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A CLASS OF OPTIMAL EIGHTH ORDER ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS WITHOUT DERIVATIVES

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ABSTRACT. The objective of this paper is to develop new class of optimal iterative methods that do not need any derivative evaluations for solving nonlinear equations. Those new methods consist of an approximation of the eighth order and require four function evaluations per iteration which support the Kung-Traub assumption on optimal order for without memory schemes. Lastly, to show those new methods' performance and effectiveness, they are compared numerically with other similar methods in high-precision computation.

1. INTRODUCTION

In numerical analysis, one of the most important computational challenges is finding the real roots of the nonlinear equation $f(x) = 0$. Many practical problems that are encountered in Mathematics, Chemistry, Physics, and Engineering require the solution of nonlinear equations. It is rare to obtain the exact solution in most of the cases of nonlinear equations [11]. When this occurs, the iterative methods can be used to find the approximate solution [2,3]. The iterative methods can be classified into two schemes, single-point and multi-point. Those two schemes are also sub-divided into two categories, with and without derivative.

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Because multi-point methods have high order of efficiency index $EI = \rho^{\frac{1}{n+1}}$, since ρ is order convergent method and have high order of convergence, they are considered to be more important than single-point methods. In some cases, there is either no derivative for $f(x)$ or it is difficult to compute $f(x)$ derivative. Kung and Traub [4] assumed that convergence order of 2^n could be achieved with optimal iterative method requiring $n+1$ function evaluations per iteration. So, it is generally preferred to utilize methods that do not use $f(x)$ derivative because they reduce the number of function evaluations for each iteration. A small number of derivative-free methods have been presented in research articles [7–9] and [11–14].

One of the most important issues in solving nonlinear equations with iterative methos is to choose a good initial approximation. So, to assure convergence of the solution, it must be ensured that the initial approximation and the solution are close enough. There are multiple methods that can be used to find good initial approximation.

Definition 1.1. *Suppose that x_{n-1}, x_n and x_{n+1} are three consecutive iterations closer to root α. Methods are approximated by computational order of convergence (COC) [5], since*

$$COC \approx \frac{\ln \mid f(x_{n+1})/f(x_n)}{\ln \mid f(x_n)/f(x_{n-1})}, \tag{1.1}$$

The goal of this paper is to develop two derivative-free methods without memory of order eight by using a number of four function evaluations per iteration, which are extended methods of Mirzaee and Hamzeh's derived method [10].

2. Eighth-order iterative method

Here, we consider a general optimal eighth order derivative free scheme [8] in the following way

$$\begin{aligned} w_n &= f(x)^2 + x_n, \\ y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \end{aligned}$$

(2.1)

$$z_n = y_n - f(y_n)\frac{A(t_n)}{f[y_n, w_n]},$$

$$x_{n+1} = z_n - \frac{f(z_n)(B(t_n, u_n) + G(s_n))}{f[z_n, w_n]}, n = 0, 1, 2, \ldots \tag{2.2}$$

where $B(t_n, u_n), G(s_n)$ are wight function has been considered in [8], [12],

$$t = \frac{f(y_n)}{f(x_n)}, \quad s = \frac{f(z_n)}{f(x_n)}, \quad u = \frac{f(z_n)}{f(y_n)},$$

and

$$f[x_n, w_n] = \frac{f(w_n) - f(x_n)}{w_n - x_n}, \quad f[y_n, w_n] = \frac{f(w_n) - f(y_n)}{w_n - y_n},$$

$$f[z_n, w_n] = \frac{f(w_n) - f(z_n)}{w_n - z_n}.$$

Here, to approximate the value of derivative, we use forward difference $f'(x_n) = f[x_n, w_n] = \frac{f(w_n)-f(x_n)}{w_n - x_n}$, where $w_n = f(x)^2 + x_n$.

