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UNCERTAINTY PRINCIPLES FOR A NONCOMMUTATIVE HYPERGROUP

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ABSTRACT. Let G be a locally compact hypergroup and let K be a compact subhypergroup of G. (G, K) is a Gelfand pair if $M_c(G//K)$, the algebra of measures with compact support on the double coset $G//K$, is commutative for the convolution. In this paper, assuming that (G, K) is a Gelfand pair, we establish uncertainty principles for the pair (G, K).

1. INTRODUCTION

Hypergroups generalize locally compact groups. They appear when the Banach space of all bounded Radon measures on a locally compact space carries a convolution having all properties of a group convolution apart from the fact that the convolution of two point measures is a probability measure with compact support and not necessarily a point measure. The intention was to unify harmonic analysis on duals of compact groups, double coset spaces $G//H$ (H a compact subgroup of a locally compact group G), and commutative convolution algebras associated with product linearization formulas of special functions. The notion of hypergroup has been sufficiently studied (see for example [2, 5, 7, 8]). Harmonic analysis and probability theory on commutative hypergroups are well developed meanwhile

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where many results from group theory remain valid (see [1]). When G is a commutative hypergroup, the convolution algebra $M_c(G)$ consisting of measures with compact support on G is commutative. The typical example of commutative hypergroup is the double coset $G//K$ when G is a locally compact group, K is a compact subgroup of G such that (G, K) is a Gelfand pair. In [5], R. I. Jewett has shown the existence of a positive measure called Plancherel measure on the dual space $\widehat{G}$ of a commutative hypergroup G; he has also established many properties of the Fourier and the inverse Fourier transform. In [11], Michael Voit relying on these results, has established an uncertainty principle for a commutative hypergroup G. When the hypergroup G is not commutative, it is possible to involve a compact sub-hypergroup K of G leading to a commutative subalgebra of $M_c(G)$. In fact, if K is a compact sub-hypergroup of a hypergroup G, the pair (G, K) is said to be a Gelfand pair if $M_c(G//K)$ the convolution algebra of measures with compact support on $G//K$ is commutative. The notion of Gelfand pairs for hypergroups is well-known (see [3, 9, 10]). When (G, K) i a Gelfand pair; it has been shown in [4] the existence of a Plancherel measure on $\widehat{G}$. The goal of this paper is to extend Voit's work by obtaining a quantitative uncertainty princilple for Gelfand pair associated with noncommutative hypergroup. This result will generate a certain qualitative uncertainty principle. In the next section, we give notations and setup useful for the remainder of this paper. In section 3, we give some properties of the Fourier transform and it reverse. Finally, thanks to these properties, we prove a quantitative and a qualitative uncertainty principles for the paire (G, K).

2. Notations and preliminaries

We use the notations and setup of this section in the rest of the paper without mentioning. Let G be a locally compact space. We denote by:

- $C(G)$ (resp. $M(G)$) the space of continuous complex valued functions (resp. the space of Radon measures) on G,
- $C_b(G)$ (resp. $M_b(G)$) the space of bounded continuous functions (resp. the space of bounded Radon measures) on G,
- $\mathcal{K}(G)$ (resp. $M_c(G)$) the space of continuous functions (resp. the space of Radon measures) with compact support on G,
- $C_0(G)$ the space of elements in $C(G)$ which are zero at infinity,

- $\mathfrak{C}(G)$ the space of compact sub-space of G,
- δ_x the point measure at $x \in G$,
- $spt(f)$ the support of the function f.
- $spt(\mu)$, the support of the measure μ.

Let us notice that the topology on $M(G)$ is the cône topology [5] and the topology on $\mathfrak{C}(G)$ is the topology of Michael [6].

Definition 2.1. G is said to be a *hypergroup* if the following assumptions are satisfied.

