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ON NIL SKEW GENERALIZED POWER SERIES REFLEXIVE RINGS

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ABSTRACT. Let R be a ring and $(S, \leq)$ a strictly ordered monoid. In this paper, we deal with a new approaches to reflexive property for rings by using nilpotent elements. In this direction we introduce the notions of (S,ω)-reflexive and (S,ω)-nil-reflexive. Examples are given that, (S,ω)-nil-reflexive is not (S,ω)-reflexive. Under some suitable conditions, we proved that, if R is a right APP-ring, then R is (S,ω)-reflexive and R be a semiprime ring with the ACC on left annihilator ideals, $(S,\leq)$ an $a.n.u.p.$-monoid, then R is (S,ω)-reflexive. Also, we proved that, R is (S,ω)-nil-reflexive if and only if R/I is $(S,\overline{\omega})$-nil-reflexive, R is (S,ω)-nil-reflexive if and only if $T_n(R)$ is (S,ω)-nil-reflexive and we will show that, if R is a right Noetherian ring, then R is (S,ω)-nil-reflexive. Moreover, we investigate ring extensions which have roles in ring theory.

1. INTRODUCTION

Throughout this article, all rings are associative with identity unless otherwise stated. The notion of Armendariz ring is introduced by Rege and Chhawchharia (see [1]). They defined a ring R to be Armendariz if $f(x)g(x) = 0$ implies $a_ib_j = 0$, for all polynomials $f(x) = a_0+a_1x+a_2x^2+\cdots+a_mx^m$, $g(x) = b_0+b_1x+b_2x^2+\cdots+b_nx^n \in R[x]$. Mason introduced the reflexive property for ideals, and this concept

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was generalized by some authors, defining idempotent reflexive right ideals and rings, completely reflexive rings, weakly reflexive rings (see namely, [2], [3], [4]). Let R be a ring and I be a right ideal of R. In [3], I is called a reflexive right ideal if for any $x, y \in R, xRy \subseteq I$ implies $yRx \subseteq I$. The reflexive right ideal concept is also specialized to the zero ideal of a ring, namely, a ring R is called reflexive [3] if its zero ideal is reflexive.

An ideal I of a ring is called semiprime if $aRa \subseteq I$ implies $a \in I$ for $a \in R$ and R is called semiprime if 0 is a semiprime ideal. Note that every semiprime ideal is reflexive by a simple computation, and so every ideal of a fully idempotent ring ($i.e., I^2 = I$ for every ideal I) is reflexive by [5]. Reflexive rings are generalized to weakly reflexive rings in [3]. The ring R is said to be weakly reflexive if $arb = 0$ implies bra is nilpotent for $a, b \in R$ and all $r \in R$. In [3], a ring R is called completely reflexive if for any $a, b \in R, ab = 0$ implies $ba = 0$. Completely reflexive rings are called reversible by Cohn in [6] and also studied in [7]. The rings without nonzero nilpotent elements are said to be reduced rings. Reduced rings are completely reflexive and every completely reflexive ring is semicommutative, i.e. according to [1], a ring R is called semicommutative if for all $a, b \in R, ab = 0$ implies $aRb = 0$. This is equivalent to the definition that any left (right) annihilator of R is an ideal of R. In [8], semicommutativity of rings is generalized to nil-semicommutativity of rings. A ring R is called nil-semicommutative if $a, b \in R$ satisfy that ab is nilpotent, then $arb \in nil(R)$ for any $r \in R$ where $nil(R)$ denotes the set of all nilpotent elements of R. Clearly, every semicommutative ring is nil-semicommutative.

Let R be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega : S \to$End(R) a monoid homomorphism. For $s \in S$, let ω_s denote the image of s under ω, that is, $\omega_s = \omega(s)$. Let H be the set of all functions $f : S \to R$ such that the support supp$(f) = \{s \in S : f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in H$ the set

$$X_s(f, g) = \{(u, v) \in supp(f) \times supp(g) : s = uv\}$$

is finite. Thus one can define the product $fg : S \to R$ of $f, g \in H$ as follows:

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)\omega_u(g(v))$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, H becomes a ring, called the ring of skew generalized power series with coefficients in R and exponents in S, see [9] and denoted by $[[R^{S,\leq},\omega]]$ (or by $R[[S,\omega]]$ when there is no ambiguity concerning the order $\leq$).

We will use the symbol 1 to denote the identity elements of the monoid S, the ring R, and the ring $[[R^{S,\leq},\omega]]$ as well as the trivial monoid homomorphism $1: S \to$End(R) that sends every element of S to the identity endomorphism. A subset $P \subseteq R$ will be called S-invariant if for every $s \in S$ it is ω_s-invariant (that is, $\omega_s(P) \subseteq P$). To each $r \in R$ and $s \in S$, we associate elements $c_r, e_s \in [[R^{S,\leq},\omega]]$ defined by

$$c_r(x) = \begin{cases} r, & x = 1, \\ 0, & x \in S\backslash\{1\}, \end{cases} \quad , \quad e_s(x) = \begin{cases} 1, & x = s, \\ 0, & x \in S\backslash\{s\}. \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $[[R^{S,\leq},\omega]]$ and $s \mapsto e_s$, is a monoid embedding of S into the multiplicative monoid of the ring $[[R^{S,\leq},\omega]]$, and $e_s c_r = c_{\omega_s(r)} e_s$.

Motivated by the works on reflexivity, in this note we study new two concepts of reflexive property, namely, skew generalized power series reflexive (S,ω)-reflexive, and nilpotent property of it ((S,ω)-nil-reflexive). Examples are given that, (S,ω)-nil-reflexive is not (S,ω)-reflexive. We proved that If R is a right APP-ring, then R is (S,ω)-reflexive. Also we prove that, R is (S,ω)-nil-reflexive if and only if R/I is $(S,\overline{\omega})$-nil-reflexive, and R is (S,ω)-nil-reflexive if and only if $T_n(R)$ is (S,ω)-nil-reflexive, when n is a positive integer. If R be a semiprime ring with the ACC on left annihilator ideals, $(S,\leq)$ an $a.n.u.p.$-monoid and $\omega : S \to$Aut(R) a monoid homomorphism, then R is (S,ω)-reflexive. Moreover, we proved that for right Noetherian ring, then R is (S,ω)-nil-reflexive. Some results of (S,ω)-nil-reflexive we discussed.