Theorem 2.1. *Let $\alpha \in D$ be exact zero of an adequately differentiable function $f : D \subseteq R \to R$ for an open interval D which contains t_0 as an initial approximation of α. Then the order of convergence of method defined in (2.2) is eight if $A(0) = A'(0) = 1, A''(0) = 6, A'''(0) = 54$ and $A^{(4)} < |\infty|$, $B(0,0) = 1, B_t(0,0) = 1, B_u(0,0) = 1, B_{tt}(0,0) = 6, B_{tu}(0,0) = 0, B_{ttt}(0,0) = 60$, and the error equation is given by*

$$e_{n+1} = \left(\left(\frac{1}{2}\right) c_2 c_3^3 B_{uu}(0,0) - \left(\frac{1}{2}\right) c_2^3 c_3^2 B_{ttu}(0,0) - \left(\frac{1}{24}\right) A^{(4)}(0) c_2^5 c_3 - c_2^2 c_3 c_4 \right.$$
$$\left. - f'(\alpha)^2 c_2^4 c_3 - c_2 c_3^3 - 7 c_2^5 c_3 + 8 c_2^3 c_3^2\right) e^8 + O\left(e^9\right).$$

Proof. Suppose α be the exact zero of $f(x)$. We introduced the error equation at n^{th} iteration as $x_n = \alpha + e_n$. Using Taylor series in each term involved in (2.2) about the exact zero α, we get

$$f(x) = f'(\alpha)\left(c_2 e^2 + c_3 e^3 + c_4 e^4 + c_5 e^5 + c_6 e^6 + c_7 e^7 + c_8 e^8 + c_9 e^9 + c_{10} e^{10} + c_{11} e^{11}\right), \tag{2.3}$$

where $c_i = \frac{f^{(i)}(\alpha)}{i! f'(\alpha)}, i = 2, 3, \ldots$. Moreover, we obtained

$$w_n = \alpha + e + f'(\alpha)^2\left[e^2 + 2c_2 e^3 + \left(c_2^2 + 2c_3\right) e^4 + \left(2c_2 c_3 + 2c_4\right) e^5\right] + \cdots$$
$$+ f'(\alpha)^2\left(2c_2 c_6 + 2c_3 c_5 + c_4^2 + 2c_7\right) e^8 + O\left(e^9\right), \tag{2.4}$$

$$\begin{aligned} d_n =&\alpha + e + f'(\alpha)^2 \left[e^2 + 2c_2e^3 + \left(c_2^2 + 2c_3\right) e^4 + \left(2c_2c_3 + 2c_4\right) e^5 + \cdots \right.\\ &+ f'(\alpha)^2 \left(2c_2c_6 + 2c_3c_5 + c_4^2 + 2c_7\right) e^8 + O\left(e^9\right) - \alpha, \end{aligned} \tag{2.5}$$

and then

$$\begin{aligned} f\left(w_n\right) = & f'(\alpha)\left[e + \left(c_2 \ + f'(\alpha)^2\right) e^2 + \left(c_3 + 4f'(\alpha)^2c_2\right) e^3 \right.\\ & + \left.\left(c_4 + 5f'(\alpha)^2c_3 + 5f'(\alpha)^2c_2^2 + f'(\alpha)^4c_2\right) e^4\right] + \cdots \\ & + \left(18c_4f'(\alpha)^3c_2c_3 + 42c_3^2f'(\alpha)^5c_2 + 24c_3f'(\alpha)^5c_2^3\right.\\ & + 15c_3f'(\alpha)^7c_2^2 + 9c_5f'(\alpha)^3c_2^2 + 44c_5f'(\alpha)^5c_2 + 48c_4f'(\alpha)^5c_2^2 \\ & + 36c_4c_3f'(\alpha)^5 + 24c_4f'(\alpha)^7c_2 + 3f'(\alpha)^3c_3^3 + 6c_3^2f'(\alpha)^7 \\ & + f'(\alpha)^5c_2^5 + 15c_6fd(\alpha)^5 + 10c_5f'(\alpha)^7 + c_4f'(\alpha)^9 + 9f'(\alpha)^3c_4^2 \\ & \left. + 9f'(\alpha)^3c_7 + 18f'(\alpha)^3c_2c_6 + 18f'(\alpha)^3c_3c_5 + f'(\alpha)c_8\right) e^8 \\ & + O\left(e^9\right). \end{aligned} \tag{2.6}$$

