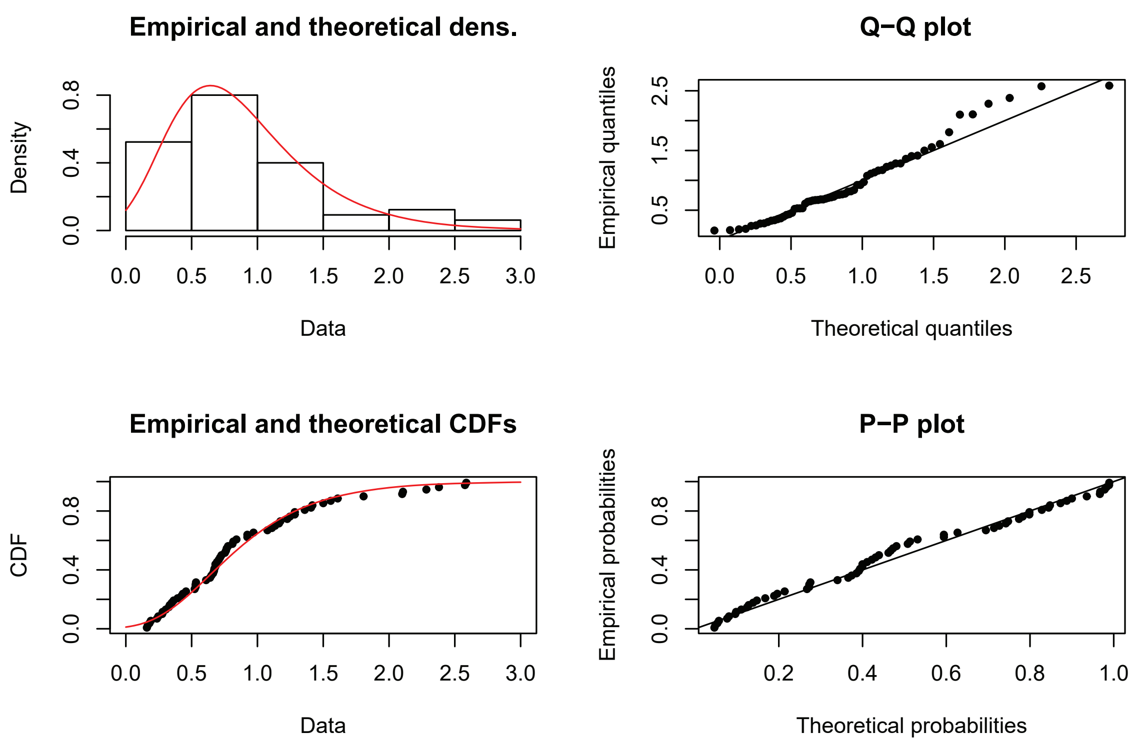


FIGURE 11. Estimated pdf, Q-Q plot, estimated cdf and P-P plot of CMR among people who injected drugs.

As shown Figure 11, the distribution of the crude mortality rate among people who inject drugs has almost the same form as theoretical quantiles and theoretical probabilities of the New-Weibull-Rayleigh distribution.

- Classical tests: The NWR distribution has been compared with five distributions Rayleigh-Rayleigh ($Ra - Ra$) Gamma Rayleigh ($GaRa$), Marshal Olkin

Rayleigh ($MORa$), Truncated-Exponential Skew Symmetric Rayleigh ($TESRa$), and Rayleigh (Ra) distributions. The various information criteria used and mentioned above are calculated from the the CMR among drug users who inject drugs to compare NWR, $Ra-Ra$, $GaRa$, $MORa$, $TESRa$ and Ra distributions, in Table 18:

TABLE 18. The AIC, BIC, $CAIC$, $HQIC$ and KS of the NWR model based on data set.

Model	AIC	BIC	$CAIC$	$HQIC$	KS
NWR	**41.54**	**48.06**	**41.93**	**44.11**	**0.0896**
$Ra-Ra$	240.98	241.17	245.33	242.70	0.1645
$GaRa$	244.74	244.93	249.08	246.45	0.2333
$MORa$	242.09	242.28	246.44	243.81	0.1756
$TESRa$	243.38	243.58	247.73	245.10	0.1511
Ra	249.30	249.36	251.47	250.16	0.1102

The NWR model has the lower Information Criteria value, as shown in Table 18, which enables us to conclude that it is the model that describes the Crude Mortality Rate among people who injected drugs.

- Modified NRR test: We suppose the null hypothesis H_0 that the data used in this example are adjusted by the NWR distribution. For that, we calculate the vector of MLEs $\varsigma\left(\widehat{\alpha},\widehat{\beta},\widehat{\lambda}\right)$ of the NWR model

$$\widehat{\varsigma}=\left(\widehat{\alpha},\widehat{\beta},\widehat{\lambda}\right)^T=(0.7594,3.4829,0.2950)^T.$$

Under H_0, we opt for $r=8$ grouping intervals, so we calculate the estimated classes limits $\widehat{a}_i$ and the corresponding v_i and p_j in Table 19:

TABLE 19. Values of $\widehat{a}_j$, v_j and p_j.

$\widehat{a}_j$	1.0597	1.4268	1.8042	2.2472	2.8246	3.6854	5.3442	107.760
v_j	12	5	4	12	9	7	9	7
p_j	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125

Consequently, we get the statistic value $Y^2 = 7.4822$. Next, we compare the value of Y^2 to $(r-1)$ chi-square critical χ^2 for different significance values $\epsilon = 1\%; 5\%; 10\%$:

$$Y^2 < \chi^2_{1\%} = 20.0902; \quad Y^2 < \chi^2_{5\%} = 15.5073; \quad Y^2 < \chi^2_{10\%} = 13.3615.$$

From these calculations it can be concluded that the NWR distribution provides a better fit for the the crude mortality rate among people who inject drugs.

5.4. **Failure time of aluminium cells dataset.**

We suppose the following censored data can be adjusted by NWR distribution, these data (Table 20) represent aluminium reduction cells of Whitmore [7], who considered the times of failures for 20 aluminium reduction cells, and the numbers of failures in $1,000$ days units

TABLE 20. Data set of failure time of 20 aluminium cells.

$0.468, 0.725, 0.838, 0.853, 0.965, 1.139, 1.142, 1.304, 1.317,$
$1.427, 1.554, 1.658, 1.764, 1.776, 1.990, 2.010, 2.224, 2.279*, 2.244*, 2.286*,$

* represent censored data.

- Graphical analysis: Using the times of failures of 20 aluminium reduction cells to compare the empirical distributiont to the theoretical quantiles and theoretical probabilities calculated from the censored NWR distribution, the QQ-plot, PP-plot, the estimated cdf and estimated pdf are showed in Figure 12

From Figure 12, the first impressions we see that the NWR distribution accurately models the failure time of 20 aluminium cells.

- Classical tests: We compute the statistic for the Information Criteria AIC, BIC, $CAIC$, $HQIC$, and KS to compare the NWR, R, IR, $Ra-Ra$, and WR distributions using the faiulre time of 20 aluminium cells. The results are shown in Table 21

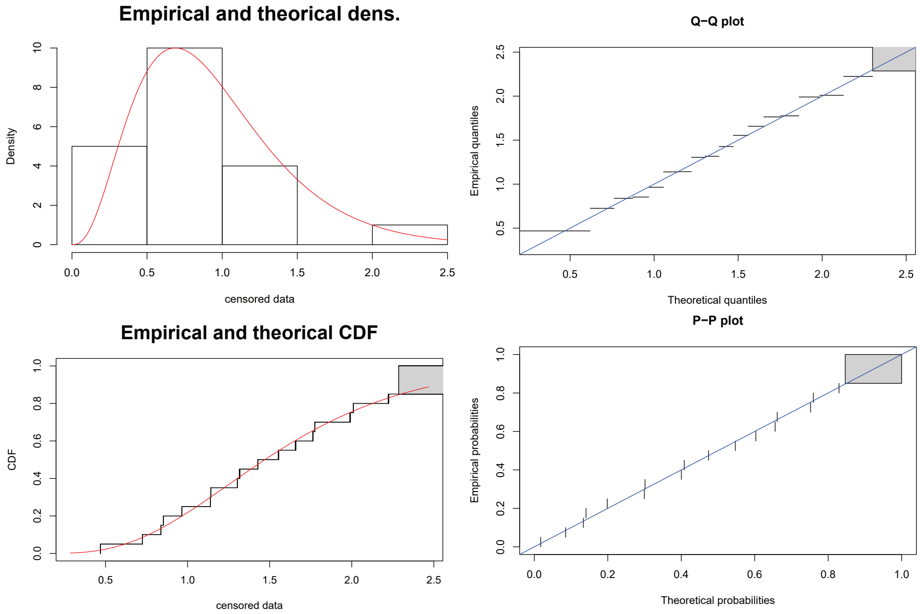


FIGURE 12. Estimated pdf, Q-Q plot, estimated cdf and P-P plot of failure time of 20 aluminium cells.