In what follows, we will write monoids multiplicatively unless otherwise indicated. If R is a ring and X is a nonempty subset of R, then the left (right) annihilator of X in R is denoted by $\ell_R(X)(r_R(X))$, and we will denote by End(R) the monoid of ring endomorphisms of R, and by Aut(R) the group of ring automorphisms of R. Any concept and notation not defined here can be found in Ribenboim ([10]– [12]), Elliott and Ribenboim [13]. $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers and the ring of integers, and for a positive integer $n, \mathbb{Z}_n$ is the ring of integers modulo $n.$ For a positive integer $n,$ let $Mat_n(R)$ denote the ring of all $n \times n$ matrices and $T_n(R)$ the ring of all $n \times n$ upper triangular matrices with entries in $R.$ We write $R[x], P(R),$ and $S_n(R),$ for the polynomial ring over a ring $R,$ the prime radical of $R,$ and the subring consisting of all upper triangular matrices over a ring R with equal main diagonal entries.

2. ON SKEW GENERALIZED POWER SERIES REFLEXIVE RINGS

In the following we discus some results for (S, ω)-reflexive rings which is an extend to the definition of S-reflexive rings by [14]. Clark defined quasi-Baer rings in [15]. A ring R is called quasi-Baer if the left annihilator of every left ideal of R is generated by an idempotent. Note that this definition is left-right symmetric. Some results of a quasi-Baer ring can be found in [15] and [16] and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. A ring R is called a right (resp., left) PP-ring if every principal right (resp., left) ideal is projective (equivalently, if the right (resp., left) annihilator of an element of R is generated (as a right (resp., left) ideal) by an idempotent of R). A ring R is called a PP-ring (also called a Rickart ring [17, p. 18]) if it is both right and left $PP.$ We say a ring R is a left APP-ring if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R.$

As a generalization of quasi-Baer rings, Birkenmeier, Kim and Park in [18] introduced the concept of principally quasi-Baer rings. A ring R is called left principally quasi-Baer (or simply left $p.q.$-Baer) if the left annihilator of a principal left ideal of R is generated by an idempotent. Similarly, right $p.q.$-Baer rings can be defined.

A ring is called $p.q.$-Baer if it is both right and left $p.q.$-Baer. Observe that biregular rings and quasi-Baer rings are $p.q.$-Baer. For more details and examples of left $p.q.$-Baer rings, see [18], [19] and [20]. We say a ring R is a left APP-ring if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R.$ This concept is a common generalization of left $p.q.$-Baer rings and right PP-rings.

According to [21], a ring R is called quasi-Armendariz if whenever polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m, g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n \in R[x]$ satisfy $f(x)R[x]g(x) = 0,$ then $a_iRb_j = 0$ for each $i, j.$ It was proved in [7, Proposition

2.4] that if R is an Armendariz ring, then R is completely reflexive if and only if $R[x]$ is completely reflexive. In [22], a ring R is called (S,ω)-quasi-Armendariz, if whenever $f,g \in [[R^{S,\leq},\omega]] \equiv A$, $fAg = 0$ implies $f(u)R\omega_u(g(v)) = 0$ for all $u,v \in S$. We start by the first concept in this paper.

Definition 2.1. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid, and $\omega : S \to$End(R) a monoid homomorphism. The ring R is called skew generalized power series reflexive $((S,\omega)$-reflexive$)$, if whenever $f[[R^{S,\leq},\omega]]g = 0$ for $f,g \in [[R^{S,\leq},\omega]]$, then $g[[R^{S,\leq},\omega]]f = 0$.*

Let $S = (\mathbb{N} \cup \{0\}, +)$ and $\leq$ is the usual order. Then, $[[R^{S,\leq},\omega]] \cong R[[x]]$. Let ω be the trivial order. Then the ring R is (S,ω)-reflexive, if and only if, R is power series reflexive.

The following result appeared in [23].

Definition 2.2. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to$End(R) a monoid homomorphism. The ring R is said to be S-compatible (resp. S-rigid) if ω_s is compatible (resp. rigid) for every $s \in S$; to indicate the homomorphism ω, we will sometimes say that R is (S,ω)-compatible (resp. (S,ω)-rigid).*

The following results appeared in [23] and [24].

Lemma 2.1. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Then $[[R^{S,\leq},\omega]]$ is reduced if and only if R is reduced.*

Let R be a ring, $(S_1,\leq_1),(S_2,\leq_2),\ldots,(S_n,\leq_n)$ be strictly ordered monoid, and $\omega^i : S_i \to End(R)$ be a monoid homomorphism for every i. Define $\omega : S \to End(R)$ as

$$\omega(s_1,s_2,\ldots,s_n) = \omega_{s_1}\omega_{s_2}\cdots\omega_{s_n}.$$

That is,

$$\omega_{(s_1,s_2,\ldots,s_n)} = \omega_{s_1}\omega_{s_2}\cdots\omega_{s_n}.$$

Then ω is well-defined.

Lemma 2.2. *If R is S_i-compatible for each i, then R is S-compatible.*

A ring R is symmetric if for all $a,b,c \in R$ we have $abc = 0$ implies that $acb = 0$. A ring R is called reversible if for all $a,b \in R$ we have $ab = 0$ if and only if $ba = 0$. Reversible rings were defined by Cohn in [6]. Reversible rings are clearly reflexive.

It is shown by [4, Lemma 2.1] that a ring R is reflexive if and only if $IJ = 0$ implies $JI = 0$ for all ideals I, J of R. These arguments naturally give rise to extending the study of symmetric ring property to the lattice of ideals. A generalization of symmetric rings was defined by Camillo, Kwak and Lee in [25]. A ring R is called ideal-symmetric if $IJK = 0$ implies $IKJ = 0$ for all ideals I, J, K of R. It is obvious that semiprime rings are ideal-symmetric.