So, we find the value

$$\begin{aligned} f\left[x_n, w_n\right] = & f'(\alpha) + 2f'(\alpha)c_2e + \left(3f'(\alpha)c_3 + f'(\alpha)^3c_2\right) e^2 \\ & + \left(4f'(\alpha)c_4 + 3f'(\alpha)^3c_3 + 2f'(\alpha)^3c_2^2\right) e^3 \\ & + \left(5f'(\alpha)c_5 + 6f'(\alpha)^3c_4 + 8f'(\alpha)^3c_2c_3 + c_3f'(\alpha)^5\right.\\ & \left. + f'(\alpha)^3c_2^3\right) e^4 + \cdots \\ & + \left(20c_4f'(\alpha)^3c_2c_3 + 12c_3^2f'(\alpha)^5c_2 + 4c_3f'(\alpha)^5c_2^3\right.\\ & + 32f'(\alpha)^3c_2c_6 + 12c_5f'(\alpha)^3c_2^2 + 26f'(\alpha)^3c_3c_5 \\ & + 40c_5f'(\alpha)^5c_2 + 24c_4f'(\alpha)^5c_2^2 + 20c_4c_3f'(\alpha)^5 + 6c_4f'(\alpha)^7c_2 \\ & + 21f'(\alpha)^3c_7 + 20c_6f'(\alpha)^5 + 5c_5f'(\alpha)^7 + 12f'(\alpha)^3c_4^2 \\ & \left. + 3f'(\alpha)^3c_3^3 + 8f'(\alpha)c_8\right) e^7 + O\left(e^8\right). \end{aligned} \tag{2.7}$$

From (2.3) and (2.7), we have

$$\begin{aligned} y_n = & \alpha + c_2e^2 + \left(f'(\alpha)^2c_2 - 2c_2^2 + 2c_3\right) e^3 \\ & + \left(-f'(\alpha)^2c_2^2 + 3f'(\alpha)^2c_3 + 4c_2^3 - 7c_2c_3 + 3c_4\right) e^4 + \cdots \\ & + \left(162c_4f'(\alpha)^2c_2c_3 + 64c_2^7 - 348c_2^2c_3c_4 + 118c_2c_3c_5\right.\\ & + 162f'(\alpha)^2c_2^4c_3 - 99f'(\alpha)^2c_2^3c_4 - 189f'(\alpha)^2c_2^2c_3^2 - 304c_2^5c_3 \end{aligned}$$

$$
\begin{aligned}
&+ 176c_2^4c_4 + 408c_2^3c_3^2 - 92c_2^3c_5 - 135c_2c_3^3 + 44c_2^2c_6 + 64c_2c_4^2 \\
&+ 75c_3^2c_4 - f'(\alpha)^6c_2^4 - 36f'(\alpha)^2c_2^6 + 39c_3^2f'(\alpha)^4c_2 \\
&- 43c_3f'(\alpha)^4c_2^3 + 7c_3f'(\alpha)^6c_2^2 + 54c_5f'(\alpha)^2c_2^2 - 10c_5f'(\alpha)^4c_2 \\
&+ 27c_4f'(\alpha)^4c_2^2 - 43c_4c_3f'(\alpha)^4 - 5c_4f'(\alpha)^6c_2 + 36f'(\alpha)^2c_3^3 \\
&- 6c_3^2f'(\alpha)^6 + 10f'(\alpha)^4c_2^5 + 20c_6f'(\alpha)^4 + 5c_5f'(\alpha)^6 - 19c_2c_7 \\
&- 27c_3c_6 - 31c_4c_5 - 30f'(\alpha)^2c_4^2 + 21f'(\alpha)^2c_7 - 24f'(\alpha)^2c_2c_6 \\
&-51f'(\alpha)^2c_3c_5 + 7c_8\big)\, e^8 + O\left(e^9\right), \qquad (2.8)
\end{aligned}
$$