(H1) There is a binary operator $*$ named convolution on $M_b(G)$ under which $M_b(G)$ is an associative algebra such that:

i) the mapping $(\mu, \nu) \longmapsto \mu * \nu$ is continuous from $M_b(G) \times M_b(G)$ in $M_b(G)$.

ii) $\forall x, y \in G$, $\delta_x * \delta_y$ is a measure of probability with compact support.

iii) the mapping: $(x, y) \longmapsto$ supp$(\delta_x * \delta_y)$ is continuous from $G \times G$ in $\mathfrak{C}(G)$.

(H2) There is a unique element e (called neutral element) in G such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x, \forall x \in G$.

(H3) There is an involutive homeomorphism: $x \longmapsto \overline{x}$ from G in G, named involution, such that:

i) $(\delta_x * \delta_y)^- = \delta_{\overline{y}} * \delta_{\overline{x}}, \forall x, y \in G$ with $\mu^-(f) = \mu(f^-)$ where $f^-(x) = f(\overline{x}), \forall f \in C(G)$ and $\mu \in M(G)$.

ii) $\forall x, y, z \in G$, $z \in$ supp$(\delta_x * \delta_y)$ if and only if $x \in$ supp$(\delta_z * \delta_{\overline{y}})$.

The hypergroup G is commutative if $\delta_x * \delta_y = \delta_y * \delta_x, \forall x, y \in G$. For $x, y \in G$, $x * y$ is the support of $\delta_x * \delta_y$ and for $f \in C(G)$,

$$f(x * y) = (\delta_x * \delta_y)(f) = \int_G f(z) d(\delta_x * \delta_y)(z).$$

The convolution of two measures μ, ν in $M_b(G)$ is defined by: $\forall f \in C(G)$

$$(\mu * \nu)(f) = \int_G \int_G (\delta_x * \delta_y)(f) d\mu(x) d\nu(y) = \int_G \int_G f(x * y) d\mu(x) d\nu(y),$$

For μ in $M_b(G)$, $\mu^* = (\overline{\mu})^-$. So $M_b(G)$ is a *-Banach algebra.

Definition 2.2. $H \subset G$ is a sub-hypergroup of G if the following conditions are satisfied.

(1) H is non empty and closed in G,
(2) $\forall x \in H, \overline{x} \in H$,
(3) $\forall x, y \in H$, $\text{supp}(\delta_x * \delta_y) \subset H$.