TABLE 21. The AIC, BIC, $CAIC$, $HQIC$ and KS of the NWR model based on data set.

Model	AIC	BIC	$CAIC$	$HQIC$	KS
NWR	**37.31**	**38.30**	**38.81**	**37.90**	**0.1095**
R	38.84	39.84	39.06	39.04	0.1486
IR	44.56	45.56	44.79	44.76	0.2099
$Ra-Ra$	38.69	40.68	39.40	39.08	0.1150
WR	39.12	42.11	40.62	39.70	0.1150

Table 21 reveals us that the NWR model has the lower Information Criteria value. So, we can determine that model NWR is the best model for our study.

- Modified NRR test: We use the modified NRR statistic test obtained previously for censored data. Using R statistical software, we compute the vector of the censored maximum likelihood estimators $\widehat{\varsigma}$ of the NWR distribution:

$$\widehat{\varsigma} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}\right)^T = (2.1308, 0.3515, 1.4651)^T.$$

We choose $k = 4$ grouping intervals (I_j) of 20 failure time of aluminium cells, the elements of the Y_n^2 statistic of the NRR test are presented in Table 22:

TABLE 22. Elements of the modified NRR test in censored case.

$\widehat{a}_j$	0.8577	1.3429	1.7532	2.2860
$\widehat{U}_j$	4	5	3	5
$\widehat{e}_j$	4.49108	4.49108	4.49108	4.49108
$\widehat{C}_{1j}$	-0.0692	0.0595	0.0601	0.1129
$\widehat{C}_{2j}$	0.6595	1.01080	0.6058	0.929
$\widehat{C}_{3j}$	0.1443	0.2242	0.1405	0.2210

We, then, deduce the value of $Y_n^2 = 0.8154$ and compare it to the chi-square statistic test χ_ϵ^2 for different significance levels $\epsilon = 1\%$; $\epsilon = 5\%$ and $\epsilon = 10\%$ we obtain:

$$Y_n^2 < \chi_{1\%}^2(3) = 13.2767; \quad Y_n^2 < \chi_{5\%}^2(3) = 9.4877; \quad Y_n^2 < \chi_{10\%}^2(3) = 7.7794.$$

The results obtained affirm that the NWR model can fit the failure time of 20 aluminium cells. Therefore, we can conclude that the NWR distribution effectively models reliability data.

5.5. **Valve seat dataset.**

We aim to investigate the null hypothesis H_0 that the NWW distribution adjusts the time to replacement of valve seats for 41 diesel engines. This dataset presented in Table **??** is available in the survival package of the statistical software R, see [26].

TABLE 23. Data set of 41 diesel engines.

$761*, 759*, 98, 667*, 326, 653, 653, 667*, 665*, 84, 667*, 87, 663*, 646, 653*, 92, 653*,$
$651*, 258, 328, 377, 621, 650*, 61, 539, 648*, 254, 276, 298, 640, 644*, 76, 538,$
$642*, 635, 641*, 349, 404, 561, 649*, 631*, 596*, 120, 479, 614*, 323,$
$449, 582*, 139, 139, 589*, 593*, 573, 589*, 165, 408, 604, 606*, 249, 594*,$
$344, 497, 613*, 265, 586, 595*, 166, 206, 348, 389*, 601*, 410, 581, 601*, 611*,$
$608*, 587*, 36, 603*, 202, 563, 570, 585*, 587*, 578*, 578*, 586*, 585*, 582*.$

* represent censored data.

- Graphical analysis: Using these data, we plot in Figure 13 the estimated pdf and the estimated cdf corresponding to the NWW distribution, and we plot the PP-plot and the QQ-plot of the NWW distribution comparing to the empirical distribution of 41 diesel engines.

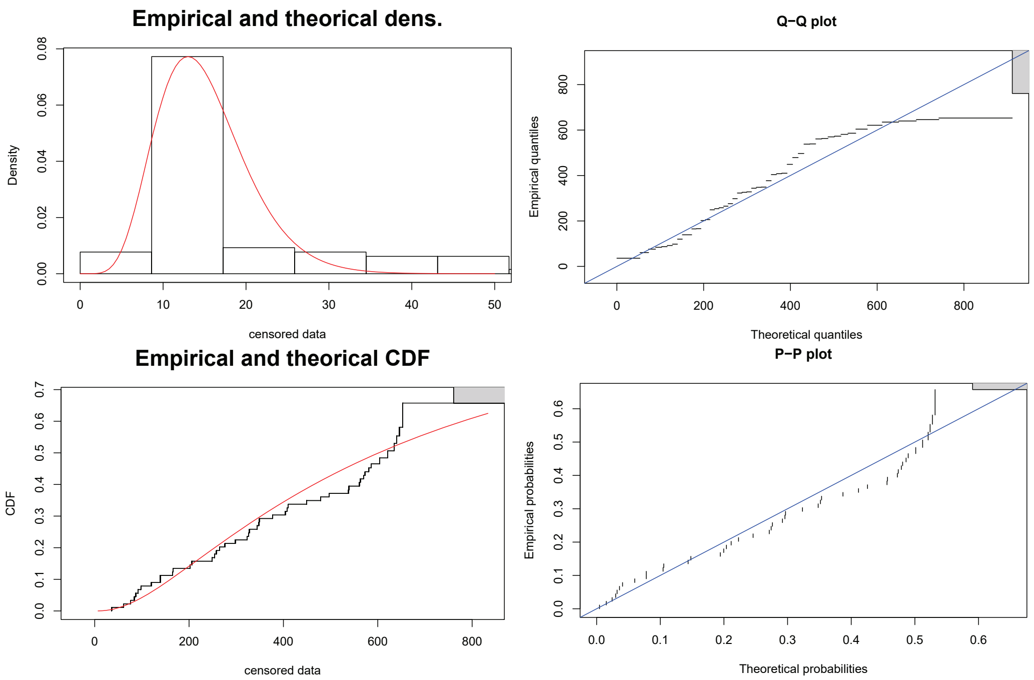


FIGURE 13. Estimated pdf, Q-Q plot, estimated cdf and P-P plot of 41 diesel engines.

From Figure 13 we can infer that the NWW distribution correctly represents the time to replacement of valve seats for 41 diesel engines.

- Classical tests: The failure time of valve seats for 41 diesel is utilized to calculate the various information criteria used and indicated above to compare the NWW, WW, $TLWW$, and IW distributions in Table 24.

TABLE 24. The AIC, BIC, $CAIC$, $HQIC$ and KS of the NWW model based on data set.

Model	AIC	BIC	$CAIC$	$HQIC$	KS
NWW	**1011.80**	**1021.75**	**1012.27**	**1015.81**	**0.1960**
WW	1177.93	1187.88	1178.41	1181.94	0.2140
$TLWW$	1240.29	1250.25	1240.77	1244.31	0.2832
IW	1306.29	1313.76	1306.58	1309.30	0.2862

Figure Table 24 demonstrates that the NWR model has the smallest information criterion values. Consequently, it can be said that the NWR model can fit the failure time of valve seats for 41 diesel.

- Modified NRR test: We use the modified NRR statistic test obtained previously. Using the statistical R software we compute the censored MLEs $\widehat{\vartheta}$ of the NWW distribution

$$\widehat{\vartheta} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}, \widehat{\gamma}\right)^T = (19.6, 0.8425, 1.0005, 0.2)^T .$$

We choose $r = 6$ grouping intervals (I_j) of these observations, the elements of the Y_n^2 statistic of the NRR test are presented in Table 25:

TABLE 25. Elements of the NRR test in the censored case.

$\widehat{a}_j$	76.1399	145.6574	226.6525	323.7033	452.215	761.00
$\widehat{U}_j$	2	7	4	7	11	17
$\widehat{e}_j$	5.8739	5.8739	5.8739	5.8739	5.8739	5.8739
$\widehat{C}_{1j}$	-0.0013	-0.0031	-0.0009	-0.0007	-0.0003	0.0010
$\widehat{C}_{2j}$	0.13157	0.4127	0.2031	0.3167	0.4541	0.6118
$\widehat{C}_{3j}$	0.1147	0.3550	0.1724	0.2668	0.3811	0.5113
$\widehat{C}_{4j}$	0.5964	2.0463	1.1234	1.8911	2.8734	4.2114

Then, the value of test criterion $Y_n^2 = 6.4797$. We calculate the statistic of the chi-square test for different significance levels $\epsilon = 1\%, \epsilon = 5\%$ and $\epsilon = 10\%$, we obtain:

$$Y_n^2 < \chi^2_{1\%}(5) = 16.8118 \; ; Y_n^2 < \chi^2_{5\%}(5) = 12.5915 \; ; Y_n^2 < \chi^2_{10\%}(5) = 10.6446.$$

The results obtained affirm the time to replacement of valve seats for 41 diesel engines can perfectly be modelled by the New-Weibull-Weibull distribution. and as a result, we can state that the two models investigated are capable of adequately describing censored data.