Theorem 2.1. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Assume that R is S-compatible (S, ω)-quasi-Armendariz. Then we have:*

(i) *R is reflexive if and only if $[[R^{S,\leq}, \omega]]$ is reflexive.*
(ii) *R is ideal-symmetric if and only if $[[R^{S,\leq}, \omega]]$ is ideal-symmetric.*

Proof. We only prove (ii), because the proof of the other case is similar. Assume that R is ideal-symmetric and $f_1, f_2, f_3 \in H = [[R^{S,\leq}, \omega]]$ are such that $f_1 H f_2 H f_3 = 0$. Since R is an S-compatible (S, ω)-quasi-Armendariz, we have $f_1(u) R f_2(v) R f_3(w) = 0$ for all $u, v, w \in S$. Since R is ideal-symmetric, we have $f_1(u) R f_3(w) R f_2(v) = 0$. Now, by compatibility of R implies that, $f_1 H f_3 H f_2 = 0$. Hence H is ideal-symmetric. Conversely, suppose that H is ideal-symmetric. Let $aRbRc = 0$ for all $a, b, c \in R$. Since R is an S-compatible, $c_a H c_b H c_c = 0$. Thus $c_a H c_c H c_b = 0$ and $aRcRb = 0$ for all $a, b, c \in R$. Therefore, R is ideal-symmetric. $\square$

Proposition 2.1. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is S-compatible and semicommutative ring, then R is (S, ω)-Armendariz if and only if R is (S, ω)-reflexive.*

Proof. The proof is clear. $\square$

An ideal I of R is said to be right s-unital if, for each $a \in I$ there exists an element $e \in I$ such that $ae = a$. Note that if I and J are right s-unital ideals, then so is $I \cap J$ (if $a \in I \cap J$, then $a \in aIJ \subseteq a(I \cap J)$).

The following result follows from Tominaga [26, Theorem 1].

Lemma 2.3. *An ideal I of a ring R is left (resp. right) s-unital if and only if for any finitely many elements $a_1, a_2, \ldots, a_n \in I$, there exists an element $e \in I$ such that $a_i = ea_i (resp.\ a_i = a_i e)$ for each $i = 1, 2, \ldots, n$.*

According to [27], a ring R with a monomorphism α is called α-weakly rigid if for each $a, b \in R, aRb = 0$ if and only if $a\alpha(Rb) = 0.$ For any positive integer $n,$ a ring R is α-weakly rigid if and only if, the $n \times n$ upper triangular matrix ring $T_n(R)$ is $\overline{\alpha}$-weakly rigid. Also if R is a semiprime α-weakly rigid ring, then the ring of polynomials $R[X],$ for X an arbitrary nonempty set of indeterminates, is a semiprime α-weakly rigid ring. For every prime ring R and any automorphism $\alpha,$ the rings $T_n(R), R[X]$ and the power series ring $R[[X]]$ are α-weakly rigid rings.

Definition 2.3. *[28] Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. We say R is S-weakly rigid if ω_s is weakly rigid monomorphism for every $s \in S$.*

Theorem 2.2. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is right APP-ring and ω_s is an automorphism for each $s \in S$ and R is S-weakly rigid, then R is (S, ω)-reflexive.*

Proof. Let $0 \neq f, g \in [[R^{S,\leq}, \omega]]$ with $f[[R^{S,\leq}, \omega]]g = 0.$ By Ribenbiom [11], there exists a strict total order $\leq'$ on S, which is finer than $\leq$ (that is, for all $u_0, v_0 \in S, u_0 \leq v_0$ implies $u_0 \leq' v_0$). We will use transfinite induction on the strictly totally ordered set $(S, \leq')$ to show that $g[[R^{S,\leq}, \omega]]f = 0.$ For any $u, v \in S,$ let u_0 and v_0 denote the minimum elements of $supp(f)$ and $supp(g)$ in the $\leq'$ order, respectively. If $u \in supp(f)$ and $v \in supp(g)$ are such that $u + v = u_0 + v_0,$ then $u_0 \leq' u$ and $v_0 \leq' v.$ If $u_0 <' u,$ then $u_0 + v_0 <' u + v = u_0 + v_0,$ a contradiction. Thus $u = u_0.$ Similarly, $v = v_0.$ Hence

$$0 = (fc_tg)(u_0 + v_0) = \sum_{(u,v) \in X_{u_0+v_0}(f, c_tg)} f(u)\omega_u(tg(v)) = f(u_0)\omega_{u_0}(tg(v_0)).$$

So by rigidness $f(u_0)Rg(v_0) = 0,$ and $g(v_0)Rf(u_0) = 0,$ Now, let $\lambda \in S$ with $u_0 + v_0 \leq \lambda$ and assume that for any $u \in supp(f)$ and any $v \in supp(g),$ if $u + v < \lambda,$ then $f(u)Rg(v) = 0.$ We claim that $f(u)Rg(v) = 0,$ for each $u \in supp(f)$ and each $v \in supp(g)$ with $u + v = \lambda.$ For convenience, we write $X_\lambda(f, g) = \{(u, v) \mid u + v = \lambda, u \in supp(f), v \in supp(g)\}$ as $\{(u_i, v_i) \mid i = 1, 2, \ldots, n\}$ such that $u_1 < u_2 < \cdots < u_n,$ where n is a positive integer (Note that if $u_1 = u_2,$ then from $u_1 + v_1 = u_2 + v_2$ we have $v_1 = v_2,$ and then $(u_1, v_1) = (u_2, v_2)).$ Since $f[[R^{S,\leq}, \omega]]g = 0,$ for any $t \in R$

we have:

$$0 = (fc_t g)(\lambda) = \sum_{(u,v)\in X_\lambda(f,c_t g)} f(u)\omega_u(tg(v)) = \sum_{i=1}^{n} f(u_i)\omega_{u_i}(tg(v_i)). \tag{2.1}$$