Let us now consider a hypergroup G provided with a left Haar measure μ_G and K a compact sub-hypergroup of G with a normalized Haar measure ω_K. Let us put $M_{\mu_G}(G)$ the space of measures in $M_b(G)$ which are absolutely continuous with respect to μ_G. $M_{\mu_G}(G)$ is a closed self-adjoint ideal in $M_b(G)$. For $x \in G$, the double coset of x with respect to K is $K * \{x\} * K = \{k_1 * x * k_2; k_1, k_2 \in K\}$. We write simply KxK for a double coset and recall that $KxK = \bigcup_{k_1, k_2 \in K} \text{supp}(\delta_{k_1} * \delta_x * \delta_{k_2})$. All double coset form a partition of G and the quotient topology with respect to the corresponding equivalence relation equips the double cosets space $G//K$ with a locally topology ([1], page 53). The natural mapping $p_K : G \longrightarrow G//K$ defined by: $p_K(x) = KxK$, $x \in G$ is an open surjective continuous mapping. A function $f \in C(G)$ is said to be invariant by K or $K-invariant$ if $f(k_1 * x * k_2) = f(x)$ for all $x \in G$ and for all $k_1, k_2 \in K$. We denote by $C^\natural(G)$, (resp. $\mathcal{K}^\natural(G)$) the space of continuous functions (resp. continuous functions with compact support) which are $K-$invariant. For $f \in C^\natural(G)$, one defines the function $\widetilde{f}$ on $G//K$ by $\widetilde{f}(KxK) = f(x)\ \forall x \in G$. $\widetilde{f}$ is well defined and it is continuous on $G//K$. Conversely, for all continuous function φ on $G//K$, the function $f = \varphi \circ p_K \in C^\natural(G)$. One has the obvious consequence that the mapping $f \longmapsto \widetilde{f}$ sets up a topological isomorphism between the topological vector spaces $C^\natural(G)$ and $C(G//K)$ (see [9, 10]). So, for any f in $C^\natural(G)$, $f = \widetilde{f} \circ p_K$. Otherwise, we consider the K-projection $f \longmapsto f^\natural$ (by identifying $f^\natural$ and $\widetilde{f^\natural}$) from $C(G)$ into $C(G//K)$ where for $x \in G, f^\natural(x) = \int_K \int_K f(k_1 * x * k_2) d\omega_K(k_1) d\omega_K(k_2)$. If $f \in \mathcal{K}(G)$, then $f^\natural \in \mathcal{K}(G//K)$. For a measure $\mu \in M(G)$, one defines $\mu^\natural$ by $\mu^\natural(f) = \mu(f^\natural)$ for $f \in \mathcal{K}(G)$. μ is said to be $K-$invariant if $\mu^\natural = \mu$ and we denote by $M^\natural(G)$ the set of all those measures. Considering these properties, one defines a hypergroup operation on $G//K$ by: $\delta_{KxK} * \delta_{KyK}(\widetilde{f}) = \int_K f(x * k * y) d\omega_K(k)$ (see [9] and [1]). This defines uniquely the convolution $(KxK) * (KyK)$ on $G//K$. The involution is defined by: $\overline{KxK} = K\overline{x}K$ and the neutral element is K. Let us put $m = \int_G \delta_{KxK} d\mu_G(x)$, m is a left Haar measure on $G//K$. We say that (G, K) is a Gelfand pair if the convolution algebra $M_c(G//K)$ is commutative. $M_c(G//K)$ is topologically isomorphic to $M_c^\natural(G)$. Considering the convolution product on $\mathcal{K}(G)$,

$\mathcal{K}(G)$ is a convolution algebra and $\mathcal{K}^\natural(G)$ is a subalgebra. Thus (G,K) is a Gelfand pair if and only if $\mathcal{K}^\natural(G)$ is commutative ([3], theorem 3.2.2).

3. UNCERTAINTY PRINCIPLE

Let put $\widehat{G}$ the space of continuous, bounded function ϕ on G such that:

(i) ϕ is K- invariant,
(ii) $\phi(e)=1$,
(iii) $\int_K \phi(x*k*y)dw_K(k)=\phi(x)\phi(y)\ \forall x,y\in G$,
(iv) $\phi(\overline{x})=\overline{\phi(x)}\ \forall x\in G$.

The function $\mathbf{1}: x\longmapsto 1$ belongs to $\widehat{G}$.

Equipped with the topology of uniform convergence on compacta, $\widehat{G}$ is a locally compact Hausdorff space. $\widehat{G}$ is the *dual space* of the hypergroup G (see [4]).

3.1. Fourier transform and inverse Fourier transform.

For β belongs to $M_b(G)$, the Fourier transform of β, is the mapping

$$\widehat{\beta}:\widehat{G}\longrightarrow\mathbb{C} \text{ defined by : } \widehat{\beta}(\phi)=\int_G \phi(\overline{x})d\beta(x).$$

The Fourier transform of $f\in\mathcal{K}(G)$ is defined by

$$\widehat{f}(\phi)=\widehat{f\mu_G}(\phi)=\int_G \phi(\overline{x})f(x)d\mu_G(x),$$

$\left\{\widehat{f};\ f\in\mathcal{K}(G)\right\}$ is a sup-norm dense subspace of $C_0(\widehat{G})$. (For more detail on the Fourier transform, see [4]).