$$
\begin{aligned}
f\left(y_n\right) = {} & f'(\alpha)c_2e^2 + \left(f'(\alpha)^3c_2 - 2c_2^2f'(\alpha) + 2f'(\alpha)c_3\right) e^3 \\
&+ \left(5f'(\alpha)c_2^3 - f'(\alpha)^3c_2^2 + 3f'(\alpha)^3c_3 - 7c_2f'(\alpha)c_3 + 3f'(\alpha)c_4\right) e^4 \\
&+ \big(20c_4f'(\alpha)^3c_2c_3 + 134c_2c_3f'(\alpha)c_5 - 455c_2^2c_3f'(\alpha)c_4 + 297c_2^4f'(\alpha)c_4 \\
&+ 73c_2c_4^2f'(\alpha) + 75c_3^2f'(\alpha)c_4 - 552c_2^5f'(\alpha)c_3 + 582c_2^3c_3^2f'(\alpha) \\
&- 147c_2c_3^3f'(\alpha) - 264f'(\alpha)^3c_2^2c_3^2 - 171c_4f'(\alpha)^3c_2^3 - 19c_2f'(\alpha)c_7 \qquad (2.9) \\
&+ 303c_2^4c_3f'(\alpha)^3 + 54c_2^2f'(\alpha)c_6 - 27c_3fd'(\alpha)c_6 \\
&- 134c_2^3f'(\alpha)c_5 - 31c_4f'(\alpha)c_5 - 86c_2^6f'(\alpha)^3 - 3f'(\alpha)^7c_2^4 + 144f'(\alpha)c_2^7
\end{aligned}
$$

$$
\begin{aligned}
f\left[y_n, w_n\right] = {} & f'(\alpha) + f'(\alpha)c_2e + \left(f'(\alpha)c_3 + f'(\alpha)^3c_2 + c_2^2f'(\alpha)\right) e^2 \\
&+ \big(f'(\alpha)c_4 + 2f'(\alpha)^3c_3 + 3f'(\alpha)^3c_2^2 - 2f'(\alpha)c_2^3 \\
&+3c_2f'(\alpha)c_3\big)\, e^3 + \cdots \\
&+ \big(-10c_4f'(\alpha)^3c_2c_3 - 30c_2c_3f'(\alpha)c_5 + 92c_2^2c_3f'(\alpha)c_4 \qquad (2.10) \\
&- 74c_2^4f'(\alpha)c_4 - 16c_2c_4^2f'(\alpha) - 7c_3^2f'(\alpha)c_4 + 112c_2^5f'(\alpha)c_3 \\
&- 94c_2^3c^2f'(\alpha) + 111c_2c_3^3f'(\alpha) + 33f'(\alpha)^3c_2^2c_3^2 + 36c_4f'(\alpha)^3c_2^3 \\
&+ 7c_2f'(\alpha)c_7 - 51c_2^4c_3f'(\alpha)^3 - 17c_2^2f'(\alpha)c_6 + 7c_3f'(\alpha)c_6 \\
&+ 37c_2^3f'(\alpha)c_5 + 7c_4f'(\alpha)c_5 + 16c_2^6f'(\alpha)^3 + f'(\alpha)^7c_2^4 \\
&- 32f'(\alpha)c_2^7 + 7c_3^2f'(\alpha)^5c_2 + 15c_3f'(\alpha)^5c_2^3 - 3c_3f'(\alpha)^7c_2^2 \\
&- 8c_5f'(\alpha)^3c_2^2 + 40c_5f'(\alpha)^5c_2 + 16c_4f'(\alpha)^5c_2^2 + 35c_4c_3f'(\alpha)^5
\end{aligned}
$$

$$+ 8c_4 f'(\alpha)^7 c_2 + 32 f'(\alpha)^3 c_2 c_6 + 35 f'(\alpha)^3 c_3 c_5 + 3 f'(\alpha)^3 c_3^3 + c_3^2 f'(\alpha)^7 - 4 f'(\alpha)^5 c_2^5 + 10 c_6 f'(\alpha)^5 + 4 c_5 f'(\alpha)^7 + 18 f'(\alpha)^3 c_4^2 + 6 f'(\alpha)^3 c_7 + f'(\alpha) c_8\big) e^7 + O\left(e^8\right).$$