Note that $u_1v_i \prec u_iv_i = \lambda$ for any $i = 2$, hence by induction hypothesis, $f(u_1)R\omega_{u_1}(g(v_i)) = 0$. Since ω_{u_1} is automorphism, there exists $y_1 \in R$ such that $\omega_{u_1}(y_1) = f(u_1)$. Then $\omega_{u_1}(y_1Rg(v_i)) = f(u_1)R\omega_{u_1}(g(v_i)) = 0$. Consequently $y_1Rg(v_i) = 0$, for $i = 2$. Hence $g(v_i) \in r_R(y_1R)$, for $i = 2$. By hypothesis, $r_R(y_1R)$ is left s-unital, and hence by Lemma 3.3, there exists $e_{u_1} \in r_R(y_1R)$ such that $g(v_i) = e_{u_1}g(v_i)$, for $i = 2$. Let $t' \in R$ be an arbitrary element. Since $y_1Re_{u_1} = 0, f(u_1)\omega_{u_1}(Re_{u_1}g(v_1)) = 0$. Hence $f(u_1)\omega_{u_1}(t'e_{u_1}g(v_1)) = 0$. Take $t = t'e_{u_1}$ in (2.1). Hence

$$\sum_{i=2}^{n} f(u_i)\omega_{u_i}(tg(v_i)) = 0. \tag{2.2}$$

Now, (2.1) and (2.2), imply that $f(u_1)\omega_{u_1}(Rg(v_1)) = 0$. Since and ω_{u_1} is automorphism, we have, $f(u_1)R\omega_{u_1}(g(v_1)) = 0$.

Next, note that $u_2v_i \prec u_iv_i = w$ for any $i = 3$, so by induction hypothesis, $f(u_2)R\omega_{u_2}((g(v_i))) = 0$. Since ω_{u_2} is an automorphism, there exists $y_2 \in R$ such that $\omega_{u_2}(y_2) = f(u_2)$. So $y_2Rg(v_i) = 0$, for $i = 3$. Hence $g(v_i) \in r_R(y_2R)$, for $i = 3$. By hypothesis, $r_R(y_2R)$ is left s-unital, and hence by using again Lemma 3.3, there exists $e_{u_2} \in r_R(y_2R)$ such that $g(v_i) = e_{u_2}g(v_i)$, for $i = 3$. Let $t' \in R$ be an arbitrary element. Since $y_2Re_{u_2} = 0, f(u_2)\omega_{u_2}(Re_{u_2}g(v_2)) = 0$. Hence $f(u_2)\omega_{u_2}(t'e_{u_2}g(v_2)) = 0$. Take $t = t'e_{u_2}$ in (2.2). We get

$$\sum_{i=3}^{n} f(u_i)\omega_{u_i}(tg(v_i)) = 0. \tag{2.3}$$

Now, (2.2) and (2.3), imply that $f(u_2)\omega_{u_2}(Rg(v_2)) = 0$. Since ω_{u_2} is an automorphism, we have $f(u_2)R\omega_{u_2}(g(v_2)) = 0$. Continuing this process, we can deduce $f(u_{n-1})\omega_{u_{n-1}}(Rg(v_{n-1})) = 0, \dots, f(u_2)\omega_{u_2}(Rg(v_2)) = 0, f(u_1)\omega_{u_1}(Rg(v_1)) = 0$. Thus $f(u)R\omega_u(g(v)) = 0$. Therefore, by transfinite induction, for any $u \in supp(f)$ and any $v \in supp(g)$ with $uv = w$, and we have $g(v)Rf(u) = 0$. Thus $g[[R^{S,\leq}, \omega]]f = 0$, and the proof is complete. $\square$

Corollary 2.1. *Let R be an S-compatible ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If I is a finitely generated left ideal of R then for all $a \in l_R(I), a \in al_R(I)$. So R is (S,ω)-reflexive.*

Proof. By Theorem 2.2 and [29, Proposition 2.6]. □

Corollary 2.2. *Let R be an S-compatible ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is a Baer ring. Then R is (S,ω)-reversible if and only if R is (S,ω)-reflexive.*

It is obvious that commutative rings are symmetric and symmetric rings are reversible, but the converses do not hold by [30, Examples I.5 and II.5] and [31, Examples 5 and 7]. Every reversible ring is semicommutative, but the converse need not hold by [7, Lemma 1.4 and Example 1.5]. On the other hand, it can be easily checked that reversible rings are reflexive, and hence there exists a reflexive and semicommutative ring which is not symmetric by [31, Examples 5 and 7]. However, we have the following which is a direct consequence of routine computations.

Proposition 2.2. *Let R be a reduced ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Then R is semicommutative and (S,ω)-reversible if and only if R is (S,ω)-reflexive.*

A ring R is called semiprime if for any $a \in R$, $aRa = 0$, implies $a = 0$. Let R be a ring and $(S,\leq)$ a strictly totally ordered monoid. A ring R is called (S,ω)-semiprime if $f[[R^{S,\leq},\omega]]f = 0$, then $f = 0$ for each $f \in [[R^{S,\leq},\omega]]$.

The following result appeared in [32]

Lemma 2.4. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Then R is a semiprime ring if and only if $[[R^{S,\leq},\omega]]$ is a semiprime ring.*

One can find the next definition in [23].

Definition 2.4. *Let $(S,\leq)$ be an ordered monoid. We say that $(S,\leq)$ is an artinian narrow unique product monoid (or an $a.n.u.p.$ monoid, or simply $a.n.u.p.$) if for every two artinian and narrow subsets X and Y of S there exists a $u.p.$ element in the product XY. We say that $(S,\leq)$ is quasitotally ordered (and that $\leq$ is a quasitotal order on S) if $\leq$ can be refined to an order $\preceq$ with respect to which S is a strictly totally ordered monoid.*

For any ordered monoid $(S, \leq)$, the following chain of implications holds:

S is commutative, torsion-free, and cancellative

$\Downarrow$

$(S, \leq)$ is quasitotally ordered

$\Downarrow$

$(S, \leq)$ is $a.n.u.p.$

$\Downarrow$

S is $u.p.$

The converse of the bottom implication holds if $\leq$ is the trivial order. For more details, examples, and interrelationships between these and other conditions on ordered monoids, we refer the reader to [33].

