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COMPARISON BETWEEN THE KRYLOV SUBSPACE METHOD AND THE TRUNCATION METHOD FOR IDENTIFYING AN UNKNOWN SOURCE IN THE HEAT EQUATION

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ABSTRACT. In this paper, we are concerned with the problem of approximating a solution of an inverse parabolic problem. In order to overcome the instability of the original problem, we use the troncature spectral method to construct a stable approximate solution. To calculate the stabilized solution, we use a numerical procedure based on the Krylov subspace method. This algorithm provides us a practical and simple method to calculate numerically the stabilized solution. Some Numerical tests are presented to illustrate the accuracy and efficiency of this method.

1. FORMULATION OF THE PROBLEM

Throughout this paper H denotes a complex separable Hilbert space endowed with the inner product $\langle .,. \rangle$ and the norm $\|.\|$, $\mathcal{L}(H)$ stands for the Banach algebra of bounded linear operators on H.

Let $A : \mathcal{D}(A) \subset H \longrightarrow H$ be a positive, self-adjoint operator with compact resolvent, so that A has an orthonormal basis of eigenvectors $(\phi_n) \subset H$ with real

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eigenvalues $(\lambda_n) \subset \mathbb{R}_+$, i.e.,

$$A\phi_n = \lambda_n \phi_n, n \in \mathbb{N}^*, \quad \langle \phi_i, \phi_j \rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases},$$

$$0 < \nu \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \quad \lim_{n\to\infty} \lambda_n = \infty,$$

$$\forall h \in H,\ h = \sum_{n=1}^{\infty} h_n \phi_n, \quad h_n = \langle h, \phi_n \rangle.$$

In this paper, we consider the following inverse source problem of determining the unknown source term f and the temperature distribution $u(t)$ for $0 \leq t < T$, in the following parabolic problem

$$\begin{cases} u'(t) + Au(t) = f, & 0 < t < T, \\ u(0) = 0,\ u(T) = g, \end{cases} \tag{1.1}$$

where $0 < T < \infty$ and g is a given H-valued function.

The main difficult of these problems is that they are ill-posed (the solution, if it exists, does not depend continuously on the data). Thus, the numerical simulation is very difficult and some special regularization are required.

To our knowledge, the literatures devoted to this class of problems is very large, but the Krylov subspace method applied to theses problems is quite scarce, except the papers [3]. This work is an extension of this method for a class of inverse source problem of parabolic type. For the regularizing effect and some theoretical developments, we refer the reader to [1, 3–6].

2. Analysis of the problem

It is well known that the operator $(-A)$ generates a C_0- semigroup $S(t) = e^{tA}$. This can be used to express the solution u of the direct problem in a closed form.

Theorem 2.1. *For all $f \in H$, the direct problem* (1.1) *admits an unique solution $u \in C([0, T]; H)$ given by*

$$u(t) = R(t)f = \int_0^t e^{-(t-s)A} f ds = (I - e^{-tA})A^{-1}f = \sum_{k=1}^{\infty} \left(\frac{1 - e^{-t\lambda_k}}{\lambda_k} \right) < f, \tag{2.1}$$

$\phi_k > \phi_k$. *Using the final condition*

$$u(T) = g = \sum_{k=1}^{\infty} < g, \phi_k > \phi_k = \sum_{k=1}^{\infty} \left(\frac{1 - e^{-T\lambda_k}}{\lambda_k} \right) < f, \quad \phi_k > \phi_k. \tag{2.2}$$

we get

$$f = \sum_{k=1}^{\infty} \left(\frac{\lambda_k}{1 - e^{-T\lambda_k}} \right) < g, \quad \phi_k > \phi_k. \tag{2.3}$$

From this representation we see that f is unstable. This follows from the highfrequency

$$\sigma_k = \sum_{k=1}^{\infty} \left(\frac{\lambda_k}{1 - e^{-T\lambda_k}} \right) \longrightarrow \infty, \quad k \longrightarrow \infty.$$

$$\lambda_k = \sum_{k=1}^{\infty} \left(\frac{\lambda_k}{1 - e^{-T\lambda_k}} \right) \leq \sum_{k=1}^{\infty} \left(\frac{\lambda_k}{1 - e^{-T\nu}} \right) \tag{2.4}$$