Then, from (2.3) and (2.8), we get

$$A(t) = A(0) + A'(0)t + \left(\frac{1}{2}\right) A''(0)t^2 + A'''(0)\frac{t^3}{3!} + A^{(4)}(0)\frac{t^4}{4!}. \tag{2.11}$$

Substituting the values of t in (2.2), we obtain

$$\begin{aligned} z_n = {} & \alpha + (-A(0)c_2 + c_2)\, e^2 + \cdots \\ & + \big(162 c_4 f'(\alpha) c_2 c_3 - 68 A(0) c_2^2 c_6 + 34 A(0) c_3 c_6 \\ & + 142 A(0) c_2^3 c_5 + \cdots + 74 A'(0) c_2^2 c_6 + 114 A'(0) c_2 c_4^2 \\ & + 136 A'(0) c_3^2 c_4 - 18 A'(0) f'(\alpha)^6 c_2^4 - 439 A'(0) f'(\alpha)^2 c_2^6 \\ & - 2118 A'(0) c_2^5 c_3 + 872 A'(0) c_2^4 c_4 + 1936 A'(0) c_2^3 c_3^2 \\ & - 289 A'(0) c_2^3 c_5 + 104 A'(0) f'(\alpha)^2 c_3^3 - 6 A'(0) c_3^2 f'(\alpha)^6 \\ & + 130 A'(0) f'(\alpha)^4 c_2^5 + 470 A(0) c_2^2 c_3 c_4 - 162 A(0) c_2 c_3 c_5 \\ & - 248 f'(\alpha)^2 A(0) c_4 c_2 c_3 + 500 A'(0) c_4 f'(\alpha)^2 c_2 c_3 \\ & - 81 A''(0) c_4 f'(\alpha)^2 c_2 c_3 + 64 c_2^7 - 348 c_2^2 c_3 c_4 + 118 c_2 c_3 c_5 \\ & + 162 f'(\alpha)^2 c_2^4 c_3 - 99 f'(\alpha)^2 c_2^3 c_4 - 189 f'(\alpha)^2 c_2^2 c_3^2 \\ & - 135 c_2 c_3^3 + 44 c_2^2 c_6 + 64 c_2 c_4^2 + 75 c_3^2 c_4 - f'(\alpha)^6 c_2^4 \\ & - 36 f'(\alpha)^2 c_2^6 - 304 c_2^5 c_3 + 176 c_2^4 c_4 + 408 c_2^3 c_3^2 - 92 c_2^3 c_5 \\ & + 54 c_5 f'(\alpha)^2 c_2^2 - 10 c_5 f'(\alpha)^4 c_2 + 27 c_4 f'(\alpha)^4 c_2^2 \\ & - 43 c_4 c_3 f'(\alpha)^4 - 5 c_4 f'(\alpha)^6 c_2 + 39 c_3^2 f'(\alpha)^4 c_2 \\ & - 43 c_3 f'(\alpha)^4 c_2^3 + 7 c_3 f'(\alpha)^6 c_2^2 + 36 f'(\alpha)^2 c_3^3 - 6 c_3^2 f'(\alpha)^6 \\ & + 10 f'(\alpha)^4 c_2^5 + 20 c_6 f'(\alpha)^4 + 5 c_5 f'(\alpha)^6 - 51 f'(\alpha)^2 c_3 c_5 \\ & - 24 f'(\alpha)^2 c_2 c_6 - 30 f'(\alpha)^2 c_4^2 + 21 f'(\alpha)^2 c_7 - 19 c_2 c_7 \\ & - 27 c_3 c_6 - 31 c_4 c_5 + 7 c_8\big)\, e^8 + O\left(e^9\right). \end{aligned} \tag{2.12}$$