Theorem 2.3. *Let R be a semiprime ring with the ACC on left annihilator ideals, $(S, \leq)$ an $a.n.u.p.$-monoid and $\omega : S \to$Aut(R) a monoid homomorphism. If R is S-weakly rigid, then R is (S, ω)-reflexive.*

Proof. Assume on the contrary that there exist f and g in $R[[S, \omega]]$ such that $fR[[S, \omega]]g = 0$ and $f(u)R\omega_u(g(v)) \neq 0$ for some $u, v \in S$. Since R is semiprime, the intersection of all minimal prime ideals of R is equal to (0). Hence there exists a minimal prime ideal P^* of R such that $f(u)R\omega_u(g(v)) \nsubseteq P^*$. Thus the sets

$$X = \{u \in S | f(u) \notin P^*\} \textit{ and } Y = \{v \in S | (\exists u \in S)\omega_u(g(v)) \notin P^*\}$$

are non-empty. Since $X \subseteq supp(f)$ and $Y \subseteq supp(g), X$ and Y are artinian and narrow subsets of S, and since S is an $a.n.u.p.$-monoid, there exists $(a, b) \in X \times Y$ such that ab is a $u.p.$-element of XY. Let r be an arbitrary element of R. Since $fc_rg = 0,$ we obtain

$$0 = (fc_rg)(ab) = f(a)\omega_a((rg(b)) + \sum_{(u,v) \in X_{ab}(f, c_rg) \setminus \{(a,b)\}} f(u)\omega_u(rg(v)). \tag{2.4}$$

Observe that if $(u, v) \in X_{ab}(f, c_rg) \setminus (a, b)$, then since ab is a $u.p.$-element of XY, we have $u \notin X$ or $v \notin Y$, and thus $f(u)\omega_u(Rg(v)) \subseteq P^*$. So (2.4) implies that $f(a)\omega_a(Rg(b)) \subseteq P^*$. Because ω_a is surjective, we have $f(a)R\omega_a(g(b)) \subseteq P^*$. Since $a \in X$, it follows that $\omega_a(g(b)) \in P^*$. On the other hand, since R is semiprime with the ACC on left annihilator ideals, $l_R(R\omega_a(g(b))R) = \cap_{i=1}^{n} P_i$ such that P_i is minimal prime which $R\omega_a(g(b))R \nsubseteq P_i^*$, for each $1 \leq i \leq n$, by [34, Lemma 11.40, Theorem 11.43]. Now, if $l_R(R\omega_a(g(b))R) \subseteq P^*$ which contradicts P^* being a minimal prime. Therefore $l_R(R\omega_a(g(b))R) \nsubseteq P^*$. Hence $yR\omega_a(g(b)) = 0$ for some $y \in R\backslash P^*$. Because R is S-weakly rigid, we have $yR\omega_u(g(b)) = 0$ for every $u \in S$, thus $b \notin Y$. This final contradiction, we have $g(v)Rf(u) = 0$. Thus R is (S, ω)-reflexive. □

Corollary 2.3. *Let R be a left PP ring or a right $p.q.$-Baer ring, $(S, \leq)$ a quasitotally ordered monoid and $\omega : S \to Aut(R)$ a monoid homomorphism. Then R is (S, ω)-reflexive.*

3. On nil skew generalized power series reflexive rings

In this section, we first give the following concept, so called nil skew generalized power series reflexive, that is a generalization of skew generalized power series reflexive and study the relations between nil skew generalized power series reflexive and some certain classes of rings.

Definition 3.1. *Let $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. A ring R is called nil skew generalized power series reflexive ((S, ω)-nil-reflexive) if whenever $f, g \in [[R^{S,\leq}, \omega]]$ satisfy $fhg \in [[nil(R)^{S,\leq}, \omega]]$ implies $ghf \in [[nil(R)^{S,\leq}, \omega]]$ for each $h \in [[R^{S,\leq}, \omega]]$.*

Let $S = (\mathbb{N} \cup \{0\}, +)$ and $\leq$ is the usual order. Then, $[[R^{S,\leq}, \omega]] \cong R[[x]]$. Let ω be the trivial order. Then the ring R is (S, ω)-nil-reflexive, if and only if, R is nil power series reflexive.

In [14] Ali, show that there are nil generalized power series reflexive over which matrix rings need not be generalized power series reflexive. In the next, we provide some examples for nil skew generalized power series reflexive rings. It is show that, nil skew generalized power series reflexive need not be skew generalized power series reflexive.

Lemma 3.1. *[24, Lemma 2.5] Let $\omega : S \to$End(R) a monoid homomorphism. For each $a, b \in R$, each $s \in S$, the followings holds:*

(i) $ab \in nil(R) \Leftrightarrow a\omega_s(b) \in nil(R)$.

(ii) $ab \in nil(R) \Leftrightarrow \omega_s(a)b \in nil(R)$.

Example 1. *Let $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ monoid homomorphism. Assume that R is S-compatible. Then*

(i) *If R is a reduced ring with $nil(R)$ an ideal of R. Then R is (S,ω)-nil-reflexive.*
(ii) *For any reduced ring R, the ring $T_n(R)$ is (S,ω)-nil-reflexive. However, the ring of all 2×2 matrices over any field and satisfying the condition that $0 \leq s$ for every $s \in S$ is not (S,ω)-nil-reflexive.*
(iii) *For R be a reduced ring. Consider the ring*

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R; 1 \leq i,j \leq n \right\}.$$

Then $S_n(R)$ is not (S,ω)-reflexive, when $n \geq 4$, but $S_n(R)$ and R are (S,ω)-nil-reflexive for all $n \geq 1$.

Proof. (i) Assume that $f, g \in [[R^{S,\leq}, \omega]]$, with fhg is nilpotent for all $h \in [[R^{S,\leq}, \omega]]$. So there exists a positive integer n such that $(fhg)^n = 0$. By compatibility, therefore $(f(u)h(w)g(v))^n = 0$, for any $u, v, w \in S$. Then $f(u)h(w)g(v) \in nil(R)$ and so $g(v)h(w)f(u)g(v)h(w)f(u) \in nil(R)$. Hence $g(v)h(w)f(u)$ is nilpotent. Thus, ghf is nilpotent.