Definition 3.1. Let $\sigma\in M_b(\widehat{G})$, we call inverse Fourier transform of σ, the mapping

$$\overset{\vee}{\sigma}:G\longrightarrow\mathbb{C} \text{ defined by : } \overset{\vee}{\sigma}(x)=\int_{\widehat{G}} \phi(x)d\sigma(\phi).$$

The inverse Fourier transform of $\varphi\in L^1(\widehat{G},\pi)$ is defined by

$$\overset{\vee}{\varphi}(x)=(\varphi\pi)^\vee(x)=\int_{\widehat{G}} \phi(x)\varphi(\phi)d\pi(\phi),$$

where π is the Plancherel measure (see [4]) on $\widehat{G}$.

For $\sigma\in M_b(\widehat{G})$, $\overset{\vee}{\sigma}$ is $K-$ invariant and belongs to $C_b(G)$. $(\mathcal{K}(\widehat{G}))^\vee$ is a sup-norm dense subspace of $C_0(G)$. If $f\in\mathcal{K}^\natural(G)$ with $\widehat{f}\in L_1(\widehat{G},\pi)$, then $\widehat{f}^\vee=f$.

Let us establish some properties of the Fourier transform and the inverse Fourier transform.

Proposition 3.2. *The Fourier transform is a bijective isometry from $L_2^{\natural}(G,\mu_G)$ onto $L_2(\widehat{G},\pi)$.*

Proof. In the usual way: for $f \in L_2^{\natural}(G,\mu_G)$, $\widehat{f}$ is defined as the $L_2(\widehat{G},\pi)$-limite of a sequence $(\widehat{f_n})_n$ in $L_2(\widehat{G},\pi)$ where $(f_n)_n \subset \mathcal{K}^{\natural}(G)$ satisfies $f = \lim f_n$ in $L_2(G,\mu_G)$. Since $\widehat{f} = \widehat{f^{\natural}}$ for $f \in L_2(G,\mu_G)$, then the Fourier transform is extended to the whole space $L_2(G,\mu_G)$ and by the Plancherel theorem, it defines an isometry from $L_2^{\natural}(G,\mu_G)$ into $L_2(\widehat{G},\pi)$. Otherwise, let φ belongs to $L_2(\widehat{G},\pi)$; since $\widehat{\mathcal{K}^{\natural}(G)}$ is dense in $L_2(\widehat{G},\pi)$, then $\varphi = \lim \widehat{f_n}$ in $L_2(\widehat{G},\pi)$ where $(f_n)_n \subset \mathcal{K}^{\natural}(G)$. By the isometry, $(f_n)_n$ converges to a certain f in $L_2(G,\mu_G)$. As above, $\widehat{f} = \widehat{f^{\natural}} = \lim \widehat{f_n}$ in $L_2(\widehat{G},\pi)$, that is $\varphi = \widehat{f^{\natural}}$, so the surjection. $\square$

Proposition 3.3. *Let $1 \leq p \leq 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If $f \in L_p(G,\mu_G)$, then $\widehat{f} \in L_q(\widehat{G},\pi)$ and $\left\|\widehat{f}\right\|_q \leq \|f\|_p$.*

(ii) *If $\varphi \in L_p(\widehat{G},\pi)$, then $\overset{\vee}{\varphi} \in L_q(G,\mu_G)$ and $\left\|\overset{\vee}{\varphi}\right\|_q \leq \|\varphi\|_p$.*

Proof. (i) Let $f \in \mathcal{K}(G)$. We have

$$
\begin{aligned}
\left(\int_{\widehat{G}} \left|\widehat{f^{\natural}}(\phi)\right|^q d\pi(\phi)\right)^{\frac{1}{q}} &= \left(\int_{\widehat{G//K}} \left|\widetilde{\widehat{f^{\natural}}}(\widetilde{\phi})\right|^q d\widetilde{\pi}(\widetilde{\phi})\right)^{\frac{1}{q}} \text{ (see [4])} \\
&= \left(\int_{\widehat{G//K}} \left|\widehat{\widetilde{f^{\natural}}}(\widetilde{\phi})\right|^q d\widetilde{\pi}(\widetilde{\phi})\right)^{\frac{1}{q}} \\
&\leq \left(\int_{G//K} \left|\widetilde{f^{\natural}}(KxK)\right|^p dm(KxK)\right)^{\frac{1}{p}} \\
&= \left(\int_G \left|f^{\natural}(x)\right|^p d\mu_G(x)\right)^{\frac{1}{p}} \\
&\leq \|f\|_p .
\end{aligned}
$$

That is, $\left\|\widehat{f}\right\|_q = \left\|\widehat{f^{\natural}}\right\|_q \leq \|f\|_p$.