Now, by imposing these conditions $A(0) = A'(0) = 1, A''(0) = 6, A'''(0) = 54$ and $A^{(4)} < |\infty|$, in the above equation, we get possible order of convergence is four and by using it, we can get

$$\begin{aligned}
z_n &= \alpha - c_2c_3e^4 + \left(2c_2^2c_3 - 2f'(\alpha)^2c_2^3 - 2f'(\alpha)^2c_2c_3 - 2c_2c_4 - 2c_3^2\right)e^5 + \cdots \\
&+ \left(-72c_4f'(\alpha)^2c_2c_3 - \left(\frac{43}{8}\right)A^{(4)}(0)c_2^7\right. \\
&- \left(\frac{55}{24}\right)A^{(4)}(0)f'(\alpha)^2c_2^4c_3 + \left(\frac{71}{24}\right)A^{(4)}(0)f'(\alpha)^2c_2^6 \\
&+ \left(\frac{85}{12}\right)A^{(4)}(0)c_2^5c_3 - \left(\frac{5}{8}\right)A^{(4)}(0)c_2^4c_4 - \left(\frac{5}{3}\right)A^{(4)}(0)c_2^3c_3^2 \\
&- \left(\frac{5}{12}\right)A^{(4)}(0)f'(\alpha)^4c_2^5 + 2289c_2^7 - 47c_2^2c_3c_4 + 20c_2c_3c_5 \\
&+ 1553f'(\alpha)^2c_2^4c_3 - 5f'(\alpha)^2c_2^3c_4 - 57f'(\alpha)^2c_2^2c_3^2 - 35c_2c_3^3 \\
&+ 5c_2^2c_6 + 10c_2c_4^2 + 14c_3^2c_4 - 7f'(\alpha)^6c_2^4 - 1451f'(\alpha)^2c_2^6 \\
&- 3696c_2^5c_3 + 424c_2^4c_4 + 1175c_2^3c_3^2 - 5c_2^3c_5 - 28c_5f'(\alpha)^2c_2^2 \\
&- 10c_5f'(\alpha)^4c_2 - 60c_4f'(\alpha)^4c_2^2 - 23c_4c_3f'(\alpha)^4 \\
&- c_4f'(\alpha)^6c_2 - 72c_3^2f'(\alpha)^4c_2 - 75c_3f'(\alpha)^4c_2^3 \\
&- 23c_3f'(\alpha)^6c_2^2 - 12f'(\alpha)^2c_3^3 - c_3^2f'(\alpha)^6 + 274f'(\alpha)^4c_2^5 \\
&- 35f'(\alpha)^2c_3c_5 - 14f'(\alpha)^2c_2c_6 - 21f'(\alpha)^2c_4^2 - 5c_2c_7 \\
&\left.-13c_3c_6 - 17c_4c_5\right)e^8 + O\left(e^9\right).
\end{aligned} \tag{2.13}$$

Now, we formulate the algorithm as follows

$$f\left[z_n, w_n\right] = \frac{f\left(w_n\right) - f\left(z_n\right)}{w_n - z_n}.$$

To find the proper weight functions B and G in (2.2), providing order eight, we will use the method of undetermined coefficients and Taylor's series about zero, since t, u and s tend to zero when x tends to α. The technique of undetermined coefficients was studied in [1]. We have

$$G(s) = G(0) + G'(0)s + \left(\frac{1}{2}\right)G''(0)s^2 + G'''(0)\frac{s^3}{3!} + G^{(4)}(0)\frac{s^4}{4!}.$$

Let us write the partial derivatives of $B(t,s)$ at the origin as $B_{i,j} = \left.\frac{\partial^{i+j}G(t,s)}{\partial t^i \partial s^j}\right|_{t,s} = (0,0)$

$$\begin{aligned} B(t,u) &= B(0,0) + B_t(0,0)t + B_u(0,0)u + \left(\frac{1}{2}\right) B_{tt}(0,0)t^2 + B_{tu}(0,0)ut \\ &+ \left(\frac{1}{2}\right) B_{uu}(0,0)u^2 + \left(\frac{1}{6}\right) B_{ttt}(0,0)t^3 + \left(\frac{1}{2}\right) B_{ttu}(0,0)t^2u \\ &+ \left(\frac{1}{2}\right) B_{tuu}(0,0)tu^2 + \left(\frac{1}{6}\right) B_{uuu}(0,0)u^3. \end{aligned}$$