(ii) For a ring R, by [37], $nil(T_n(R)) = \begin{pmatrix} nil(R) & R & R & \cdots & R \\ 0 & nil(R) & R & \cdots & R \\ 0 & 0 & nil(R) & \cdots & R \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & nil(R) \end{pmatrix}$.
Let R be a reduced ring. Then $nil(R) = 0$ and so $nil(T_n(R))$ is an ideal. By (i), $T_n(R)$ is (S,ω)-nil-reflexive. For $(S,\leq)$ a strictly ordered monoid. Let the element $s \neq 1$. We show that the ring $M_2(R)$ of 2×2 matrices over R is not $(S,\overline{\omega})$-nil-reflexive, where $\overline{\omega} : S \to$End$(M_2(R))$ is a monoid homomorphism given by $\overline{\omega}_s((a_{ij})) = (\omega_s(a_{ij}))$ for all $s \in S$. Let $f = c_{E_{12}} + c_{E_{11}}e_s$ and $g = c_{E_{11}+E_{12}} - (c_{E_{21}+E_{22}})e_s$ be elements of $M_2(R)[[S,\overline{\omega}]]$. Then $fg = 0 \in nil(M_2(R))[[S,\overline{\omega}]]$. But

$f(s)\overline{\omega}_s((g(1)) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is not nilpotent. Thus $M_2(R)$ is not (S,ω)-nil-reflexive.

(iii) By the same argument as in [14, Example 3.2] that $S_n(R)$ is not (S,ω)-reflexive when $n \geq 4$. Since R is reduced, R is (S,ω)-nil-reflexive. Note that

$$nil(S_n(R)) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a \in nil(R), a_{ij} \in R; 1 \leq i,j \leq n \right\}.$$

The ring R being reduced implies that $nil(S_n(R))$ is an ideal. By (i), $S_n(R)$ is (S,ω)-nil-reflexive. □

By Example 1(ii), for n by n upper triangularmatrix ring over R. It is easy to verify the next proposition.

Proposition 3.1. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is S-compatible. Then R is (S,ω)-nil-reflexive if and only if $T_n(R)$ is $(S,\overline{\omega})$-nil-reflexive, for any positive integer n.*

Proof. Suppose that $T_n(R)$ is $(S,\overline{\omega})$-nil-reflexive. Note that R is isomorphic to the subring of $T_n(R)$. Thus R is (S,ω)-nil-reflexive, since each subring of (S,ω)-nil-reflexive ring is also (S,ω)-nil-reflexive. For the forward implication, first consider the map $\phi : T_n(R)[[S,\overline{\omega}]] \to T_n([[R[[S,\omega]])$, given by $\phi(f) = f_{ij}$ where $f_{ij}(s) = (f(s))_{ij}$ for all $s \in S$ and the $(f(s))_{ij}$ is the (i,j)th entry of $f(s)$. It is easy to show that ϕ is an isomorphism. Now, assume that $f,g \in T_n(R)[[S,\overline{\omega}]]$ such that $fhg \in nil(T_n(R)[[S,\overline{\omega}]])$. Since $nil(T_n(R)) = \{(a_{ij})|a_{ij} \in nil(R)\}$, by the above isomorphism we have $f_{ii}h_{ii}g_{ii} \in nil(R)$ for each $1 \leq i \leq n$. Since R is (S,ω)-nil-reflexive, there exists some positive integer $m_{u,w,v,i}$ such that $(f_{ii}(u)\omega_u(h_{ii}(w)g_{ii}(v)))^{m_{u,w,v,i}} = 0$ for any i and any $u,w,v \in S$. Let $m_{u,w,v} = max\{m_{u,w,v,i}|1 \leq i \leq n\}$. Then $(f_{ii}(u)\overline{\omega}_u(h_{ii}(w)g_{ii}(v)))^m = 0$, so $g_{ii}(v)\overline{\omega}_v(h_{ii}(w)f_{ii}(u))$ is nilpotent. Therefore, $T_n(R)$ is $(S,\overline{\omega})$-nil-reflexive. □

Lemma 3.2. *[14] Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. The following conditions are equivalent for a ring $R, u \in S$.*

(i) $f(u)R \subseteq nil(R)$ *for any* $f(u) \in nil(R)$.

(ii) $Rf(u) \subseteq nil(R)$ *for any* $f(u) \in nil(R)$.

The next result gives a source of nil skew generalized power series reflexive.

Proposition 3.2. *Let S be a torsion-free and cancellative monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is S-compatible, that $f(u)R \subseteq nil(R)$ for any $f(u) \in nil(R)$, for each $u \in S$. Then R is (S,ω)-nil-reflexive.*

Proof. Assume that $f, g \in [[R^{S,\leq}, \omega]]$ with $fhg \in [[nil(R)^{S,\leq}, \omega]]$ for any $h \in [[R^{S,\leq}, \omega]]$. So there exists a positive integer n such that $(fhg)^n = 0$. Therefore, $(f(u)\omega_u(h(w)g(v)))^n = 0$, for any $u, v, w \in S$. So $f(u)\omega_u(h(w)g(v)) \in nil(R)$, by hypothesis and compatibility, $f(u)g(v)R \subseteq nil(R)$. Hence $g(v)h(w)f(u) \in nil(R)$ for any $h(w) \in R$. Thus, $ghf \in [[nil(R)^{S,\leq}, \omega]]$. □

By [41, Lemma 3.1], in a semicommutative ring $R, nil(R)$ is an ideal of R. In [3, Example 2.1] shows that any semicommutative ring need not be reflexive, but in [35] show that every nil-semicommutative is nil-reflexive. Since any (S,ω)-Armendariz is (S,ω)-quasi-Armendariz. Here we have.