(ii) Let us show that $\overset{\vee}{\widetilde{\varphi}} = \widetilde{\overset{\vee}{\varphi}}$ for $\varphi \in \mathcal{K}(\widehat{G})$. In fact, since $\varphi \in \mathcal{K}(\widehat{G})$ then $\overset{\vee}{\varphi} \in C_b^{\natural}(G)$ and $\widetilde{\varphi} \in \mathcal{K}(\widehat{G//K})$. So $\overset{\vee}{\widetilde{\varphi}}$ and $\widetilde{\overset{\vee}{\varphi}}$ belong to $C_b(G//K)$. For $KxK \in G//K$, we have

$$\begin{aligned} \overset{\vee}{\widetilde{\varphi}}(KxK) &= \int_{\widehat{G//K}} \widetilde{\phi}(KxK)\widetilde{\varphi}(\widetilde{\phi})d\widetilde{\pi}(\widetilde{\phi}) \\ &= \int_{\widehat{G}} \phi(x)\varphi(\phi)d\pi(\phi) \\ &= \overset{\vee}{\varphi}(x) \\ &= \widetilde{\overset{\vee}{\varphi}}(KxK). \end{aligned}$$

If $\varphi \in \mathcal{K}(\widehat{G})$, then

$$\begin{aligned} \left(\int_G \left|\overset{\vee}{\varphi}(x)\right|^q d\mu_G(x)\right)^{\frac{1}{q}} &= \left(\int_{G//K} \left|\widetilde{\overset{\vee}{\varphi}}(KxK)\right|^q dm(KxK)\right)^{\frac{1}{q}} \\ &= \left(\int_{G//K} \left|\overset{\vee}{\widetilde{\varphi}}(KxK)\right|^q dm(KxK)\right)^{\frac{1}{q}} \\ &= \left\|\overset{\vee}{\widetilde{\varphi}}\right\|_q \text{ in } L_q(G//K, m) \\ &\leq \|\widetilde{\varphi}\|_p \text{ in } L_p(\widehat{G//K}, \widetilde{\pi}) \\ &= \left(\int_{\widehat{G//K}} \left|\widetilde{\varphi}(\widetilde{\phi})\right|^p d\widetilde{\pi}(\widetilde{\phi})\right)^{\frac{1}{p}} \\ &= \left(\int_{\widehat{G}} |\varphi(\phi)|^p d\pi(\phi)\right)^{\frac{1}{p}} \\ &= \|\varphi\|_p \text{ in } L_p(\widehat{G}, \pi). \end{aligned}$$

So $\left\|\overset{\vee}{\varphi}\right\|_q \leq \|\varphi\|_p$ and the proof is complete. □

3.2. **A quantitative and a qualitative uncertainty principles.**

Let $1 \leq p \leq 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$.

Let us fixe the sets $T \subset G$ and $U \subset \widehat{G}$ such that $\mu_G(T) < \infty$ and $\pi(U) < \infty$. Let 1_T and 1_U be their respective indicator functions. Let us define the operators $P = P_T$ and $Q = Q_U$ by

$$Pf = 1_T.f \text{ and } Qf = (1_U.\widehat{f})^{\vee} \text{ for } f \in L_p(G, \mu_G).$$

In the following theorems we establish some properties of these operators.

