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A NOTE ON REFLEXIVE RINGS

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ABSTRACT. Mason introduced the reflexive property for ideals and then this concept was generalized by Kim and Baik, defining idempotent reflexive right ideals and rings. In this note we consider reflexive property of a special subring of the infinite upper triangular matrix ring over a ring R. We proved that, if R is a left APP-ring, then $V_n(R)$ is reflexive. We also give an example which shows that the ring $V_n(R)$ need not be left APP when R is a left APP-ring.

All rings considered here are associative with identity. Mason introduced the reflexive property for ideals, and this concept was generalized by some authors, defining idempotent reflexive right ideals and rings, completely reflexive rings, weakly reflexive rings (see namely, [1–4]). The reflexive right ideal concept is also specialized to the zero ideal of a ring, namely, a ring R is called reflexive [2] if its zero ideal is reflexive and a ring R is called completely reflexive if for any $a, b \in R, ab = 0$ implies $ba = 0$. Completely reflexive rings are called reversible by Cohn in [5] and also studied in [6]. It is clear that every reduced ring ($i.e.$ rings without nonzero nilpotent elements) are completely reflexive and every completely reflexive ring is semicommutative. The notion of Armendariz ring is introduced by Rege and Chhawchharia (see [7]). They defined a ring R

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to be Armendariz if $f(x)g(x) = 0$ implies $a_ib_j = 0$, for all polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m, g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n \in R[x]$.

In [8] A ring R is called strongly reflexive whenever $f(x), g(x) \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, then $g(x)R[x]f(x) = 0$. Clearly, every strongly reflexive ring is reflexive, but the converse is not true (see [8, Example 2.1]). Obviously, subrings and direct products of a strongly reflexive ring are strongly reflexive. The concept of quasi-Armendariz rings is another generalization of Armendariz rings. According to [9], a ring R is called a quasi-Armendariz if whenever polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m, g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, then $a_iRb_j = 0$ for each i, j. It was proved in [6, Proposition 2.4] that if R is an Armendariz ring, then R is completely reflexive if and only if $R[x]$ is completely reflexive. According to [8], if R is quasi-Armendariz, then R is a reflexive ring if and only if $R[x]$ is strongly reflexive ring.

Let R be a ring. It was shown in [4] that R is a reflexive ring if and only if $M_n(R)$ is a reflexive for all $n \geq 1$. Here we consider the following ring:

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & a_2 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_2 \\ 0 & 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_1, a_2, a_3, \ldots, a_n \in R \right\}.$$

The aim of this note, we will show that if R is a left APP-ring, then $V_n(R)$ is reflexive. We also give an example which shows that the ring $V_n(R)$ need not be left APP when R is a left APP-ring.

An ideal I of R is said to be right s-unital if, for each $a \in I$ there exists an element $x \in I$ such that $ax = a$. It follows from Tominaga ([10, Theorem 1]) that I is right s-unital if and only if for any finitely many elements $a_1, a_2, \ldots, a_n \in I$, there exists an element $x \in I$ such that $a_i = xa_i (resp.\ a_i = a_ix)$ for each $i = 1, 2, \ldots, n$. According to [11] a ring R is called a left APP-ring if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R$. Right APP-rings can be defined analogously. Recall a ring R is a left $p.q.$-Baer ring if the left annihilator of a principal left ideal of R is generated by an idempotent (see, for example, [12–14]). Clearly every left $p.q.$-Baer ring is a left APP-ring (thus the class of left APP-rings includes all biregular rings and all quasi-Baer rings). A ring R is a right PP-ring if the right annihilator of an element of R is generated by an idempotent. Right PP rings are left APP.

The following results follows from [9, 15], respectively.

Proposition 1. *Every left APP-ring is quasi-Armendariz, but not conversely.*

Lemma 1. *Let R be a left APP-ring and $a_1, \ldots, a_n, b_1, \ldots, b_m$ belong to R. If $a_iRb_j = 0$ for all i and j, then there exists $e \in R$ such that $a_i = a_ie$ and $eRb_j = 0$ for all i and j.*

Theorem 1. *Let R be a reduced ring. If R is a left APP-ring, then $V_n(R)$ is reflexive.*