Proposition 3.3. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Assume that R is (S,ω)-Armendariz and S-compatible. Then R is (S,ω)-nil-reflexive.*

Proof. Let $f, g \in [[R^{S,\leq}, \omega]]$ be such that $(fhg) \in [[nil(R)^{S,\leq}, \omega]]$ for some positive integer $n, (fhg)^n = 0$. Then $fc_rg = 0$ and hence $f(u)\omega_u(rg(v)) \in nil(R)$, for all $r \in R$ and all $u, v \in S$. Thus, $f(u)R\omega_u(g(v)) \in nil(R)$. Since R is (S,ω)-Armendariz, them R is abelian, for each $u, w, v \in S$ we have $g(v)h(w)f(u) \in nil(R)$. □

Corollary 3.1. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Assume that R is (S,ω)-Armendariz and ω_s is compatible for some $s \in S$. Then $[[R^{S,\leq}, \omega]]$ is nil-semicommutative.*

Proposition 3.4. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Assume that R is S-compatible and semicommutative ring, then R is (S,ω)-nil-Armendariz if and only if R is (S,ω)-nil-reflexive.*

Proof. The proof is clear. □

Corollary 3.2. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Assume that R is S-compatible and (S, ω)-reversible ring, then R is (S, ω)-nil-reflexive.*

Let I be an index set and R_i be a ring for each $i \in I$. Let $(S, \leq)$ be a strictly ordered monoid, if there is an injective homomorphism $f : R \to \prod_{i \in I} R_i$ such that, for each $j \in I, \pi_j f : R \to R_j$ is a surjective homomorphism, where $\pi_j : \prod_{i \in I} R_i \to R_j$ is the jth projection.

Proposition 3.5. *Let $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Assume that R is S-compatible. If R is finite subdirect product of (S, ω)-nil-reflexive rings, then R is (S, ω)-nil-reflexive ring.*

Proof. Let $I_k (k = 1, \dots, l)$ be ideals of R such that R/I_k is $(S, \overline{\omega})$-nil-reflexive ring and $\bigcap_{k=1}^{l} I_k = 0$. Let f and g be in $[[R^{S,\leq}, \omega]]$ with $fhg \in [[nil(R)^{S,\leq}, \omega]]$, for all $h \in [[R^{S,\leq}, \omega]]$. Clearly $\bar{f}\bar{h}\bar{g} \in [[nil(R/I_k)^{S,\leq}, \overline{\omega}]]$. Since R/I_k is $(S, \overline{\omega})$-nil-reflexive, by compatibility, we have $(f(u)h(w)g(v))^{r_{u,w,v,k}} \in I_k$, for each $u, w, v \in S$ and $k = 1, \dots, l$. Assume that $r_{u,w,v} = max\{r_{u,w,v,k} | k = 1, \dots, l\}$. So $(f(u)h(w)g(v))^{r_{u,w,v}} \in \bigcap_{k=1}^{l} I_k = 0$. Hence $f(u)h(w)g(v) \in nil(R)$, for each $u, w, v \in S$, then $g(v)h(w)f(u) \in nil(R)$. Thus, $ghf \in [[nil(R)^{S,\leq}, \omega]]$, and we are done. □

Lemma 3.3. *[24, Definition 2.24] Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to$End(R) a monoid homomorphism. We say that a ring R is completely S-compatible if, for any ideal I of R, R/I is S-compatible, to indicate the homomorphism ω, we will sometimes say that R is completely (S, ω)-compatible.*

Proposition 3.6. *Let $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is (S, ω)-nil-reflexive, completely S-compatible and $r_R(I)$ is S-invariant for an ideal I of R, then $R/r_R(I)$ is $(S, \overline{\omega})$-nil-reflexive.*

Proof. Let $A = r_R(I)$. Suppose that $\overline{f}, \overline{g} \in (R/A)[[S, \overline{\omega}]]$ such that $\overline{fhg} \in [[nil(R/A)^{S,\leq}, \overline{\omega}]]$, for any $\overline{h} \in [[(R/A)^{S,\leq}, \overline{\omega}]]$. Hence, we have $(fhg)(s) \in A$ and $s \in S$. Then $yfhg(s) = 0$ for all $y \in I$ and $s \in S$. It follows that $(c_y\overline{f})R[[S, \overline{\omega}]]\overline{g} = 0$ for all $y \in I$. Since R is (S, ω)-nil-reflexive, there exist some positive integer n, such that $(yf(s)R\omega_s(g(v)))^n = 0$, by Lemma 3.2 and Lemma 3.3, we obtain $yg(v)R\omega_v(f(s)) \in nil(R)$ for each $s, v \in S$ and $y \in I$. Thus $g(v)R\omega_v(f(s)) \in A$. Then $\overline{g}(v)(R/A)\overline{\omega}_v(\overline{f}(s)) = 0$ for all $s, v \in S$, and therefore R/A is $(S, \overline{\omega})$-nil-reflexive. □

By combining Theorem 2.2 and Proposition 3.6, we obtain the following corollary.

Corollary 3.3. *Let R be a right APP-ring, $(S, \leq)$ a quasitotally ordered monoid and $\omega : S \to Aut(R)$ a monoid homomorphism. Then R/A is $(S, \overline{\omega})$-nil-reflexive and R is completely S-compatible ring, where A is an S-invariant right annihilator of a principal right ideal in R.*

Let I be an index set and R_i a ring for each $i \in I$. Let $(S, \leq)$ be a strictly ordered monoid and $\omega_i : S \to End(R_i)$ a monoid homomorphism. Then the mapping $\omega : S \to End(\prod_{i\in I} R_i)$ is a monoid homomorphism given by $\omega_s(\{r_i\}_{i\in I}) = \{(\omega_i)_s(r_i)\}_i \in I$ for all $s \in S$.