Now, by imposing these conditions $B(0,0) = 1, B_t(0,0) = 1, B_u(0,0) = 1, B_{tt}(0,0) = 6, B_{tu}(0,0) = 0, B_{ttt}(0,0) = 60; G(0) = 0, G'(0) = 2$, in the above equation, we get possible order of convergence is four and by using it, we can get

$$\begin{aligned} x_{n+1} = \alpha + &\left(\left(\frac{1}{2}\right) c_2 c_3^3 B_{uu}(0,0) - \left(\frac{1}{2}\right) c_2^3 c_3^2 B_{ttu}(0,0)\right. \\ &- \left(\frac{1}{24}\right) A^{(4)}(0) c_2^5 c_3 - c_2^2 c_3 c_4 - f'(\alpha)^2 c_2^4 c_3 - c_2 c_3^3 \\ &\left. -7c_2^5 c_3 + 8c_2^3 c_3^2\right) e^8 + O\left(e^9\right). \end{aligned} \tag{2.14}$$

□

3. Numerical comparisons

In order to prove the accuracy of those proposed methods, it is important to numerically compare them with other similar studied methods. The effectiveness of those proposed methods are compared with some existing methods. Here we denote eighth order derivative free method (Eq:9) in [13], which is denoted by SKHM

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[w_n, x_n]} \\ z_n = y_n - \dfrac{f(y_n)}{g(x_n)} \dfrac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)} \\ x_{n+1} = z_n - \dfrac{f(z_n)}{c_2 - c_1 c_4} \end{cases},$$

where

$$\begin{aligned} c_1 &= f(z_n) \\ c_2 &= f[y_n, z_n] - c_3(y_n - z_n) + c_4 f(y_n) \end{aligned}.$$

$$c_3 = f[y_n, z_n, w_n] + c_4 f[y_n, w_n]$$
$$c_4 = \frac{f[y_n, z_n, x_n] - f[y_n, z_n, w_n]}{f[y_n, w_n] - f[y_n, x_n]}$$
$$f'(x_n) \approx g(x_n) = f[w_n, x_n] + 2(w_n - x_n) f[w_n, x_n, y_n] - f[y_n, w_n] + f[x_n, y_n].$$

Here

$$w_n = x_n + f(x_n), f[x_n, y_n] = \frac{f(x_n) - f(y_n)}{x_n - y_n}$$

and

$$f[w_n, x_n, y_n] = \frac{f[w_n, x_n] - f[x_n, y_n]}{w_n - y_n},$$

and method (Eq:8) [13], which is denoted by SKHM2

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[w_n, x_n]}, \\ z_n = y_n - \dfrac{f(y_n)}{g(x_n)} \dfrac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)(m_1 + m_2 + m_3)}{m_1 f[w_n, x_n] + m_2 f[y_n, x_n] + m_3 f[z_n, x_n]}, \end{cases}$$

where

$$m_1 = f(y_n) f(z_n)(z_n - y_n),$$
$$m_2 = f(w_n) f(z_n)(w_n - z_n),$$
$$m_3 = f(w_n) f(y_n)(y_n - w_n).$$

Then, we also compare our proposed eighth order derivative free methods with the similar scheme given in [7], which is denoted by TOM1

$$w_n = x_n + \beta f(x_n),$$
$$y_n = x_n - \left(\frac{f(x_n)^2}{f(w_n) - f(x_n)} \right),$$
$$z_n = y_n - \left(\frac{f[w_n, x_n] f(y_n)}{f[x_n, w_n] f[w_n, y_n]} \right),$$
$$x_{n+1} = z_n - \left(1 - \frac{f(z_n)}{f(w_n)}\right)^{-1} \left(1 + \frac{2 f(y_n)^3}{f(w_n)^2 f(x_n)}\right) \cdot \left(\frac{f(z_n)}{f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n]} \right),$$