Now by using the Picard condition and (2.4)*, we deduce the following result.*

$$\| f \|^2 = \sum_{k=1}^{\infty} \left(\frac{\lambda_k}{1 - e^{-T\lambda_k}} \right)^2 |< g, \phi_k >|^2 < +\infty \Leftrightarrow \sum_{k=1}^{\infty} \lambda_k^2 \, |< g, \phi_k >|^2 < +\infty$$

$$\Leftrightarrow g \in D(A).$$

Corollary 2.1. *The inverse problem* (2.2) *is uniquely solvable if, and only if,*

$$g \in D(A) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^2 \, |< g, \phi_k >|^2 < +\infty\}. \tag{2.5}$$

3. Regularisation by truncature methode and error estomates

It is well known that the ill posed problem is usually sensitive to the regularization parameter and the a priori bound is usually difficult to be obtained precisely in practice. In order to overcome the ill-posedness of problem (2.2), we modify the solution by filtering the high frequencies using a suitable method and instead consider (2.3) only for $k \leq N$.

Definition 3.1. *For* $N > 0$*, the regularized solution of problem* (2.3) *is given by*

$$f_N = \sum_{k=1}^{N} \left(\frac{\lambda_k}{1 - e^{-T\lambda_k}} \right) < g, \quad \phi_k > \phi_k. \tag{3.1}$$

Remark 3.1. *If the parameter N is large, f_N is close to the exact solution f. On the other hand, if the parameter N is fixed, f_N is bounded. So the positive integer N plays the role of regularization parameter. Since the data g are based on (physical) observations and are not known with complete accuracy, we assume that g and g_δ satisfy $\| gg_\delta \| \leq \delta$, where g_δ denotes the measured data and δ denotes the level noisy. Let f^δ denote by solution of problem* (2.2) *with measured data g_δ,*

$$f^\delta = \sum_{k=1}^{\infty} \left(\frac{\lambda_k}{1 - e^{-T\lambda_k}} \right) < g_\delta, \quad \phi_k > \phi_k. \tag{3.2}$$

Definition 3.2. *We denote by f_N^δ the regularized of problem* (2.2) *with measured data g_δ, i.e,*

$$f_N^\delta = \sum_{k=1}^{N} \left(\frac{\lambda_k}{1 - e^{-T\lambda_k}} \right) < g_\delta, \quad \phi_k > \phi_k. \tag{3.3}$$

As usual, in order to obtain convergence rate, we assume that there exists an a priori bound for problem (2.2). *We assume the following a priori bound on the unknown source f:*

$$f \in D(A^p) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2p} |< h, \phi_k >|^2 < +\infty\}, p > 0.$$

and

$$\| A^p f \|^2 = \{\sum_{k=1}^{\infty} \lambda_k^{2p} |< f, \phi_k >|^2 < +\infty\} \leq E^2.$$

Theorem 3.1. *If $f \in B(p, E) = \{h \in D(A^p) : \|A^p h\| \leq E\}$, $p > 0$, and if we choose $\lambda_{N+1} \approx \left(\frac{E}{\delta}\right)^{1/(1+p)}$, then we have the error bound*

$$\|f - f_N^\delta\| \leq C \delta^{\frac{p}{p+1}} E^{\frac{1}{p+1}}, \quad where \quad C = (1 + M) = 1 + \frac{1}{1 - e^{-\tau\lambda_1}}. \tag{3.4}$$

Proof. Putting

$$\omega_k = \frac{\lambda_k}{1 - e^{-\tau\lambda_k}} \leq \frac{\lambda_k}{1 - e^{-\tau\lambda_1}} = M\lambda_k,$$
$$g_k = \langle g, \phi_k \rangle, \quad g_k^\delta = \langle g_\delta, \phi_k \rangle,$$

from direct computations, we have,

$$\|f - f_N^\delta\| = \|f - f_N + f_N - f_N^\delta\| \leq \|f - f_N\| + \|f_N - f_N^\delta\| = \Delta_1 + \Delta_2 \tag{3.5}$$

$$\Delta_1^2 = \|f - f_N\|^2 = \left\| \sum_{k=1}^{\infty} f_k \xi_k - \sum_{k=1}^{N} f_k \xi_k \right\|^2 = \sum_{k=N+1}^{\infty} |f_k|^2 \tag{3.6}$$

$$\Delta_2^2 = \|f_N - f_N^\delta\|^2 = \left\| \sum_{k=1}^{N} \omega_k g_k \xi_k - \sum_{k=1}^{N} \omega_k g_k^\delta \xi_k \right\|^2 = \sum_{k=1}^{N+1} \omega_k^2 |g_k - g_k^\delta|^2 \tag{3.7}$$