6. Concluding remarks

Two new distributions based on the Weibull distribution named New-Weibull-Weibull and New-Weibull-Rayleigh distributions are considered in this article. There is a veritable interest in applying the proposed distributions due to the different hazard rate's shapes which enable us to model a several of real phenomena. The parameters estimation via the maximum likelihood method is discussed as well as the validation of the models by the modified chi-square test modified test in the completed and right-censored case. Numerical illustrations via simulation study and application with real data sets are conducted using R software, the proficiency and consistency of the maximum likelihood estimators (MLEs) and the importance of the modified chi-square test (NRR) of the proposed distributions are illustrated. In the practical application, we have used five real data set from several domains (medical, bio-animal, economic, reliability and automotive). The proposed distributions "New-Weibull-Weibull" and "New-Weibull-Rayleigh" reveals better fits more flexibility and applicability to those real data set than the other compared distributions. This flexibility enables using the New-Weibull-Weibull and the New-Weibull-Rayleigh distributions in various application areas.

Appendix 1: The New-Weibull-Weibull model

- Completed case:

The score functions of the NWW distribution

$$\frac{\partial l_{NWW}\left(x_i,\vartheta\right)}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left[-\log\left(1-\exp\left(-\lambda x_i^{\gamma}\right)\right)\right]^{\beta},$$
$$\frac{\partial l_{NWW}\left(x_i,\vartheta\right)}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left[-\log\left(1-\exp\left(-\lambda x_i^{\gamma}\right)\right)\right]$$

$$-\alpha\sum_{i=1}^{n}\log\left[K_{NWW}\left(x_i,\vartheta\right)\right]\times K_{NWW}^{\beta}\left(x_i,\vartheta\right),$$

$$\frac{\partial l_{NWW}\left(x_i,\vartheta\right)}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i^{\gamma}+\left(\beta-1\right)\sum_{i=1}^{n}\frac{x_i^{\gamma}\exp\left(-\lambda x_i^{\gamma}\right)}{Z_{NWW}\left(x_i,\vartheta\right)}$$
$$+\alpha\beta\sum_{i=1}^{n}\frac{x_i^{\gamma}\exp\left(-\lambda x_i^{\gamma}\right)K_{NWW}^{\beta-1}\left(x_i,\vartheta\right)}{\left(1-\exp\left(-\lambda x_i^{\gamma}\right)\right)}-\sum_{i=1}^{n}\frac{x_i^{\gamma}\exp\left(-\lambda x_i^{\gamma}\right)}{\left(1-\exp\left(-\lambda x_i^{\gamma}\right)\right)},$$

$$\frac{\partial l_{NWW}\left(x_i,\vartheta\right)}{\partial\gamma}=\frac{n}{\gamma}+\sum_{i=1}^{n}\log\left(x_i\right)-\lambda\sum_{i=1}^{n}x_i^{\gamma}\log\left(x_i\right)+\lambda\left(\beta-1\right)\sum_{i=1}^{n}\frac{M_{NWW}\left(x_i,\vartheta\right)}{Z_{NWW}\left(x_i,\vartheta\right)}$$
$$-\lambda\sum_{i=1}^{n}\frac{M_{NWW}\left(x_i,\vartheta\right)}{\left(1-\exp\left(-\lambda x_i^{\gamma}\right)\right)}\left[1-\alpha\beta K_{NWW}^{\beta-1}\left(x_i,\vartheta\right)\right],$$

where,

$$\begin{aligned}
K_{NWW}\left(x,\vartheta\right) &= \left[-\log\left(1-\exp\left(-\lambda x^{\gamma}\right)\right)\right], \quad \vartheta=\left(\alpha,\beta,\lambda,\gamma\right),\\
M_{NWW}\left(x,\vartheta\right) &= x^{\gamma}\log\left(x\right)\exp\left(-\lambda x^{\gamma}\right),\\
Z_{NWW}\left(x,\vartheta\right) &= \left(1-\exp\left(-\lambda x^{\gamma}\right)\right)\times\left[\log\left(1-\exp\left(-\lambda x^{\gamma}\right)\right)\right].
\end{aligned}$$

The components of the Fisher Information Matrix $I_{NWW}=\left(\widehat{i}_{ll\prime}\right)_{4\times 4}$ are:

$$\begin{aligned}
\widehat{i}_{12} &= -\sum_{i=1}^{n}\log\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]\times\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]^{\widehat{\beta}},\\
\widehat{i}_{13} &= \widehat{\beta}\sum_{i=1}^{n}\frac{x_i^{\widehat{\gamma}}\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]^{\widehat{\beta}-1}}{\left(1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right)},\\
\widehat{i}_{14} &= \widehat{\beta}\widehat{\lambda}\sum_{i=1}^{n}\frac{M_{NWW}\left(x_i,\widehat{\vartheta}\right)\times\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]^{\widehat{\beta}-1}}{\left(1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right)},
\end{aligned}$$

$$\widehat{i}_{23}=\widehat{\alpha}\widehat{\beta}\sum_{i=1}^{n}\frac{x_i^{\widehat{\gamma}}\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\log\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]\times\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]^{\widehat{\beta}-1}}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}$$

$$
+\sum_{i=1}^{n} x_i^{\widehat{\gamma}} \exp\left(-\widehat{\lambda} x_i^{\widehat{\gamma}}\right) \left\{ \frac{1-\widehat{\alpha}\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]^{\widehat{\beta}}}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)} \right\},
$$

$$
\begin{aligned}
\widehat{i}_{24} &= \widehat{\lambda}\sum_{i=1}^{n} \frac{M_{NWW}\left(x_i,\widehat{\vartheta}\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)} \left[1-\widehat{\alpha}\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]^{\widehat{\beta}}\right] \\
&\quad +\widehat{\alpha}\widehat{\beta}\widehat{\lambda}\sum_{i=1}^{n} \frac{M_{NWW}\left(x_i,\widehat{\vartheta}\right)\log\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]^{\widehat{\beta}-1}}{1-\exp\left(-\widehat{\lambda} x_i^{\widehat{\gamma}}\right)},
\end{aligned}
$$

$$
\widehat{i}_{34} = \widehat{\lambda}\sum_{i=1}^{n} \frac{x_i^{2\widehat{\gamma}}\log\left(x_i\right)\exp\left(-2\widehat{\lambda} x_i^{\widehat{\gamma}}\right)}{\left[1-\exp\left(-\widehat{\lambda} x_i^{\widehat{\gamma}}\right)\right]^2} \left\{1-\widehat{\alpha}\widehat{\beta}\left[K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)\right.\right.
$$

$$
\left.\left. +\left(\widehat{\beta}-1\right)K_{NWW}^{\widehat{\beta}-2}\left(x_i,\widehat{\vartheta}\right)\right]\right\} - \sum_{i=1}^{n} x_i^{\widehat{\gamma}}\log\left(x_i\right)
$$

$$
+\sum_{i=1}^{n}\left[1-\widehat{\lambda} x_i^{\widehat{\gamma}}\right] \frac{M_{NWW}\left(x_i,\widehat{\vartheta}\right)}{1-\exp\left(-\widehat{\lambda} x_i^{\widehat{\gamma}}\right)} \left[\widehat{\alpha}\widehat{\beta}K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)-1\right]
$$

$$
+\left(\widehat{\beta}-1\right)\sum_{i=1}^{n} \frac{M_{NWW}\left(x_i,\widehat{\vartheta}\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)} \left[1-\widehat{\lambda} x_i^{\widehat{\gamma}} - \frac{\widehat{\lambda} x_i^{\widehat{\gamma}}\exp\left(-\widehat{\lambda} x_i^{\widehat{\gamma}}\right)}{1-\exp\left(-\widehat{\lambda} x_i^{\widehat{\gamma}}\right)} - \frac{\widehat{\lambda} x_i^{\widehat{\gamma}}\exp\left(-\widehat{\lambda} x_i^{\widehat{\gamma}}\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right],
$$

$$
\begin{aligned}
\widehat{i}_{11} &= -\frac{n}{\widehat{\alpha}^2}, \\
\widehat{i}_{22} &= -\frac{n}{\widehat{\beta}^2} - \widehat{\alpha}\sum_{i=1}^{n}\log^2\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]\times\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]^{\widehat{\beta}}, \\
\widehat{i}_{33} &= -\frac{n}{\widehat{\lambda}^2} + \sum_{i=1}^{n} \frac{x_i^{2\widehat{\gamma}}\exp\left(-\widehat{\lambda} x_i^{\widehat{\gamma}}\right)}{1-\exp(-\widehat{\lambda} x_i^{\widehat{\gamma}})} \left[1-\widehat{\alpha}\widehat{\beta}\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]^{\widehat{\beta}-1}\right]
\end{aligned}
$$