Proof. Suppose that R is left APP and $\sum_{i=1}^{\ell} A_ix^i, \sum_{j=1}^{m} B_jx^j \in V_n(R)[x]$ such that $(\sum_{i=1}^{\ell} A_ix^i)V_n(R)[x](\sum_{j=1}^{m} B_jx^j) = 0$. We will show that

$$(\sum_{j=1}^{m} B_jx^j)V_n(R)[x](\sum_{i=1}^{\ell} A_ix^i) = 0$$

for all i and j. Suppose that

$$A_i = \begin{pmatrix} a_1^i & a_2^i & a_3^i & a_4^i & \cdots \\ 0 & a_1^i & a_2^i & a_3^i & \cdots \\ 0 & 0 & a_1^i & a_2^i & \cdots \\ 0 & 0 & 0 & a_1^i & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad B_j = \begin{pmatrix} b_1^j & b_2^j & b_3^j & b_4^j & \cdots \\ 0 & b_1^j & b_2^j & b_3^j & \cdots \\ 0 & 0 & b_1^j & b_2^j & \cdots \\ 0 & 0 & 0 & b_1^j & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Set $f_p = \sum_{i=1}^{\ell} a_p^ix^i, g_p = \sum_{j=1}^{m} b_p^jx^j$ for any p with $1 \leq p$. Then from $(\sum_{i=1}^{\ell} A_ix^i)$ $V_n(R)[x](\sum_{j=1}^{m} B_jx^j) = 0$ it follows that for any $\lambda_p = \sum_{k=1}^{h} c_p^kx^k \in R[x]$ with $1 \leq p$.

$$\begin{pmatrix} f_1 & f_2 & f_3 & f_4 & \cdots \\ 0 & f_1 & f_2 & f_3 & \cdots \\ 0 & 0 & f_1 & f_2 & \cdots \\ 0 & 0 & 0 & f_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \cdots \\ 0 & \lambda_1 & \lambda_2 & \lambda_3 & \cdots \\ 0 & 0 & \lambda_1 & \lambda_2 & \cdots \\ 0 & 0 & 0 & \lambda_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} g_1 & g_2 & g_3 & g_4 & \cdots \\ 0 & g_1 & g_2 & g_3 & \cdots \\ 0 & 0 & g_1 & g_2 & \cdots \\ 0 & 0 & 0 & g_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0.$$

Note that $a_ic_kb_jx^{i+k+j} = 0$ for all i, j and k with $i + k + j = n$. Since $f\lambda g = 0$, we have the following equations:

$$a_1c_1b_1 = 0 \tag{1}$$

$$a_1c_1b_2 + a_1c_2b_1 + a_2c_1b_1 = 0 \tag{2}$$

$$a_1c_1b_3 + a_1c_2b_2 + a_1c_3b_1 + a_2c_1b_2 + a_2c_2b_1 + a_3c_1b_1 = 0 \tag{3}$$

$$\vdots$$

$$a_1c_1b_m + a_1c_2b_m + \cdots + a_1c_{m+1}b_1 + \cdots + a_mc_1b_2 + a_mc_2b_1 + a_{m+1}c_1b_1 = 0 \tag{4}$$

$$\vdots$$

$$a_1c_1b_{n-1} + a_1c_2b_{n-2} + \cdots + a_{n-2}c_2b_1 + a_{n-1}c_1b_1 = 0 \tag{5}$$

$$a_1c_1b_n + a_1c_2b_{n-1} + \cdots + a_{n-1}c_1b_2 + a_{n-1}c_2b_1 + a_nc_1b_1 = 0, \tag{6}$$

where $1 \leq m \leq n$. Note that R is reflexive and that $aRcRc = 0$ if and only if $aRc = 0$ for $a, c \in R$. We freely use these facts in the following computations. From Eq. (1), we have $a_1Rb_1 = 0$. Thus by Lemma 1, there exist $e \in R$ such that $a^i = a^ie$ and $eRb^j = 0$ for all i, j and so $f = fe$ and $eR[x]g = 0$. Hence $g_j \in r_R(dR[x])$ for $j = 2$, where $d \in R$ is an arbitrary element. By hypothesis, $r_R(dR[x])$ is s-unital and hence by Lemma 1, again there exist $e \in r_R(dR[x])$ such that $g_j = eg_j$, for $j = 2$. Since $dRe = 0$, $f_1R[x]eg_1 = 0$. Thus $f_1R[x]g_1 = 0$. Multiplying Eq. (2) by Rb_1 on the right side, we get $a_2Rb_1Rb_1 = 0$ and so $a_2Rb_1 = 0$. Then Eq. (2) implies $a_1c_1b_2 = 0$. Substitute et for c_1 in $a_1c_1b_2 = 0$ to yield $a_1(et)b_2 = 0, t \in R$ is an arbitrary element, then we have $a_1Rb_2 = 0$. Thus by Lemma 1 again, there exist $u \in R$ such that $a^i = a^iu$ and $uRb^j = 0$ for all i and j. Hence $f = fu$ and $uR[x]g = 0, uR[x]g_2 = 0$. Thus $f_1R[x]g_2 = 0$ and so $f_2R[x]g_1 = 0$.