$$\Delta_1^2 = \sum_{k=N+1}^{\infty} \lambda_k^{-2p} \lambda_k^{2p} |f_k|^2 \leq \lambda_{N+1}^{-2p} \sum_{k=N+1}^{\infty} \lambda_k^{2p} |f_k|^2 \leq \lambda_{N+1}^{-2p} E^2 \tag{3.8}$$

$$\Delta_2^2 = \sum_{k=1}^{N+1} \omega_k^2 |g_k - g_k^\delta|^2 \leq \lambda_{N+1}^2 M^2 \sum_{k=1}^{N+1} |g_k - g_k^\delta|^2 \leq \lambda_{N+1}^2 M^2 \delta^2, \tag{3.9}$$

we obtain

$$\Delta_1 + \Delta_2 \leq \lambda_{N+1}^{-p} E + \lambda_{N+1} M \delta \approx \left(\left(\frac{E}{\delta} \right)^{\frac{1}{1+p}} \right)^{-p} E + M\delta \left(\frac{E}{\delta} \right)^{\frac{1}{1+p}} = (1+M) E^{\frac{1}{1+p}} \delta^{\frac{p}{1+p}}.$$

□

4. KRYLOV SUBSPACE METHOD

Following the idea from [3], we construct a discrete approximate solution to our problem. For this, let $\mathbf{A} \in M_m(\mathbb{R})$ the discrete representation of A. From the properties of A, the matrix $\mathbf{A}$ is symmetric and positive definite. Let $\mathbf{f} \in \mathbb{R}^m$ (resp. $\mathbf{f}^\delta \in \mathbb{R}^m$) the discret representation of the continuous solution of (2.3) with the exact data (resp. with measured data) given by

$$\mathbf{f} = \mathbf{A}(I_m - e^{-T\mathbf{A}})^{-1}\mathbf{g} = \sum_{i=1}^{m} \left(\frac{\mu_i}{1 - e^{-T\mu_i}} \right) < \mathbf{g}, \quad \xi_i > \xi_i \tag{4.1}$$

$$\mathbf{f}^\delta = \mathbf{A}(I_m - e^{-T\mathbf{A}})^{-1}\mathbf{g}_\delta = \sum_{i=1}^{m} \left(\frac{\mu_i}{1 - e^{-T\mu_i}} \right) < \mathbf{g}_\delta, \quad \xi_i > \xi_i \tag{4.2}$$

where $\mathbf{g} \in \mathbb{R}^m$ (resp. $\mathbf{g}_\delta \in \mathbb{R}^m$) is the discrete representation of g (resp. g_δ) and (μ_i, ξ_i) are the eigenpairs of $\mathbf{A}$, i.e.,

$$\mathbf{A}\xi_i = \mu_i \xi_i, \qquad < \xi_i, \xi_i >= \delta_{ij},$$

$$\text{and}\quad 0 \le \mu_1 \le \mu_2 \le \mu_3 \le \cdots \le \mu_m.$$

The standard Krylov subspace method to calculate the vector (4.2) consists in generating the Krylov subspace

$$\mathcal{K}(\mathbf{A}, \mathbf{g}_\delta) = \text{span}\{\mathbf{g}_\delta, \mathbf{A}\mathbf{g}_\delta, \ldots, \mathbf{A}^{l-1}\mathbf{g}_\delta\}, \quad l \le m.$$

Let $(q_i)_{i=1}^l$ be an orthonormal basis of $\mathcal{K}(\mathbf{A}, \mathbf{g}_\delta)$, with $q_1 = \frac{\mathbf{g}_\delta}{\|\mathbf{g}_\delta\|}$. Letting $\mathbf{Q}_l = [q_1, q_2, \ldots, q_l]$ and $\mathbf{R}_l = \mathbf{Q}_l^T \mathbf{A}\mathbf{Q} \in \mathcal{M}_l(\mathbb{R})$ be the symmetric representation of A onto the space. An approximation of (4.2) in $\mathcal{K}(\mathbf{A}, \mathbf{g}_\delta)$ may be obtained by projection:

$$\mathbf{f}_l^\delta = \|\mathbf{g}_\delta\| \mathbf{Q}_l \mathbf{R}_l (I - \exp(-T\mathbf{R}_l))^{-1} e_1, \tag{4.3}$$

where e_1 is the first canonical vector of $\mathbb{R}^l$. Let $(r_j, \zeta_j)_{j=1}^l$ the eigenpairs of $\mathbf{R}_l$ (Ritz values and Ritz vector of $\mathbf{A}$), then the discret approximation (4.2) can be approximated as

$$\mathbf{f}_l^\delta = \|\mathbf{g}_\delta\| \mathbf{Q}_l \left\{ \sum_{j=1}^{l} \frac{r_j}{1 - e^{-Tr_j}} \left(\zeta_j^T e_1\right) \zeta_j \right\}. \tag{4.4}$$

5. Numerical results

In this section we give a two-dimensional numerical test to show the feasibility and efficiency of the proposed method. Numerical experiments where carried out using MATLAB.