$$+\sum_{i=1}^{n}\frac{x_i^{2\widehat{\gamma}}\exp\left(-2\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{\left[1-\exp(-\widehat{\lambda}x_i^{\widehat{\gamma}})\right]^2}\left\{1-\widehat{\alpha}\widehat{\beta}K^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)\left[1+\frac{\left(\widehat{\beta}-1\right)}{K_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]\right\}$$
$$-\left(\widehat{\beta}-1\right)\sum_{i=1}^{n}\frac{x_i^{2\widehat{\gamma}}\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\left[1+\frac{\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{1-\exp(-\widehat{\lambda}x_i^{\widehat{\gamma}})}+\frac{\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right],$$

and

$$\begin{aligned}\widehat{i}_{44} \;=\;& -\frac{n}{\widehat{\gamma}^2}-\widehat{\lambda}\sum_{i=1}^{n}x_i^{\widehat{\gamma}}\log^2(x_i)-\widehat{\lambda}\left(\widehat{\beta}-1\right)\sum_{i=1}^{n}\frac{M_{NWW}\left(x_i,\widehat{\vartheta}\right)\log(x_i)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\left[\widehat{\lambda}x_i^{\widehat{\gamma}}-1\right]\\
&+\widehat{\lambda}\sum_{i=1}^{n}\frac{1-\widehat{\alpha}\widehat{\beta}{K_{NWW}}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}\left[\widehat{\lambda}x_i^{2\widehat{\gamma}}\log^2(x_i)\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right.\\
&\left.-M_{NWW}\left(x_i,\widehat{\vartheta}\right)\log(x_i)\right]\\
&-\widehat{\lambda}^2\left(\widehat{\beta}-1\right)\sum_{i=1}^{n}\frac{{M_{NWW}}^2\left(x_i,\widehat{\vartheta}\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\left[\frac{1}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}+\frac{1}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]\\
&+\widehat{\lambda}^2\sum_{i=1}^{n}\frac{{M_{NWW}}^2\left(x_i,\widehat{\vartheta}\right)}{\left[1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right]^2}\left[1-\widehat{\alpha}\widehat{\beta}{K_{NWW}}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)\right.\\
&\left.-\widehat{\alpha}\left(\widehat{\beta}-1\right)\widehat{\beta}{K_{NWW}}^{\widehat{\beta}-2}\left(x_i,\widehat{\vartheta}\right)\right].\end{aligned}$$

Elements of the NRR statistic test of the NWW distribution

$$\frac{\partial p_j\left(\widehat{\vartheta}\right)}{\partial\widehat{\alpha}} = \left[K_{NWW}(\widehat{a}_j)\right]^{\widehat{\beta}}F_{NWW}(\widehat{a}_j)-\left[K_{NWW}(\widehat{a}_{j-1})\right]^{\widehat{\beta}}F_{NWW}(\widehat{a}_{j-1}),$$
$$\begin{aligned}\frac{\partial p_j\left(\widehat{\vartheta}\right)}{\partial\widehat{\beta}} =\;& \widehat{\alpha}\log\left[K_{NWW}(\widehat{a}_j)\right]\times\left[K_{NWW}(\widehat{a}_j)\right]^{\widehat{\beta}}F_{NWW}(\widehat{a}_j)\\
&-\widehat{\alpha}\log\left[K_{NWW}(\widehat{a}_{j-1})\right]\times\left[K_{NWW}(\widehat{a}_{j-1})\right]^{\widehat{\beta}}F_{NWW}(\widehat{a}_{j-1}),\end{aligned}$$

$$
\begin{aligned}
\frac{\partial p_j\left(\widehat{\vartheta}\right)}{\partial\widehat{\lambda}} &= \widehat{\alpha}\widehat{\beta}\left[\frac{\widehat{a}_{j-1}\left[K_{NWW}\left(\widehat{a}_{j-1}\right)\right]^{\beta-1}\times D_{NWW}\left(\widehat{a}_{j-1}\right)}{1-\exp\left(-\widehat{\lambda}x^{\widehat{a}_{j-1}^{\widehat{\gamma}}}\right)}\right. \\
&\qquad\left.-\frac{\widehat{a}_j\left[K_{NWW}\left(\widehat{a}_j\right)\right]^{\beta-1}\times D_{NWW}\left(\widehat{a}_j\right)}{1-\exp\left(-\widehat{\lambda}x^{\widehat{a}_j^{\widehat{\gamma}}}\right)}\right], \\
\frac{\partial p_j\left(\widehat{\vartheta}\right)}{\partial\widehat{\gamma}} &= \widehat{\alpha}\widehat{\beta}\widehat{\lambda}\left[\frac{\widehat{a}_{j-1}^{\widehat{\gamma}}\log\left(\widehat{a}_{j-1}\right)\left[K_{NWW}\left(\widehat{a}_{j-1}\right)\right]^{\widehat{\beta}-1}\times D_{NWW}\left(\widehat{a}_{j-1}\right)}{1-\exp\left(-\lambda x^{\widehat{a}_{j-1}^{\widehat{\gamma}}}\right)}\right. \\
&\qquad\left.-\frac{\widehat{a}_{j-1}^{\widehat{\gamma}}\log\left(\widehat{a}_j\right)\left[K_{NWW}\left(a_j\right)\right]^{\widehat{\beta}-1}\times D_{NWW}\left(\widehat{a}_j\right)}{1-\exp\left(-\widehat{\lambda}x^{\widehat{a}_j^{\widehat{\gamma}}}\right)}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
L_{NWW}\left(\widehat{\vartheta}\right) &= \left[L_1\left(\widehat{\vartheta}\right),L_2\left(\widehat{\vartheta}\right),L_3\left(\widehat{\vartheta}\right),L_4\left(\widehat{\vartheta}\right)\right] \\
&= \left[\sum_{j=1}^{r}\frac{\upsilon_j}{p_j}\frac{\partial p_j\left(\widehat{\vartheta}\right)}{\partial\widehat{\alpha}},\ \sum_{j=1}^{r}\frac{\upsilon_j}{p_j}\frac{\partial p_j\left(\widehat{\vartheta}\right)}{\partial\widehat{\beta}},\right. \\
&\qquad\left.\sum_{j=1}^{r}\frac{\upsilon_j}{p_j}\frac{\partial p_j\left(\widehat{\vartheta}\right)}{\partial\widehat{\lambda}},\ \sum_{j=1}^{r}\frac{\upsilon_j}{p_j}\frac{\partial p_j\left(\widehat{\vartheta}\right)}{\partial\widehat{\gamma}}\right]^{x}.
\end{aligned}
$$

- Censored case:

The score functions of the censored NWW distribution:

$$
\frac{\partial l_{NWW}(x_i,\vartheta)}{\partial\alpha} = \frac{r}{\alpha}-\sum_{i\in F}K_{NWW}^{\beta}\left(x_i,\vartheta\right)+\sum_{i\in C}\frac{K_{NWW}^{\beta}\left(x_i,\vartheta\right)\exp\left[-\alpha K_{NWW}^{\beta}\left(x_i,\vartheta\right)\right]}{1-\exp\left[-\alpha K_{NWW}^{\beta}\left(x_i,\vartheta\right)\right]},
$$

$$
\begin{aligned}
\frac{\partial l_{NWW}(x_i,\vartheta)}{\partial\beta} &= \frac{r}{\beta}+\sum_{i\in F}\log\left[K_{NWW}\left(x_i,\vartheta\right)\right]-\alpha\sum_{i\in F}\log\left[K_{NWW}\left(x_i,\vartheta\right)\right]K_{NWW}^{\beta}\left(x_i,\vartheta\right) \\
&+\alpha\sum_{i\in C}\frac{\log\left[K_{NWW}\left(x_i,\vartheta\right)\right]K_{NWW}^{\beta}\left(x_i,\vartheta\right)\exp\left[-\alpha K_{NWW}^{\beta}\left(x_i,\vartheta\right)\right]}{1-\exp\left[-\alpha K_{NWW}^{\beta}\left(x_i,\vartheta\right)\right]},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l_{NWW}(x_i,\vartheta)}{\partial \lambda} &= \frac{r}{\lambda} - \sum_{i\in F} x_i^{\gamma} - \sum_{i\in F} \frac{x_i^{\gamma}\exp\left(-\lambda x_i^{\gamma}\right)}{1-\exp\left(-\lambda x_i^{\gamma}\right)} + (\beta-1)\sum_{i\in F}\frac{x_i^{\gamma}\exp\left(-\lambda x_i^{\gamma}\right)}{Z_{NWW}\left(x_i,\vartheta\right)} \\
&+ \alpha\beta\frac{x_i^{\gamma}K_{NWW}^{\beta-1}\left(x_i,\vartheta\right)}{1-\exp\left(-\lambda x_i^{\gamma}\right)}\left[\sum_{i\in F}\exp\left(-\lambda x_i^{\gamma}\right) - \sum_{i\in C}\frac{h_{NWW}\left(x_i,\vartheta\right)}{1-\exp\left[-\alpha K_{NWW}^{\beta}\left(x_i,\vartheta\right)\right]}\right], \\
\frac{\partial l_{NWW}(x_i,\vartheta)}{\partial \gamma} &= \frac{r}{\gamma} + \sum_{i\in F}\log\left(x_i\right)\left[1-\lambda x_i^{\gamma}\right] + \lambda\left(\beta-1\right)\sum_{i\in F}\frac{M_{NWW}\left(x_i,\vartheta\right)}{Z_{NWW}\left(x_i,\vartheta\right)} \\
&+ \lambda\sum_{i\in F}\frac{M_{NWW}\left(x_i,\vartheta\right)}{1-\exp\left(-\lambda x_i^{\gamma}\right)}\left[\alpha\beta K_{NWW}^{\beta-1}\left(x_i,\vartheta\right)-1\right] \\
&- \alpha\beta\lambda\sum_{i\in C}\frac{x_i^{\gamma}\log\left(x_i\right)K_{NWW}^{\beta-1}\left(x_i,\vartheta\right)h_{NWW}\left(x_i,\vartheta\right)}{1-\exp\left(-\lambda x_i^{\gamma}\right)1-\exp\left[-\alpha K_{NWW}^{\beta}\left(x_i,\vartheta\right)\right]}.
\end{aligned}
$$