Theorem 3.4. *Let $1 \leq p \leq 2$; $s \geq 1$; $f \in L_p(G, \mu_G)$ and $\zeta : G \longrightarrow [0; +\infty[$ a continuous function, then*

$$\|\zeta PQf\|_s \leq [(\zeta^s \mu_G)(T)]^{1/s} \pi(U)^{1/p} \|f\|_p .$$

In particular, for $p = s$ we have the operator norm inequality

$$\|PQ\|_p \leq \mu_G(T)^{1/p}\pi(U)^{1/p}.$$

Proof. For $x \in G$, we have

$$\begin{aligned} Qf(x) &= \int_{\widehat{G}} \phi(x)1_U(\phi)\widehat{f}(\phi)d\pi(\phi) \\ &= \int_G f(y)(\int_{\widehat{G}} \phi(\overline{y})\phi(x)1_U(\phi)d\pi(\phi))d\mu_G(y) \qquad (3.1) \\ &= \langle f, \overline{k_x}\rangle \text{ with } k_x(y) = \int_{\widehat{G}} \phi(\overline{y})\phi(x)1_U(\phi)d\pi(\phi). \end{aligned}$$

Since $\phi \in \widehat{G}$, then

$$\begin{aligned} k_x(y) &= \int_{\widehat{G}} \int_K \phi(x * k * \overline{y})dk1_U(\phi)d\pi(\phi) \\ &= \int_K \int_G \int_{\widehat{G}} \phi(z)1_U(\phi)d\pi(\phi)d(\delta_x * \delta_k * \delta_{\overline{y}})(z)dw_k \\ &= \int_K \int_G 1_U^{\vee}(z)d(\delta_x * \delta_k * \delta_{\overline{y}})(z)dw_k. \end{aligned}$$

Thanks to Jensen's inequality we have

$$\begin{aligned} |k_x(y)|^q &\leq \int_K \left|\int_G 1_U^{\vee}(z)d(\delta_x * \delta_k * \delta_{\overline{y}})(z)\right|^q dw_k \\ &\leq \int_K \int_G |1_U^{\vee}(z)|^q d(\delta_x * \delta_k * \delta_{\overline{y}})(z)dw_k \qquad (3.2) \\ &= \int_K |1_U^{\vee}(x * k * \overline{y})|^q dw_k. \end{aligned}$$

Since G is unimodular, then

$$\|k_x\|_q \leq \|1_U^{\vee}\|_q .$$

So by the Hölder inequality we deduce that

$$\begin{aligned} |Qf(x)| &= \left|\langle f, \overline{k_x}\rangle\right| \\ &\leq \|f\|_p \|1_U^{\vee}\|_q . \end{aligned}$$

Otherwise, we have

$$\begin{aligned} |\zeta.PQf(x)|^s &\leq |\zeta(x)1_T(x)|^s \|Qf\|_\infty^s \text{, that is} \\ \|\zeta.PQf\|_s &\leq \|Qf\|_\infty \left(\int_G |\zeta(x)1_T(x)|^s \, d\mu_G\right)^{1/s} \\ &\leq \|Qf\|_\infty \left(\int_T |\zeta(x)|^s \, d\mu_G\right)^{1/s} \\ &\leq \|Qf\|_\infty \left((\zeta^s\mu_G)(T)\right)^{1/s} \\ &\leq \|f\|_p \|1_U^\vee\|_q \left((\zeta^s\mu_G)(T)\right)^{1/s} \\ &\leq \|f\|_p \|1_U\|_p \left((\zeta^s\mu_G)(T)\right)^{1/s} \\ &= \|f\|_p \left(\pi(U)\right)^{1/p} \left((\zeta^s\mu_G)(T)\right)^{1/s}. \end{aligned}$$

□

Theorem 3.5. *Let* $1 \leq p \leq 2$*;* $s \geq q$ *and* $f \in L_p(G, \mu_G)$*, then*

(i) $\|QPf\|_s \leq \mu_G(T)^{1/s}\pi(U)^{1/p} \|f\|_p$;
(ii) $\|QPf\|_s \leq \mu_G(T)^{1/p}\pi(U)^{1/p} \|Pf\|_s$.