We consider the following inverse problem

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}\right) u(x,t) = f(x), \quad x \in (0,\pi),\ t \in (0,1), \\ u(0,t) = u(\pi,t) = 0, \quad t \in (0,1), \\ u(x,1) = g(x), \quad x \in [0,\pi], \end{cases} \tag{5.1}$$

where $f(x)$ is the unknown source and $u(x,1) = g(x)$ is the final condition. It is easy to check that the operator

$$A = -\frac{\partial^2}{\partial x^2}, \quad \mathcal{D}(A) = H_0^1(0,\pi) \cap H^2(0,\pi) \subset H = L^2(0,\pi),$$

is positif, self-adjoint with compact resolvent (A is diagonalizable). The eigenpairs (λ_n, ϕ_n) of A are

$$\lambda_n = n^2,\ \phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), \quad n \in \mathbb{N}^*.$$

In this case, the formula (2.3) (resp. the truncadted solution (3.1)) takes the form

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{n^2}{1-e^{-n^2}}\left(\int_0^\pi g(s)\sin(sx)dx\right)\sin(nx). \tag{5.2}$$

$$f_N = \frac{2}{\pi}\sum_{n=1}^{N}\frac{n^2}{1-e^{-n^2}}\left(\int_0^\pi g(s)\sin(sx)dx\right)\sin(nx). \tag{5.3}$$

In the following, we consider an example which has an exact expression of solutions $(u(x,t), f(x))$.

Example. If $f(x) = \phi_1(x) = \sqrt{\frac{2}{\pi}}\sin(x)$, then the function

$$u(x,t) = \sqrt{\frac{2}{\pi}}(1-e^{-t})\sin(x)$$

is the exact solution of the problem (5.1). Consequently, the data function is $g(x) = u(x,1) = \sqrt{\frac{2}{\pi}}(1-e^{-1})\sin(x)$.

By using the central difference with step length $h = \frac{\pi}{N+1}$ to approximate the first derivative u_x and the second derivative u_{xx}, we can get the following semi-discret problem (ordinary differential equation):

$$\begin{cases} \left(\frac{d}{dt} - \mathbb{A}_h\right)u(x_i,t) = f(x_i), \quad x_i = ih, i = 1,\dots m,\ t \in (0,1), \\ u(x_0 = 0, t) = u(x_{N+1} = \pi, t) = 0, \quad t \in (0,1), \\ u(x_i, 0) = g(x_i), \quad x_i = ih, i = 1, \dots m, \end{cases} \tag{5.4}$$

where $\mathbb{A}_h$ is the discretisation matrix stemming from the operator $A = -\frac{d^2}{dx^2}$:

$$\mathbb{A}_h = \frac{1}{h^2}\text{Tridiag}(-1,2,-1) \in \mathcal{M}_N(\mathbb{R})$$

is a symmetric, positive definite matrix. We assume that it is fine enough so that the discretization errors are small compared to the uncertainty δ of the data; this means that $\mathbb{A}_h$ is a good approximation of the differential operator $A = -\frac{d^2}{dx^2}$, whose unboundedness is reflected in a large norm of $\mathbb{A}_h$. The eigenpairs (μ_k, e_k) of $\mathbb{A}_h$ are give by

$$\mu_k = 4\left(\frac{m+1}{\pi}\right)^2 \sin^2\left(\frac{k\pi}{2(m+1)}\right),\ e_k = \left(\sin\left(\frac{jk\pi}{m+1}\right)\right)_{j=1}^{m}, \quad k = 1\dots m.$$

Adding a random distributed perturbation (obtained by the Matlab command `randn`) to each data function, we obtain the vector g^δ:

$$g^\delta = g + \varepsilon \text{randn}(\text{size}(g)),$$

where ε indicates the noise level of the measurement data and the function "randn(.)" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$. "randn(size(g))" returns an array of random entries that is the same size as g. The bound on the measurement error δ can be measured in the sense of Root Mean Square Error **(RMSE)** according to

$$\delta = \|g^\delta - g\|_* = \left(\frac{1}{m}\sum_{i=1}^{m}\left(g(x_i) - g^\delta(x_i)\right)^2\right)^{1/2}.$$

The discret approximation of (5.4) takes the form

$$f(x_i) = \mathbb{A}_h(\mathbb{I}_m - e^{T\mathbb{A}_h})^{-1} g(x_i), \tag{5.5}$$

where $\mathbb{I}_m$ is the identity matrix.