Now Eq. (3) becomes

$$a_1c_1b_3 + a_2c_1b_2 + a_3c_1b_1 = 0.$$

Multiply this equality on the right side by Rb_1 and Rb_2 in turn, to obtain $a_3Rb_1 = 0, a_2Rb_2 = 0$ and $a_1Rb_3 = 0$. Thus by Lemma 1, there exist $h \in R$ such that $a^i = a^ih$ and $hRb^j = 0$ and so $f = fh, hR[x]g = 0$. Thus $f_3R[x]g_1 = 0$. By Lemma 1 again, there exist $w \in R$ such that $a^i = a^iw, wRb^j = 0, b^j \in r_R(wR)$ is s-unital and so $f = fw, wR[x]g = 0$. Thus $f_2R[x]g_2 = 0$ and $f_1R[x]g_3 = 0$. Summarizing, we have that

$$a_iRb_j = 0 \; for \; i + j = 2, 3, 4.$$

Inductively, we assume that $a_iRb_j = 0$ for $i + j = 2, 3, \ldots m$ with $m - 1 \leq n$. Then Eq. (4) becomes

$$a_1c_1b_{m-1} + a_2c_1b_m + a_2c_1b_{m-1} + \cdots + a_mc_1b_2 + a_{m-1}c_1b_1 = 0 \qquad (7).$$

Multiplying Eq. (7) on the right side by $Rb_1, Rb_2, \ldots,$ and Rb_m in turn, we obtain $a_{m-1}Rb_1 = 0, a_mRb_2 = 0, \ldots,$ and $a_2Rb_m = 0$, entailing $a_1Rb_{m-1} = 0$. These show that $a_iRb_j = 0$ for all i and j with $i + j = m - 1$. Consequently, $a_iRb_j = 0$ for all i and j with $1 \leq i + k \leq n$. Since R is reflexive, $b_jRa_i = 0$ for all i and k with $1 \leq i + k \leq n$. Hence there exists $r \in R$ be an arbitrary element such that $a^i = a^ir$ and $rRb^j = 0$ for all i and j. Hence $b^j \in r_R(rR)$. By hypothesis, $r_R(rR)$ is left s-unital and by Lemma 1, again which implies that $f_p = f_pr$ and $rR[x]g_p = 0$. Hence $g_p \in r_R(rR[x])$ for $p = 1, 2, \ldots$ is left s-unital. Thus by the induction hypothesis, $g_1R[x]f_1 = 0, g_1R[x]f_2 = 0, g_2R[x]f_1 = 0, \ldots, g_1R[x]f_n = 0, \ldots, g_nR[x]f_1 = 0$. This yields $g\lambda f = 0$, proving that $V_n(R)$ is reflexive. □

Proposition 2. *If $V_n(R)$ is reflexive then so is R.*

Proof. Suppose that $f = \Sigma a_ix^i, g = \Sigma b_jx^j$ are in $R[x]$ such that $fR[x]g = 0$. Then for any $\lambda \in R[x]$,

$$\begin{pmatrix} f & 0 & 0 & 0 & \cdots \\ 0 & f & 0 & 0 & \cdots \\ 0 & 0 & f & 0 & \cdots \\ 0 & 0 & 0 & f & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 & 0 & \cdots \\ 0 & \lambda & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} g & 0 & 0 & 0 & \cdots \\ 0 & g & 0 & 0 & \cdots \\ 0 & 0 & g & 0 & \cdots \\ 0 & 0 & 0 & g & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0.$$

Thus

$$\begin{pmatrix} b_j & 0 & 0 & 0 & \cdots \\ 0 & b_j & 0 & 0 & \cdots \\ 0 & 0 & b_j & 0 & \cdots \\ 0 & 0 & 0 & b_j & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} (V_n(R)) \begin{pmatrix} a_i & 0 & 0 & 0 & \cdots \\ 0 & a_i & 0 & 0 & \cdots \\ 0 & 0 & a_i & 0 & \cdots \\ 0 & 0 & 0 & a_i & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0.$$

for all i and j, which implies that $b_jRa_i = 0$ for all i, j. □

Corollary 1. *Let R be a ring. If R is quasi-Armendraiz, then $V_n(R)$ is reflexive.*

The following example shows that the left APP property of R does not imply the left APP property of $V_n(R)$.

Example 1. *Let F be a field and consider the ring $V_n(F)$. Let*

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

belong to $V_n(F)$. Then $V_n(F)B = \left\{ \begin{pmatrix} 0 & b & b & b & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mid b \in F \right\}$. Thus it is easy to see that

$$l_{V_n(F)}(V_n(F)B) = \left\{ \begin{pmatrix} 0 & x_2 & x_3 & x_4 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mid x_i \in F \right\}.$$

Now let

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in l_{V_n(F)}(V_n(F)B).$$

If $V_n(F)$ is left APP, then there exists $C \in l_{V_n(F)}(V_n(F)B)$ such that $A = AC$. But this contradicts with the fact

$$AC = A \begin{pmatrix} 0 & c_2 & c_3 & \cdots \\ 0 & 0 & c_3 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_3 & c_4 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus $V_n(F)$ is not left APP.

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