Proposition 3.7. *Let I be an index set and R_i a reducsd ring for each $i \in I$. Assume that $(S, \leq)$ is a strictly ordered monoid and $\omega_i : S \to End(R_i)$ a monoid homomorphism, for each $i \in I$. If each R_i is (S, ω_i)-nil-reflexive, then $R = \prod_{i\in I} R_i$ is (S, ω)-nil-reflexive ring.*

Proof. Let $f, g \in [[R^{S,\leq}, \omega]]$ with $fhg \in [[nil(R)^{S,\leq}, \omega]]$ for any $h \in [[R^{S,\leq}, \omega]]$. By a similar argument as in [14, Proposition 2.18], we can see that there exists an isomorphism of rings $\varphi : [[R^{S,\leq}, \omega]] \to [[\prod_{i\in I}(R)_i^{S,\leq}, \omega_i]]$ defined by $\varphi(f) = (f_i)_{i\in I}$ where $f_i = \pi_i o f$. Thus, in $[[R_i^{S,\leq}, \omega_i]]$ we have $f_i[[R_i^{S,\leq}, \omega_i]]g_i = 0$ for all $i \in I$. Since each R_i is (S, ω_i)-nil-reflexive, there exist some positive integer m, such that $(f_i h_i g_i)^m = 0$, for any $h_i \in [[R_i^{S,\leq}, \omega_i]]$, and we have $(f_i(u)R_i(\omega_{iu}(h_i(w)(g_i(v)))))^{m_{u,w,v}}$ $\in nil(R_i)$ for all $u, w, v \in S$, $m_{u,w,v} = max\{m_{u,w,v,i}|1 \leq i \leq n\}$. and for all $i \in I$. by reduced ring, it follows that $g(v)R\omega_v(f(u)) =\in nil(R)$ for all $u, v \in S$. Thus, $ghf \in [[nil(R)^{S,\leq}, \omega]]$ and hence R is (S, ω)-nil-reflexive. □

Proposition 3.8. *Let R be an abelian ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If the endomorphism ω_s is idempotentstabilizing, for every $s \in S$, then the following statements are equivalent:*

(i) *R is (S, ω)-nil-reflexive;*
(ii) *eR and $(1 - e)R$ are (S, ω)-nil-reflexive, for each idempotent $e \in R$;*
(iii) *Re and $R(1 - e)$ are (S, ω)-nil-reflexive, for each idempotent $e \in R$.*

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear.

$(iii) \Rightarrow (i)$ Let $f, g \in R[[S,\omega]]$ with $fhg \in nil(R)[[S,\omega]]$, for all $h \in R[[S,\omega]]$ Let e be an idempotent of R. It is easy to see that c_e is an idempotent element of $R[[S,\omega]]$ and $c_e g = gc_e$ for every $g \in R[[S,\omega]]$. Then $(c_e f)(c_e h)(c_e g) \in nil(eR)[[S,\omega]]$ and $((1-c_e)f)((1-c_e)h)((1-c_e)g) \in nil((1-e)R)[[S,\omega]]$ (as c_e is central). Since $eR, (1-e)R$ are (S,ω)-nil-reflexive, we have $ef(u)\omega_u(eh(w)(eg(v))) \in nil(R)$ and $(1-e)f(u)\omega_u((1-e)h(w)((1-e)g(v))) \in nil(R)$ for all $u, w, v \in S$. On the other hand, since ω_s is idempotent-stabilizing, one can see that $ef(u)\omega_u(eh(w)eg(v))) = ef(u)\omega_u(h(v)(g(v))) = e(f(u)h(v)g(v))$. Similarly, we have $(1-e)f(u)\omega_u((1-e)h(w)((1-e)g(v))) = (1-e)f(u)\omega_u(h(w)(g(v))) = (1-e)(f(u)h(w)g(v))$. Hence $ef(u)h(w)g(v)$ and $(1-e)f(u)h(w)(g(v)) \in nil(R)$ for all $u, w, v \in S$. It follows that $f(u)h(w)(g(v)) \in nil(R)$, so $g(v)h(w)f(u) \in nil(R)$. Thus $ghf \in nil(R)[[S,\omega]]$. $\square$

In the next, we investigate the relations between a ring R and R/I for some ideal I of R in terms of nil skew generalized power series reflexivity. By Theorem 2.1, symmetric ring is nil skew generalized power series reflexive.

Theorem 3.1. *Let $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Suppose that R is completely S-compatible. If I be an ideal of R contained in $nil(R)$. Then R is (S,ω)-nil-reflexive if and only if R/I is $(S,\overline{\omega})$-nil-reflexive.*

Proof. Since I is nil, we have $nil(R/I) = nil(R)/I$. Hence, by completely compatibility, $fhg \in [[nil(R)^{S,\leq}, \omega]]$ if and only if $\overline{fh\overline{g}} \in [[nil(R/I)^{S,\leq}, \overline{\omega}]]$. Also, $acb \in nil(R)$ if and only if $\overline{ac}\overline{b} \in nil(R/I)$. Therefore, R is (S,ω)-nil-reflexive if and only if R/I is $(S,\overline{\omega})$-nil-reflexive, as desired. $\square$

Now we give some characterizations of nil generalized power series reflexivity by using the prime radical of a ring.

Corollary 3.4. *Let $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Suppose that R is completely S-compatible. A ring R is (S,ω)-nil-reflexive if and only if $R/P(R)$ is $(S,\overline{\omega})$-nil-reflexive.*

Proof. Since every element of $P(R)$ is nilpotent, it follows from Theorem 3.1. $\square$

We denote the the unique maximal nil ideal and the set of all nilpotent elements of R by $N^*(R)$ and $N(R)$ respectively. Recall that $K\ddot{o}the's$ conjecture means that the sum of two nil left ideals is nil. In [38], a ring R is called NI if $N^*(R) = N(R)$.

IFP rings are NI and NI rings are nil-IFP, are nil-reflexive, but not conversely each case. $K\ddot{o}the's$ conjecture holds clearly in NI rings.

Proposition 3.9. *Let $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R be a ring in which $K\ddot{o}the's$ conjecture holds, and assume that aR is nil for all $a \in N(R)$. Then R is (S,ω)-nil-reflexive.*

Proof. It directly comes from [39, Lemma 1.3]. □

Proposition 3.10. *Let $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Assume that R be a nil-IFP and right Noetherian ring. Then R is (S,ω)-nil-reflexive.*

Proof. It is well-known that $K\ddot{o}the's$ conjecture holds in right Noetherian rings. Thus the result follows from Proposition 3.9. □

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