In our numerical computations we always take $m = 900$ and consider only the cases when $\varepsilon = 0.1$ (aggressive noise). The cut-off frequency $N = 1, 2, 3$ and the Krylov subspace dimension $l = 500$.

To calculate the solution approached by the method of Krylov, we generate in first step the Ritz spectral band with $(l = 500)$, then we truncate the solution for $s = 1, 2, 3$, i.e.,

(5.6)

$$\mathbf{f}_l g^\delta = \|\mathbf{g}_\delta\|\mathbf{Q}_l\left\{\sum_{j=1}^{l}\frac{r_j}{1 - e^{-Tr_j}}\left(\zeta_j^T e_1\right)\zeta_j\right\} \approx \|\mathbf{g}_\delta\|\mathbf{Q}_l\left\{\sum_{j=1}^{s}\frac{r_j}{1 - e^{-Tr_j}}\left(\zeta_j^T e_1\right)\zeta_j\right\}.$$

The relative error $RE(f)$ is given by:

$$RE(f) = \frac{\|\mathbf{f}_l^\delta - f\|_*}{\|f\|_*}, \quad \text{and} \quad RE(f) = \frac{\|f_N^\delta - f\|_*}{\|f\|_*}.$$

Conclusion and discussion

Numerical results are shown in Figures $1-6$ and Tables $1-2$. In this paper, we have proposed a comparative study between the truncation method and Krylov subspace method to approximate an inverse source problem of parabolic type. The comparison is based on numerical experiments.

According to the numerical tests, we observe that the Krylov subspace method is more practical and it does not ask the exact calculation of the eigenvalues of the operator to apply the SVD method, moreover it is stable even for a strong noise. This shows that the Krylov subspace method has a nice regularizing effect and gives a better approximation with comparison to the truncation method.

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Tables

Table 1. Truncation method

m	ϵ	N	RE
900	0.1	1	$1.249619179020874e002$
900	0.1	2	$2.201835522902764e003$
900	0.1	3	$6.535303516630861e002$

In Table 1, Relative error RE for fixed $m = 900$ and for various value of N.

Table 2. Krylov subspace method

m	ϵ	s	RE
900	0.1	1	0.0117
900	0.1	2	0.0138
900	0.1	3	0.0054

In Table 2, Relative error RE for fixed $m = 900$, $l = 500$ and for various value of s.

FIGURES

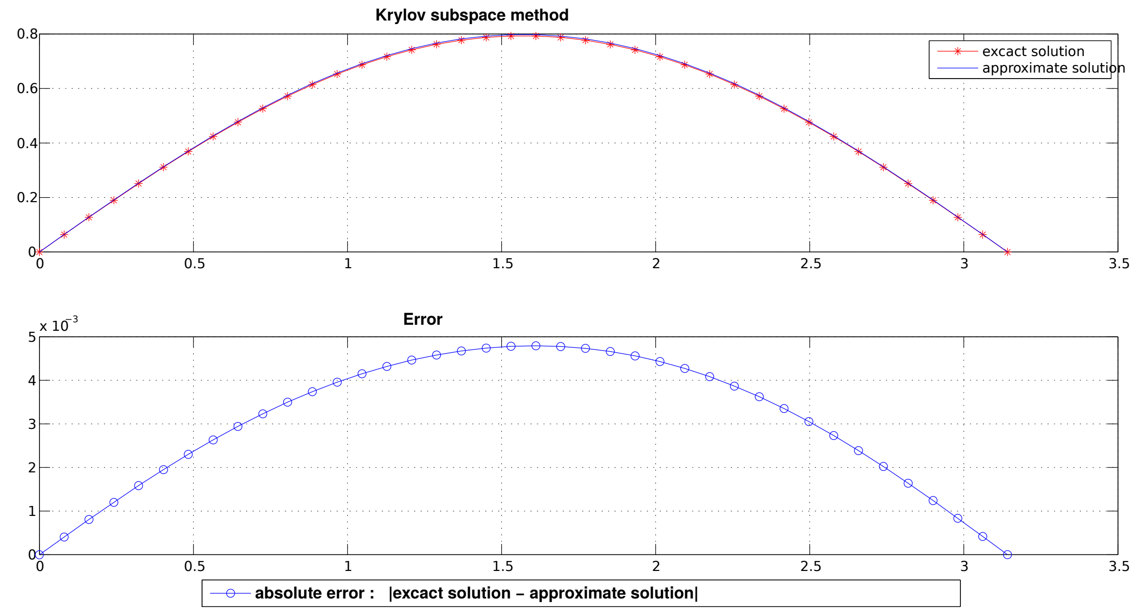

FIGURE 1. ϵ (noise level)=0.1, m=900, l=500, s=1.