Here

$$
\begin{aligned}
h_{NWW}\left(x_i,\vartheta\right) &= \exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x_i^{\gamma}\right)\right]\right\}^{\beta} - \lambda x_i^{\gamma}\right), \\
K_{NWW}\left(x_i,\vartheta\right) &= -\log\left[1-\exp\left(-\lambda x_i^{\gamma}\right)\right], \\
M_{NWW}\left(x_i,\vartheta\right) &= x_i^{\gamma}\log\left(x_i\right)\exp\left(-\lambda x_i^{\gamma}\right), Z_{NWW}\left(x_i,\vartheta\right) \\
&= \left[1-\exp\left(-\lambda x_i^{\gamma}\right)\right]\log\left[1-\exp\left(-\lambda x_i^{\gamma}\right)\right].
\end{aligned}
$$

Elements of the matrix $\widehat{C} = \left(\widehat{C}\right)_{4\times k}$

$$
\widehat{C}_{1j} = \frac{1}{n}\sum_{i:X_i\in I_j}\delta_i\left[\frac{1}{\widehat{\alpha}} - \frac{K_{NWW}^{\widehat{\beta}}\left(x_i,\vartheta\right)}{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\vartheta\right)\right]}\right],
$$

$$
\widehat{C}_{2j} = \frac{1}{n}\sum_{i:X_i\in I_j}^{n}\delta_i\left[\frac{1}{\widehat{\beta}} + \log\left[K_{NWW}\left(x_i,\vartheta\right)\right] - \widehat{\alpha}\frac{\log\left[K_{NWW}\left(x_i,\vartheta\right)\right]K_{NWW}^{\widehat{\beta}}\left(x_i,\vartheta\right)}{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\vartheta\right)\right]}\right],
$$

$$
\begin{aligned}
\widehat{C}_{3j} &= \frac{1}{n}\sum_{i:X_i\in I_j}\delta_i\left[\frac{1}{\widehat{\lambda}} - x_i^{\widehat{\gamma}} + \widehat{\alpha}\widehat{\beta}\frac{x_i^{\widehat{\gamma}}K_{NWW}^{\widehat{\beta}-1}\left(x_i,\vartheta\right)h_{NWW}\left(x_i,\vartheta\right)}{V_{NWW}\left(x_i,\vartheta\right)}\right] \\
&+ \frac{1}{n}\sum_{i:X_i\in I_j}\delta_i\left[+\frac{x_i^{\widehat{\gamma}}\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}\left[\widehat{\alpha}\widehat{\beta}K_{NWW}^{\widehat{\beta}-1}\left(x_i,\vartheta\right)-1\right]\right.
\end{aligned}
$$

$$+\left(\widehat{\beta}-1\right)\frac{x_i^{\widehat{\gamma}}\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{Z_{NWW}\left(x_i,\vartheta\right)}\Bigg]$$

and

$$\widehat{C}_{4j}=\frac{1}{n}\sum_{i:X_i\in I_j}\delta_i\left[\frac{1}{\widehat{\gamma}}+\widehat{\lambda}\left(\widehat{\beta}-1\right)\frac{M_{NWW}\left(x_i,\vartheta\right)}{Z_{NWW}\left(x_i,\vartheta\right)}\right.$$

$$\left.+\widehat{\lambda}\frac{M_{NWW}\left(x_i,\vartheta\right)}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}\left[\widehat{\alpha}\widehat{\beta}K_{NWW}^{\widehat{\beta}-1}\left(x_i,\vartheta\right)-1\right]\right]$$

$$+\sum_{i:X_i\in I_j}^{n}\delta_i\left[\log\left(x_i\right)\left[1-\widehat{\lambda}x_i^{\widehat{\gamma}}\right]\right.$$

$$\left.+\widehat{\alpha}\widehat{\beta}\widehat{\lambda}\frac{x_i^{\widehat{\gamma}}\log\left(x_i\right)K_{NWW}^{\widehat{\beta}-1}\left(x_i,\vartheta\right)h_{NWW}\left(x_i,\vartheta\right)}{\left[1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right]\left\{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\vartheta\right)\right]\right\}}\right].$$

The components of the Information Fisher Matrix $\widehat{\mathbf{I}}=(i_{ll'})_{4\times 4}$ are giver by

$$\widehat{i}_{11}=\frac{1}{n}\sum_{i=1}^{n}\delta i\left[\frac{-1}{\widehat{\alpha}^2}+\frac{K_{NWW}^{2\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]}{\left\{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]\right\}^2}\right],$$

$$\widehat{i}_{22}=\frac{1}{n}\sum_{i=1}^{n}\delta i\left[\frac{-1}{\widehat{\beta}^2}-\widehat{\alpha}\frac{\log^2\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)}{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]}\right]$$

$$+\frac{1}{n}\sum_{i=1}^{n}\delta i\left[\widehat{\alpha}^2\frac{\log^2\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]K_{NWW}^{2\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]}{\left\{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]\right\}^2}\right],$$

$$\widehat{i}_{33}=\frac{1}{n}\sum_{i=1}^{n}\delta i\left\{\frac{-1}{\widehat{\lambda}^2}+\frac{x_i^{2\widehat{\gamma}}\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}\left[1-\frac{\left(\widehat{\beta}-1\right)}{\log\left[1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right]}\right.\right.$$

$$\left.\left.-\widehat{\alpha}\widehat{\beta}\frac{K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}\right]\right\}$$

$$-\frac{\left(\widehat{\beta}-1\right)}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{2\widehat{\gamma}}\exp\left(-2\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\left[\frac{1}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}+\frac{1}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}\right]$$
$$+\frac{\widehat{\alpha}^2\widehat{\beta}^2}{n}\sum_{i=1}^{n}\delta i\frac{K_{NWW}^{2\widehat{\beta}-2}\left(x_i,\widehat{\vartheta}\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)}\left[\frac{1}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}+\frac{h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]$$
$$-\frac{\widehat{\alpha}\widehat{\beta}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{2\widehat{\gamma}}K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)\left[1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right]}\left[1+\frac{\left(\widehat{\beta}-1\right)\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{K_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]$$
$$+\frac{1}{n}\sum_{i=1}^{n}\delta i\left[\frac{x_i^{2\widehat{\gamma}}\exp\left(-2\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{\left[1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right]^2}\left[1-\widehat{\alpha}\widehat{\beta}\left(\widehat{\beta}-1\right)K_{NWW}^{\widehat{\beta}-2}\left(x_i,\widehat{\vartheta}\right)\right]\right]$$

$$\widehat{i}_{44}=\frac{1}{n}\sum_{i=1}^{n}\delta i\left\{\frac{-1}{\widehat{\gamma}^2}-\widehat{\lambda}\sum_{i\in F}x_i^{\widehat{\gamma}}\log^2\left(x_i\right)+\widehat{\lambda}\left(1-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right.$$
$$\left.\times\left[\widehat{\alpha}\widehat{\beta}\frac{K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)-1}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}+\frac{\left(\widehat{\beta}-1\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]\left[M_{NWW}\left(x_i,\widehat{\vartheta}\right)\log\left(x_i\right)\right]\right\}$$
$$+\frac{\widehat{\lambda}^2}{n}\sum_{i=1}^{n}\delta i\frac{M_{NWW}^2\left(x_i,\widehat{\vartheta}\right)}{\left[1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right]^2}\left\{1-\widehat{\alpha}\widehat{\beta}K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)\left[1+\frac{\left(\widehat{\beta}-1\right)}{K_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]\right\}$$
$$-\frac{\widehat{\lambda}^2\left(\widehat{\beta}-1\right)}{n}\sum_{i=1}^{n}\delta i\frac{M_{NWW}^2\left(x_i,\widehat{\vartheta}\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\left[\frac{1}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}+\frac{1}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]$$
$$+\frac{\widehat{\alpha}\widehat{\beta}\widehat{\lambda}^2}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{2\widehat{\gamma}}\log^2\left(x_i\right)K_{NWW}^{\widehat{\beta}-2}\left(x_i,\widehat{\vartheta}\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)\left[1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right]}$$
$$\times\left[\widehat{\alpha}\widehat{\beta}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)-\left(\widehat{\beta}-1\right)\right]$$
$$+\frac{\widehat{\alpha}\widehat{\beta}\widehat{\lambda}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{\widehat{\gamma}}\log^2\left(x_i\right)K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)}$$