Proof. (i) By (3.1), we have $QPf(x) = Q1_T f(x) = \left\langle 1_T f, \overline{k_x}\right\rangle = \left\langle f, \overline{1_T k_x}\right\rangle \; \forall x \in G$. Using (3.2), we have

$$\begin{aligned} \|1_T k_x\|_q &= \left(\int_G 1_T(y) \, |k_x(y)|^q \, d\mu_G(y)\right)^{1/q} \\ &\leq \left(\int_G 1_T(y) \int_K |1_U^\vee|^q \, (x * k * \overline{y}) dw_k d\mu_G(y)\right)^{1/q} \\ &\leq \left(\int_K \int_G 1_T(y) \, |1_U^\vee|^q \, (x * k * \overline{y}) d\mu_G(y) dw_k\right)^{1/q} \\ &= \left(\int_K (|1_U^\vee|^q * 1_T)(x * k) dw_k\right)^{1/q}. \end{aligned}$$

Thanks to the Hölder inequality and using the ineqality above, we have

$$\begin{aligned} \|QPf\|_s &= \left(\int_G |QPf(x)|^s \, d\mu_G\right)^{1/s} \\ &\leq \left(\int_G \left(\|f\|_p \|1_T k_x\|_q\right)^s d\mu_G(x)\right)^{1/s} \\ &\leq \|f\|_p \left(\int_G \left(\int_K (|1_U^\vee|^q * 1_T)(x * k) dw_k\right)^{s/q} d\mu_G(x)\right)^{1/s} \\ &\leq \|f\|_p \left(\int_G \left((|1_U^\vee|^q * 1_T)(x)\right)^{s/q} d\mu_G(x)\right)^{1/s} \text{, since } G \text{ is unimodular} \\ &= \|f\|_p \left(\| |1_U^\vee|^q * 1_T \|_{s/q}\right)^{1/q}. \end{aligned}$$

Since $|1_U^\vee|^q \in L_1(G,\mu_G)$ and $1_T \in L_{s/q}(G,\mu_G)$, then

$$\begin{aligned}\|QPf\|_s &\leq \|f\|_p \left(\|(1_U^\vee)^q\|_1 \, \|1_T\|_{s/q}\right)^{1/q} \\ &= \|f\|_p \, \|1_U^\vee\|_q \, \mu_G(T)^{1/s} \\ &\leq \|f\|_p \, \|1_U\|_p \, \mu_G(T)^{1/s} \\ &= \|f\|_p \, \pi(U)^{1/p} \mu_G(T)^{1/s}.\end{aligned}$$

(ii) Using $P^2 = P$, we can replace f by Pf in the first inequality. This leads to

$$\|QPf\|_s \leq \|Pf\|_p \, \pi(U)^{1/p} \mu_G(T)^{1/s}.$$

Moreover, let us put $r = s/p$ and r' such that $1/r + 1/r' = 1$. Since $|Pf|^p \in L_r(G,\mu_G)$ and $1_T \in L_{r'}(G,\mu_G)$, we have

$$\begin{aligned}\|Pf\|_p^p &= \textstyle\int_G 1_T(x) \, |Pf(x)|^p \, d\mu_G(x) \\ &= \|1_T \, |Pf|^p\|_1 \\ &\leq \|1_T\|_{r'} \, \||Pf|^p\|_r \\ &= \|1_T\|_{r'} \, \|Pf\|_s^p \, .\end{aligned}$$

This implies

$$\begin{aligned}\|Pf\|_p &\leq \|Pf\|_s \left(\|1_T\|_{r'}\right)^{1/p} \\ &= \|Pf\|_s \, \mu_G(T)^{1/pr'} \\ &= \|Pf\|_s \, \mu_G(T)^{1/p-1/s}.\end{aligned}$$

Hence

$$\begin{aligned}\|QPf\|_s &\leq \|Pf\|_s \, \mu_G(T)^{1/p-1/s} \pi(U)^{1/p} \mu_G(T)^{1/s} \\ &= \|Pf\|_s \, \mu_G(T)^{1/p} \pi(U)^{1/p},\end{aligned}$$

and the proof is complete. □

Let T, U and p giving as above; for $\varepsilon, \delta \geq 0$, let us remind those definitions.