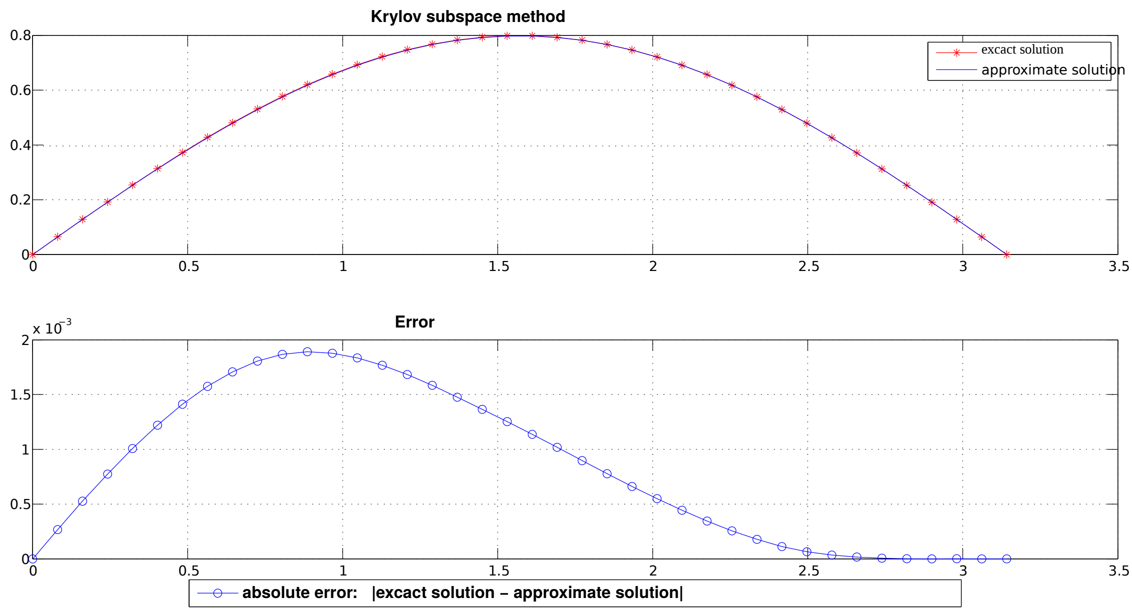

FIGURE 2. ϵ (noise level)=0.1, m=900, l=500, s=2.