$$\times\left\{1-\widehat{\lambda}x_i^{\widehat{\gamma}}\left[\frac{1}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}-\widehat{\alpha}\widehat{\beta}\frac{K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]\right\}$$

$$\widehat{i}_{12}=\frac{1}{n}\sum_{i=1}^{n}\delta i\frac{\log\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)}{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]}$$
$$\times\left[\widehat{\alpha}\frac{K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]}{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]}-1\right]$$

$$\widehat{i}_{13}=\frac{\widehat{\beta}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{\widehat{\gamma}}K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}\left[\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)+\frac{h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]}\right]$$
$$-\frac{\widehat{\alpha}\widehat{\beta}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{\widehat{\gamma}}K_{NWW}^{2\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)}\left[\frac{1}{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]}\right]$$

$$\widehat{i}_{14}=\frac{\widehat{\beta}\widehat{\lambda}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{\widehat{\gamma}}\log\left(x_i\right)K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}$$
$$\times\left[\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)+\frac{h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]}\right]$$
$$-\frac{\widehat{\alpha}\widehat{\beta}\widehat{\lambda}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{\widehat{\gamma}}\log\left(x_i\right)K_{NWW}^{2\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)}$$
$$\times\left[\frac{1}{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]}\right],$$

$$\begin{aligned}\widehat{i}_{23} &= \frac{1}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{\widehat{\gamma}}\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)} \\ &- \frac{\widehat{\alpha}^2\widehat{\beta}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{\widehat{\gamma}}\log\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]K_{NWW}^{2\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)\left[1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]\right]} \\ &+ \frac{\widehat{\alpha}}{n}\sum_{i=1}^{n}\delta i\left\{x_i^{\widehat{\gamma}}K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)\left[\frac{\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}+\frac{h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]\right. \\ &\left.\times\left\{1+\widehat{\beta}\log\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]\right\}\right\}\end{aligned}$$

$$\begin{aligned}\widehat{i}_{24} &= \frac{1}{n}\sum_{i=1}^{n}\delta i\left[\widehat{\lambda}\frac{M_{NWW}\left(x_i,\widehat{\vartheta}\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right] \\ &- \frac{\widehat{\alpha}^2\widehat{\beta}\widehat{\lambda}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{\widehat{\gamma}}\log\left(x_i\right)\log\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]K_{NWW}^{2\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)\left\{1-\exp\left[-\widehat{\alpha}K_{NWW}^{\widehat{\beta}}\left(x_i,\widehat{\vartheta}\right)\right]\right\}} \\ &+ \frac{\widehat{\alpha}\widehat{\lambda}}{n}\sum_{i\in F}\delta i\left\{K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)\left[\frac{M_{NWW}\left(x_i,\widehat{\vartheta}\right)}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}+\frac{x_i^{\widehat{\gamma}}\log\left(x_i\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]\right. \\ &\left.\times\left\{1+\widehat{\beta}\log\left[K_{NWW}\left(x_i,\widehat{\vartheta}\right)\right]\right\}\right\}\end{aligned}$$

$$\begin{aligned}\widehat{i}_{34} &= \frac{1}{n}\sum_{i=1}^{n}\delta i\left\{M_{NWW}\left(x_i,\widehat{\vartheta}\right)\left[\frac{\widehat{\alpha}\widehat{\beta}K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)-1}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}+\frac{\left(\widehat{\beta}-1\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]\right. \\ &\left.\times\left(1-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)-\left[x_i^{\widehat{\gamma}}\log\left(x_i\right)\right]\right\} \\ &+ \frac{\widehat{\lambda}}{n}\sum_{i=1}^{n}\delta i\left\{\frac{x_i^{2\widehat{\gamma}}\log\left(x_i\right)\exp\left(-2\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{\left[1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)\right]^2}\left[1-\widehat{\alpha}\widehat{\beta}K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)\right.\right.\end{aligned}$$

$$-\widehat{\alpha}\left(\widehat{\beta}-1\right)\widehat{\beta}K_{NWW}^{\widehat{\beta}-2}\left(x_i,\widehat{\vartheta}\right)\Big]\Big\}$$

$$-\frac{\widehat{\lambda}\left(\widehat{\beta}-1\right)}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{2\widehat{\gamma}}\log\left(x_i\right)\exp\left(-2\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}$$

$$\times\left[\frac{1}{Z_{NWW}\left(x_i,\widehat{\vartheta}\right)}+\frac{1}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}\right]$$

$$+\frac{\widehat{\alpha}^2\widehat{\beta}^2\widehat{\lambda}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{2\widehat{\gamma}}\log\left(x_i\right)K_{NWW}^{2\widehat{\beta}-2}\left(x_i,\widehat{\vartheta}\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)}$$

$$\times\left[\frac{\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}+\frac{h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)}\right]$$

$$+\frac{\widehat{\alpha}\widehat{\beta}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^{\widehat{\gamma}}\log\left(x_i\right)K_{NWW}^{\widehat{\beta}-1}\left(x_i,\widehat{\vartheta}\right)h_{NWW}\left(x_i,\widehat{\vartheta}\right)}{V_{NWW}\left(x_i,\widehat{\vartheta}\right)}$$

$$\times\left\{1-\lambda x_i^{\widehat{\gamma}}\left[\frac{1+\left(\widehat{\beta}-1\right)\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)K_{NWW}^{-1}\left(x_i,\widehat{\vartheta}\right)}{1-\exp\left(-\widehat{\lambda}x_i^{\widehat{\gamma}}\right)}\right]\right\}.$$

Here

$$V_{NWW}\left(x_i,\vartheta\right)=\left[1-\exp\left(-\lambda x_i^{\gamma}\right)\right]\times\left[1-\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x_i^{\gamma}\right)\right]\right\}^{\beta}\right)\right].$$

Appendix 2: the New-Weibull-Rayleigh model

- Completed case:

The scores function of the NWR distribution

$$\frac{\partial l_{NWR}\left(x_i,\varsigma\right)}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left[-\log\left(1-\exp\left(-\lambda x_i^2\right)\right)\right]^{\beta},$$

$$
\begin{aligned}
\frac{\partial l_{NWR}\left(x_i,\varsigma\right)}{\partial\beta} &= \frac{n}{\beta}+\sum_{i=1}^{n}\log\left[-\log\left(1-\exp\left(-\lambda x_i^2\right)\right)\right] \\
&\quad -\alpha\sum_{i=1}^{n}\log\left[K\left(x_i,\varsigma\right)\right]\times K_{NWR}^{\beta}\left(x_i,\varsigma\right), \\
\frac{\partial l_{NWR}\left(x_i,\varsigma\right)}{\partial\lambda} &= \frac{n}{\lambda}-\sum_{i=1}^{n}x_i^2+\left(\beta-1\right)\sum_{i=1}^{n}\frac{x_i^2\exp\left(-\lambda x_i^2\right)}{Z\left(x_i,\varsigma\right)} \\
&\quad +\alpha\beta\sum_{i=1}^{n}\frac{x_i^2\exp\left(-\lambda x_i^2\right)K^{\beta-1}\left(x_i,\varsigma\right)}{1-\exp\left(-\lambda x_i^2\right)}-\sum_{i=1}^{n}\frac{x_i^{\gamma}\exp\left(-\lambda x_i^2\right)}{1-\exp\left(-\lambda x_i^2\right)}.
\end{aligned}
$$

Here

$$
\begin{aligned}
K\left(x,\varsigma\right) &= \left[-\log\left(1-\exp\left(-\lambda x^2\right)\right)\right], \quad \varsigma=\left(\alpha,\beta,\lambda\right), \\
Z\left(x,\varsigma\right) &= \left(1-\exp\left(-\lambda x^2\right)\right)\times\left[\log\left(1-\exp\left(-\lambda x^2\right)\right)\right].
\end{aligned}
$$