Definition 3.6. $f \in L_p(G,\mu_G)$ is called ε-concentrated on T if $\|f - Pf\|_p \leq \varepsilon \, \|f\|_p$ and δ-bandlimited to U if there exists $f_U \in L_p(G,\mu_G)$ with $\left\{\phi \in \widehat{G}; \widehat{f_U}(\phi) \neq 0\right\} \subset U$ and $\|f - f_U\| \leq \delta \, \|f\|_p$.

Let us put $A_f = \{x \in G; f(x) \neq 0\}$ and $B_f = \left\{\phi \in \widehat{G}; \widehat{f}(\phi) \neq 0\right\}$. Thanks to the results above, we establish the following results which are uncertainty principles.

Theorem 3.7.

(i) *Let* $1 \leq p \leq 2$ *and* $\varepsilon, \delta \geq 0$. *If* $f \in L_p(G, \mu_G)$ *with* $f \neq 0$ *is* ε*-concentrated on* T *and* δ*-bandlimited to* U*, then*

$$\|PQ\|_p \geq \frac{1-\varepsilon-\delta}{1+\delta}.$$

(ii) *Let* f *be a* $K-$*invariant function belongs to* $L_1(G, \mu_G) \cap L_2(G, \mu_G)$. *If* $\mu_G(A_f)\pi(B_f) < 1$, *then* $f = 0$.

Proof. (i)Using $f_U = Qf_U$ and $\|P\|_p = \sup_{\|f\|_p \leq 1} \|Pf\|_p \leq 1$, we have

$$\begin{aligned} \|f\|_p - \|PQf\|_p &\leq \|f - PQf\|_p \\ &\leq \|f - Pf\|_p + \|Pf - Pf_U\|_p + \|PQf_U - PQf\|_p \\ &\leq \varepsilon \|f\|_p + \delta \|f\|_p + \|PQ\|_p \delta \|f\|_p \\ &= (\varepsilon + \delta + \delta \|PQ\|_p) \|f\|_p . \end{aligned}$$

Thus $\|PQf\|_p \geq (1 - \varepsilon - \delta - \delta \|PQ\|_p) \|f\|_p$, hence $\|PQ\|_p \geq \frac{1-\varepsilon-\delta}{1+\delta}$.

(ii) By their definitions, $P_{A_f} f = f$ and $1_{B_f}\widehat{f} = \widehat{f}$. So $Q_{B_f} f = (1_{B_f}\widehat{f})^{\vee} = (\widehat{f})^{\vee} = f$ since f is $K-$invariant.

If $\mu_G(A_f) = 0$, then $f = 0$. If $\pi(B_f) = 0$, then $\widehat{f} = 0$, thus $f = 0$.

Let suppose $\mu_G(A_f) \neq 0$ and $\pi(B_f) \neq 0$. $\mu_G(A_f)\pi(B_f) < 1 \Longrightarrow \mu_G(A_f) < \infty$ and $\pi(B_f) < \infty$. Since $P_{A_f} f = f$ and $Q_{B_f} f = f$, then f is $0-$concentrated on A_f and $0-$bandlimited to B_f. If $f \neq 0$, by theorem 3.4 and (i), we have $1 \leq \|PQ\|_2 \leq \mu_G(T)^{1/2}\pi(U)^{1/2} < 1$, which is absurd, hence $f = 0$. □

Remark 3.8. In the theorem above, (i) is a quantitative uncertainty principle and (ii) is a qualitative one.

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