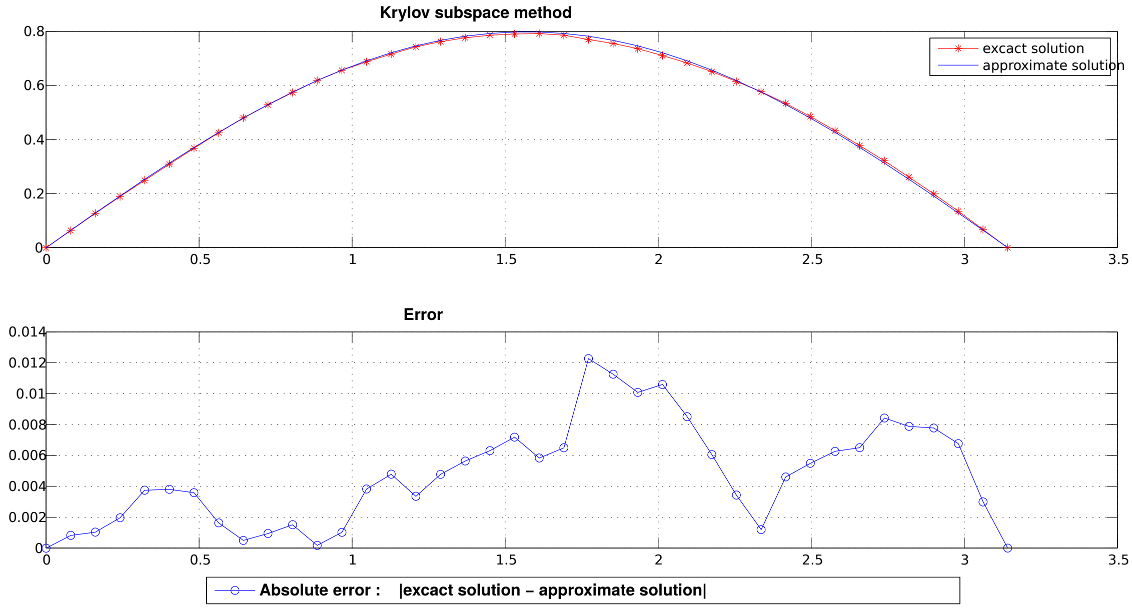

FIGURE 3. ϵ (noise level)=0.1, m=900, l=500, s=3.

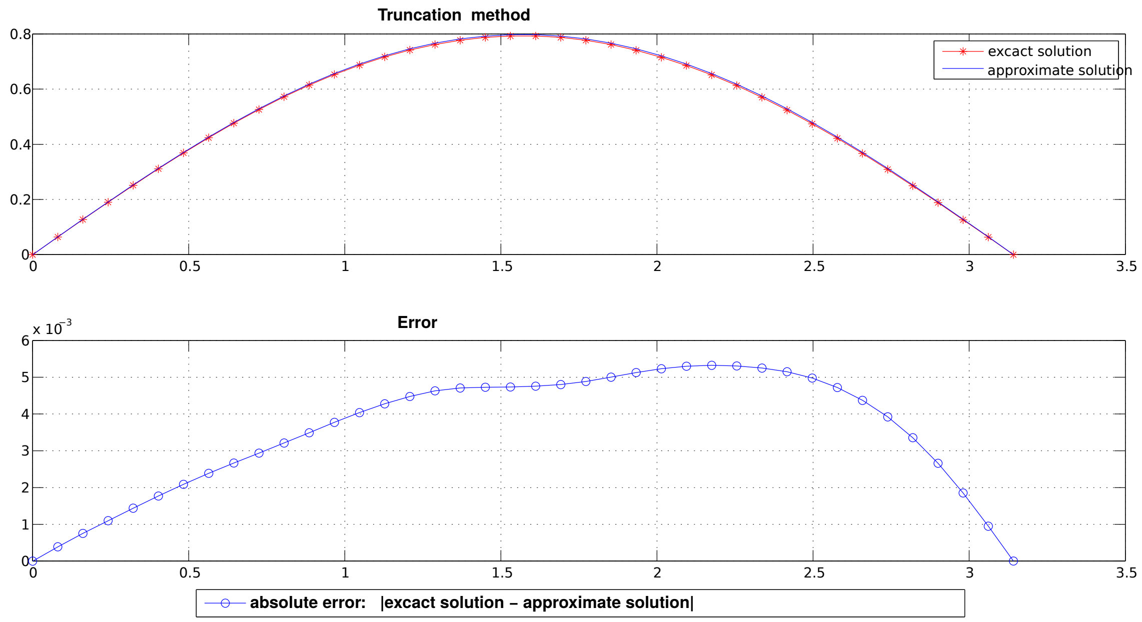

FIGURE 4. ϵ (noise level)=0.1, m=900, N=1.