Elements of the NRR statistic test for the NWR distribution

$$
\begin{aligned}
\frac{\partial p_j\left(\widehat{\varsigma}\right)}{\partial\widehat{\alpha}} &= \left[K\left(\widehat{a}_j\right)\right]^{\widehat{\beta}}F_{NWR}\left(\widehat{a}_j\right)-\left[K\left(\widehat{a}_{j-1}\right)\right]^{\widehat{\beta}}F_{NWR}\left(\widehat{a}_{j-1}\right), \\
\frac{\partial p_j\left(\widehat{\varsigma}\right)}{\partial\widehat{\beta}} &= \widehat{\alpha}\log\left[K\left(\widehat{a}_j\right)\right] \\
&\quad \times\left[K\left(\widehat{a}_j\right)\right]^{\widehat{\beta}}F_{NWR}\left(\widehat{a}_j\right)-\widehat{\alpha}\log\left[K\left(\widehat{a}_{j-1}\right)\right]\times\left[K\left(\widehat{a}_{j-1}\right)\right]^{\widehat{\beta}}F_{NWR}\left(\widehat{a}_{j-1}\right), \\
\frac{\partial p_j\left(\widehat{\varsigma}\right)}{\partial\widehat{\lambda}} &= \widehat{\alpha}\widehat{\beta}\left[\frac{\widehat{a}_{j-1}\left[K\left(\widehat{a}_{j-1}\right)\right]^{\beta-1}\times h\left(\widehat{a}_{j-1}\right)}{1-\exp\left(-\widehat{\lambda}x^{\widehat{a}_{j-1}^2}\right)}-\frac{\widehat{a}_j\left[K\left(\widehat{a}_j\right)\right]^{\beta-1}\times h\left(\widehat{a}_j\right)}{1-\exp\left(-\widehat{\lambda}x^{\widehat{a}_j^2}\right)}\right],
\end{aligned}
$$

where

$$
h\left(x,\varsigma\right)=\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x^2\right)\right]\right\}^{\beta}-\lambda x^2\right), \quad \varsigma=\left(\alpha,\beta,\lambda\right).
$$

The vector $\mathbf{L}_{NWR}=\left(L_1,\ldots,L_s\right)^x$ is given by

$$
\begin{aligned}
\mathbf{L}_{NWR}\left(\widehat{\varsigma}\right) = \Bigg[& L_1\left(\widehat{\varsigma}\right)=\sum_{j=1}^{r}\frac{\upsilon_j}{p_j}\frac{\partial p_j\left(\widehat{\varsigma}\right)}{\partial\widehat{\alpha}},\ L_2\left(\widehat{\varsigma}\right)=\sum_{j=1}^{r}\frac{\upsilon_j}{p_j}\frac{\partial p_j\left(\widehat{\varsigma}\right)}{\partial\widehat{\beta}}, \\
& L_3\left(\widehat{\varsigma}\right)=\sum_{j=1}^{r}\frac{\upsilon_j}{p_j}\frac{\partial p_j\left(\widehat{\varsigma}\right)}{\partial\widehat{\lambda}}\Bigg].
\end{aligned}
$$

The components of the Information Fisher Matrix $\mathbf{I}_{NWR} = \left(\widehat{i}_{ll\prime}\right)_{3\times 3}$ are

$$\widehat{i}_{12} = -\sum_{i=1}^{n} \log\left[K\left(x_i, \widehat{\varsigma}\right)\right] \times \left[K\left(x_i, \widehat{\varsigma}\right)\right]^{\widehat{\beta}},$$

$$\widehat{i}_{13} = \widehat{\beta} \sum_{i=1}^{n} \frac{x_i^2 \exp\left(-\widehat{\lambda} x_i^2\right)\left[K\left(x_i, \widehat{\varsigma}\right)\right]^{\widehat{\beta}-1}}{\left(1 - \exp\left(-\widehat{\lambda} x_i^2\right)\right)}$$

$$\widehat{i}_{23} = \widehat{\alpha}\widehat{\beta} \sum_{i=1}^{n} \frac{x_i^2 e^{-\widehat{\lambda} x_i^2} \log\left[K\left(x_i, \widehat{\varsigma}\right)\right] \times \left[K\left(x_i, \widehat{\varsigma}\right)\right]^{\widehat{\beta}-1}}{1 - \exp\left(-\widehat{\lambda} x_i^2\right)}$$

$$+ \sum_{i=1}^{n} x_i^{\widehat{\gamma}} e^{-\widehat{\lambda} x_i^2} \left\{\frac{1 - \widehat{\alpha}\left[K\left(x_i, \widehat{\varsigma}\right)\right]^{\widehat{\beta}}}{Z\left(x_i, \widehat{\varsigma}\right)}\right\},$$

$$\widehat{i}_{11} = -\frac{n}{\widehat{\alpha}^2},$$

$$\widehat{i}_{22} = -\frac{n}{\widehat{\beta}^2} - \widehat{\alpha} \sum_{i=1}^{n} \log^2\left[K\left(x_i, \widehat{\varsigma}\right)\right] \times \left[K\left(x_i, \widehat{\varsigma}\right)\right]^{\widehat{\beta}},$$

$$\widehat{i}_{33} = -\frac{n}{\widehat{\lambda}^2} + \sum_{i=1}^{n} \frac{x_i^4 e^{-\widehat{\lambda} x_i^2}}{1 - e^{-\widehat{\lambda} x_i^2}} \left[1 - \widehat{\alpha}\widehat{\beta}\left[K\left(x_i, \widehat{\varsigma}\right)\right]^{\widehat{\beta}-1}\right]$$

$$+ \sum_{i=1}^{n} \frac{x_i^4 e^{-\widehat{\lambda} x_i^2}}{\left[1 - e^{-\widehat{\lambda} x_i^2}\right]^2} \left\{1 - \widehat{\alpha}\widehat{\beta} K^{\widehat{\beta}-1}\left(x_i, \widehat{\varsigma}\right)\left[1 + \frac{\left(\widehat{\beta} - 1\right)}{K\left(x_i, \widehat{\varsigma}\right)}\right]\right\}$$

$$-\left(\widehat{\beta} - 1\right) \sum_{i=1}^{n} \frac{x_i^4 e^{-\widehat{\lambda} x_i^2}}{Z\left(x_i, \widehat{\varsigma}\right)} \left[1 + \frac{e^{-\widehat{\lambda} x_i^2}}{1 - \exp(-\widehat{\lambda} x_i^{\widehat{\gamma}})} + \frac{e^{-\widehat{\lambda} x_i^2}}{\left(x_i, \widehat{\varsigma}\right)}\right].$$

- Censored case:

The score functions of the censored NWR model

$$\frac{\partial l(x_i, \varsigma)}{\partial \alpha} = \frac{r}{\alpha} - \sum_{i \in F} K^{\beta}\left(x_i, \varsigma\right) + \sum_{i \in C} \frac{K^{\beta}\left(x_i, \varsigma\right) \exp\left[-\alpha K^{\beta}\left(x_i, \varsigma\right)\right]}{1 - \exp\left[-\alpha K^{\beta}\left(x_i, \varsigma\right)\right]},$$

$$\frac{\partial l(x_i, \varsigma)}{\partial \beta} = \frac{r}{\beta} + \sum_{i \in F} \log\left[K\left(x_i, \varsigma\right)\right] - \alpha \sum_{i \in F} \log\left[K\left(x_i, \varsigma\right)\right] K^{\beta}\left(x_i, \varsigma\right)$$

$$+\alpha\sum_{i\in C}\frac{\log\left[K\left(x_i,\varsigma\right)\right]K^{\beta}\left(x_i,\varsigma\right)\exp\left[-\alpha K^{\beta}\left(x_i,\varsigma\right)\right]}{1-\exp\left[-\alpha K^{\beta}\left(x_i,\varsigma\right)\right]},$$

$$\frac{\partial l(x_i,\varsigma)}{\partial\lambda}=\frac{r}{\lambda}-\sum_{i\in F}x_i^2-\sum_{i\in F}\frac{x_i^2\exp\left(-\lambda x_i^2\right)}{1-\exp\left(-\lambda x_i^2\right)}+(\beta-1)\sum_{i\in F}\frac{x_i^2\exp\left(-\lambda x_i^2\right)}{Z\left(x_i,\varsigma\right)}$$

$$+\alpha\beta\frac{x_i^2K^{\beta-1}\left(x_i,\varsigma\right)}{1-\exp\left(-\lambda x_i^2\right)}\left[\sum_{i\in F}\exp\left(-\lambda x_i^2\right)-\sum_{i\in C}\frac{h\left(x_i,\varsigma\right)}{1-\exp\left[-\alpha K^{\beta}\left(x_i,\varsigma\right)\right]}\right].$$