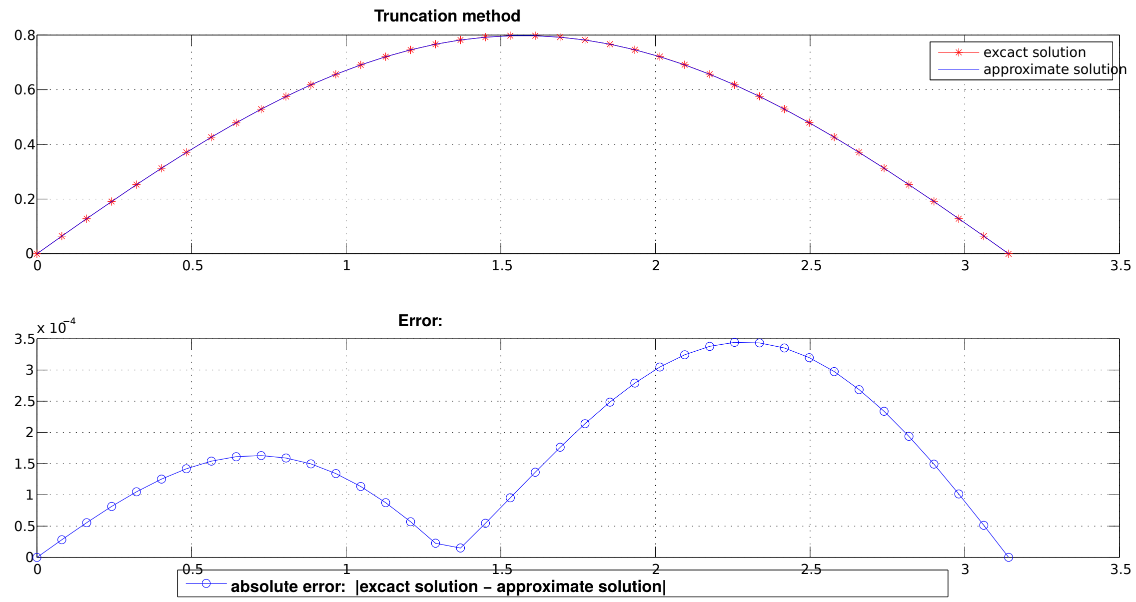

FIGURE 5. ϵ (noise level)=0.1, m=900, N=2.

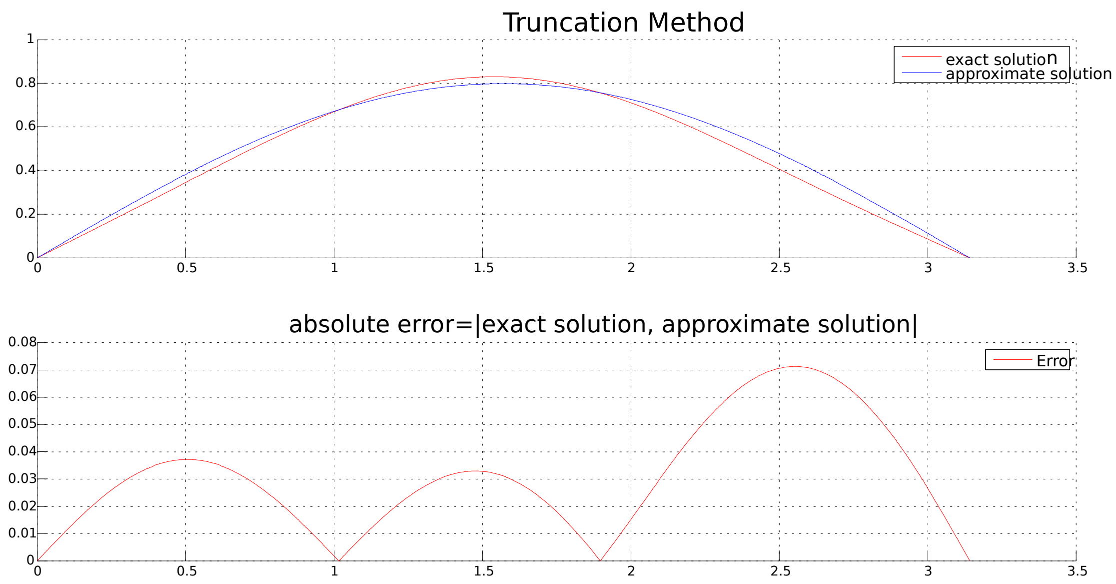

FIGURE 6. ϵ (noise level)=0.1, m=900, N=3.

REFERENCES

[1] P. BRIANZI, P. FAVATI, O. MENCHI, F. ROMANI: *A framework for studying the regularizing properties of Krylov subspace methods*, Inverse Problems, **22** (2006), 1007-1021.

[2] COULIBALY: O.S. COULIBALY: *Stabilization and numerical approximation for a classe of inverse PDEs Problem*, Ph.D. Thesis, FST, Bamako, Mali 2017.

[3] L. ELDEN: *A numerical solution of a Cauchy problem for an elliptic equation by Krylov subspaces*, Inverses Problems 25 (2009) 065002 (22pp).

[4] N.J. HIGHAM: *Matrix Functions, Theory and Application*, SIAM - Philadelphia, USA 2008.

[5] J. LIESEN, Z. STRAKOS: *Krylov Subspace Methods: Principles and Analysis*, Oxford University Press 2013.

[6] V. KOVTUN: *Regularization Based on Krylov Subspace Iterations*, Ph.D. Thesis, Praha 2013.

[7] X.X. LI: *X.X. Li, F. Yang, The truncation method for identifying the heat source dependent on a spatial vari- able*, Computers and Mathematics with Applications, **62** (2011), 2497-2505.

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