Elements of the matrix $\widehat{C}=\left(\widehat{C}\right)_{s\times k}$ of the censored NWR distribution

$$\widehat{C}_{1j}=\frac{1}{n}\sum_{i:X_i\in I_j}\delta_i\left[\frac{1}{\widehat{\alpha}}-\frac{K_{NWR}^{\widehat{\beta}}\left(x_i,\varsigma\right)}{1-\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\varsigma\right)\right]}\right],$$

$$\widehat{C}_{2j}=\frac{1}{n}\sum_{i:X_i\in I_j}^{n}\delta_i\left[\frac{1}{\widehat{\beta}}+\log\left[K_{NWR}\left(x_i,\varsigma\right)\right]-\widehat{\alpha}\frac{\log\left[K_{NWR}\left(x_i,\varsigma\right)\right]K_{NWR}^{\widehat{\beta}}\left(x_i,\varsigma\right)}{1-\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\varsigma\right)\right]}\right],$$

$$\widehat{C}_{3j}=\frac{1}{n}\sum_{i:X_i\in I_j}\delta_i\left[\frac{1}{\widehat{\lambda}}-x_i^2+\widehat{\alpha}\widehat{\beta}\frac{x_i^2K_{NWR}^{\widehat{\beta}-1}\left(x_i,\varsigma\right)h_{NWR}\left(x_i,\varsigma\right)}{V_{NWR}\left(x_i,\varsigma\right)}\right]$$

$$+\frac{1}{n}\sum_{i:X_i\in I_j}\delta_i\left[+\frac{x_i^2\exp\left(-\widehat{\lambda}x_i^2\right)}{1-\exp\left(-\widehat{\lambda}x_i^2\right)}\left[\widehat{\alpha}\widehat{\beta}K_{NWR}^{\widehat{\beta}-1}\left(x_i,\varsigma\right)-1\right]+\left(\widehat{\beta}-1\right)\frac{x_i^2\exp\left(-\widehat{\lambda}x_i^2\right)}{Z_{NWR}\left(x_i,\varsigma\right)}\right]$$

The Information Fisher Matrix $\widehat{\mathbf{I}}_{NWR}=(i_{ll'})_{3\times3}$

$$\widehat{i}_{11}=\frac{1}{n}\sum_{i=1}^{n}\delta i\left[\frac{-1}{\widehat{\alpha}^2}+\frac{K_{NWR}^{2\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\right]}{\left\{1-\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\right]\right\}^2}\right]$$

$$\widehat{i}_{22}=\frac{1}{n}\sum_{i=1}^{n}\delta i\left[\frac{-1}{\widehat{\beta}^2}-\widehat{\alpha}\frac{\log^2\left[K_{NWR}\left(x_i,\widehat{\varsigma}\right)\right]K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)}{1-\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\right]}\right]$$

$$+\frac{1}{n}\sum_{i=1}^{n}\delta i\left[\widehat{\alpha}^2\frac{\log^2\left[K_{NWR}\left(x_i,\widehat{\varsigma}\right)\right]K_{NWR}^{2\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\right]}{\left\{1-\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\right]\right\}^2}\right]$$

$$
\begin{aligned}
\widehat{i}_{33} &= \frac{1}{n}\sum_{i=1}^{n}\delta i\left\{\frac{-1}{\widehat{\lambda}^2}+\frac{x_i^4\exp\left(-\widehat{\lambda}x_i^2\right)}{1-\exp\left(-\widehat{\lambda}x_i^2\right)}\left[1-\frac{\left(\widehat{\beta}-1\right)}{\log\left[1-\exp\left(-\widehat{\lambda}x_i^2\right)\right]}\right.\right.\\
&\quad\left.\left.-\widehat{\alpha}\widehat{\beta}\frac{K_{NWR}^{\widehat{\beta}-1}\left(x_i,\widehat{\varsigma}\right)}{1-\exp\left(-\widehat{\lambda}x_i^2\right)}\right]\right\}\\
&\quad-\frac{\left(\widehat{\beta}-1\right)}{n}\sum_{i=1}^{n}\delta i\frac{x_i^4\exp\left(-2\widehat{\lambda}x_i^2\right)}{Z_{NWR}\left(x_i,\widehat{\varsigma}\right)}\left[\frac{1}{Z_{NWR}\left(x_i,\widehat{\varsigma}\right)}+\frac{1}{1-\exp\left(-\widehat{\lambda}x_i^2\right)}\right]\\
&\quad+\frac{\widehat{\alpha}^2\widehat{\beta}^2}{n}\sum_{i=1}^{n}\delta i\frac{K_{NWR}^{2\widehat{\beta}-2}\left(x_i,\widehat{\varsigma}\right)h_{NWR}\left(x_i,\widehat{\varsigma}\right)}{V_{NWR}\left(x_i,\widehat{\varsigma}\right)}\left[\frac{1}{1-\exp\left(-\widehat{\lambda}x_i^2\right)}+\frac{h_{NWR}\left(x_i,\widehat{\varsigma}\right)}{V_{NWR}\left(x_i,\widehat{\varsigma}\right)}\right]\\
&\quad-\frac{\widehat{\alpha}\widehat{\beta}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^4K_{NWR}^{\widehat{\beta}-1}\left(x_i,\widehat{\varsigma}\right)h_{NWR}\left(x_i,\widehat{\varsigma}\right)}{V_{NWR}\left(x_i,\widehat{\varsigma}\right)\left[1-\exp\left(-\widehat{\lambda}x_i^2\right)\right]}\left[1+\frac{\left(\widehat{\beta}-1\right)\exp\left(-\widehat{\lambda}x_i^2\right)}{K_{NWR}\left(x_i,\widehat{\varsigma}\right)}\right]\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\delta i\left[\frac{x_i^4\exp\left(-2\widehat{\lambda}x_i^2\right)}{\left[1-\exp\left(-\widehat{\lambda}x_i^2\right)\right]^2}\left[1-\widehat{\alpha}\widehat{\beta}\left(\widehat{\beta}-1\right)K_{NWR}^{\widehat{\beta}-2}\left(x_i,\widehat{\varsigma}\right)\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
\widehat{i}_{12} &= \frac{1}{n}\sum_{i=1}^{n}\delta i\frac{\log\left[K_{NWR}\left(x_i,\widehat{\varsigma}\right)\right]K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)}{1-\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\right]}\\
&\quad\times\left[\widehat{\alpha}\frac{K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\right]}{1-\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\right]}-1\right]\\
\widehat{i}_{13} &= \frac{\widehat{\beta}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^2K_{NWR}^{\widehat{\beta}-1}\left(x_i,\widehat{\varsigma}\right)}{1-\exp\left(-\widehat{\lambda}x_i^2\right)}\left[\exp\left(-\widehat{\lambda}x_i^2\right)+\frac{h_{NWR}\left(x_i,\widehat{\varsigma}\right)}{1-\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\right]}\right]\\
&\quad-\frac{\widehat{\alpha}\widehat{\beta}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^2K_{NWR}^{2\widehat{\beta}-1}\left(x_i,\widehat{\varsigma}\right)h_{NWR}\left(x_i,\widehat{\varsigma}\right)}{V_{NWR}\left(x_i,\widehat{\varsigma}\right)}\left[\frac{1}{1-\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\right]}\right]
\end{aligned}
$$

$$\begin{aligned}\widehat{i}_{23} &= \frac{1}{n}\sum_{i=1}^{n}\delta i\frac{x_i^2\exp\left(-\widehat{\lambda}x_i^2\right)}{Z_{NWR}\left(x_i,\widehat{\varsigma}\right)}\\
&-\frac{\widehat{\alpha}^2\widehat{\beta}}{n}\sum_{i=1}^{n}\delta i\frac{x_i^2\log\left[K_{NWR}\left(x_i,\widehat{\varsigma}\right)\right]K_{NWR}^{2\widehat{\beta}-1}\left(x_i,\widehat{\varsigma}\right)h_{NWR}\left(x_i,\widehat{\varsigma}\right)}{V_{NWR}\left(x_i,\widehat{\varsigma}\right)\left[1-\exp\left[-\widehat{\alpha}K_{NWR}^{\widehat{\beta}}\left(x_i,\widehat{\varsigma}\right)\right]\right]}\\
&+\frac{\widehat{\alpha}}{n}\sum_{i=1}^{n}\delta i\left\{x_i^2K_{NWR}^{\widehat{\beta}-1}\left(x_i,\widehat{\varsigma}\right)\left[\frac{\exp\left(-\widehat{\lambda}x_i^2\right)}{1-\exp\left(-\widehat{\lambda}x_i^2\right)}+\frac{h_{NWR}\left(x_i,\widehat{\varsigma}\right)}{V_{NWR}\left(x_i,\widehat{\varsigma}\right)}\right]\right.\\
&\left.\times\left\{1+\widehat{\beta}\log\left[K_{NWR}\left(x_i,\widehat{\varsigma}\right)\right]\right\}\right\}\end{aligned}$$

. Here

$$V\left(x_i,\varsigma\right)=\left[1-\exp\left(-\lambda x_i^2\right)\right]\times\left[1-\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x^2\right)\right]\right\}\right)\right].$$

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DEPARTMENT OF MATHEMATICS
BADJI MOKHTAR UNIVERSITY
ANNABA, ALGERIA.
Email address: khaoula.meribout@univ-annaba.org

DEPARTMENT OF MATHEMATICS
BADJI MOKHTAR UNIVERSITY
ANNABA, ALGERIA.
Email address: naciraseddik@gmail.com

DEPARTMENT OF MATHEMATICS
BADJI MOKHTAR UNIVERSITY
ANNABA, ALGERIA.
Email address: goual.hafida@gmail.com