and given by in [7], which is denoted by KTM

$$\begin{aligned} w_n &= x_n + \beta f\left(x_n\right), \\ y_n &= x_n - \left(\frac{f\left(x_n\right)^2}{f\left(w_n\right) - f\left(x_n\right)}\right), \\ z_n &= y_n - \left(\frac{f\left(x_n\right) f\left(w_n\right)}{f\left(y_n\right) - f\left(x_n\right)}\right)\left[\frac{1}{f[w,x]} - \frac{1}{f[w,y]}\right], \\ x_{n+1} &= z_n - \left(\frac{f\left(w_n\right) f\left(x_n\right) f\left(y_n\right)}{f\left(z_n\right) - f\left(x_n\right)}\right) \\ &\quad \times \left\{\left(\frac{1}{f\left(z_n\right) - f\left(w_n\right)}\right)\left[\frac{1}{f[y,z]} - \frac{1}{f[w,y]}\right]\right. \\ &\quad \left. - \left(\frac{1}{f\left(y_n\right) - f\left(x_n\right)}\right)\left[\frac{1}{f[w,y]} - \frac{1}{f[w,x]}\right]\right\} \end{aligned}$$

The previous methods have been compared with our following methods according to the examples listed in Table 1.

4. Method 1 (LM1)

If the functions $A(t), G(s)$ and $B(t,u)$ are define by:

$$\begin{aligned} A(t) &= \cos(t) + t + \frac{7}{2}t^2 + 9t^3, \\ G(s) &= \sin(s) + s, \\ B(t,u) &= \cos(tu) + t + 3t^2 + 10t^3 + u, \end{aligned}$$

satisfy the conditions in (2.2), then

$$\begin{aligned} w_n &= f(x)^2 + x_n, \\ y_n &= x_n - \frac{f\left(x_n\right)}{f\left[x_n, w_n\right]}, \\ z_n &= y_n - f\left(y_n\right)\frac{\cos\left(t_n\right) + t + \frac{7}{2}t_n^2 + 9t_n^3}{f\left[y_n, w_n\right]}, \\ x_{n+1} &= z_n - \frac{f\left(z_n\right)\left(\cos\left(t_n u_n\right) + t_n + 3t_n^2 + 10t_n^3 + u_n + \sin\left(s_n\right) + s_n\right)}{f\left[z_n, w_n\right]}, \end{aligned}$$

$n = 0, 1, 2, \ldots$.

5. METHOD 2 (LM2)

If the functions $A(t), G(s)$ and $B(t,u)$ are define by:

$$A(t) = \exp(t) + \frac{53}{3}t^3 + \frac{5}{2}t^2,$$
$$G(s) = \exp(\mathrm{s}) + \mathrm{s} - 1,$$
$$B(t,u) = \exp(\mathrm{tu}) + t + u + 3t^2 + 10t^3 - tu,$$

satisfy the conditions in (2.2), then

$$w_n = f(x)^2 + x_n$$
$$y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}$$
$$z_n = y_n - f(y_n)\frac{\exp(t_n) + \frac{53}{3}t_n^3 + \frac{5}{2}t_n^2}{f[y_n, w_n]},$$
$$x_{n+1} = z_n - \frac{f(z_n)\left(\exp(t_n u_n) + t_n + u_n + 3t_n^2 + 10t_n^3 - t_n u_n + \exp(\mathrm{s}_n) + \mathrm{s}_n - 1\right)}{f[z_n, w_n]}.$$

$n = 0, 1, 2, \ldots$.

The functions of the test and their exact roots are listed in Table 1. We have used as stopping criteria that $|x_n - x_{n-1}| \leq 10^{-15}$ and $|f_n(x)| \leq 10^{-15}$. TABLES 2 and 3 presents the number of iterations to approximate the zero (IT) and contains the value of $|f_n(x)|, |x_n - x_{n-1}|$ and COC. The values of the computational order of convergence COC may be approximated as (1.1), we have observed that our proposed methods perform better when compared with other similar existing eighth order methods. The illustrative numerical results demonstrate that they agree with the theoretical results obtained in Theorem 2.1.

TABLE 1. Function and their exact roots

Function	Exact Roots
$f_1(x) = \sin(3x) + x\cos(x)$	1.197769535217117
$f_2(x) = \exp(\sin(x)) - x + 1$	2.630664147927904
$f_3(x) = \frac{1}{10}(5\cos(2x) + 5 - 2x$	1.085982678007472
$f_4(x) = a\tan(x)$	$4.588036768824585\mathrm{e} - 3256$
$f_5(x) = x^2 - (1-x)^{25}$	0.143739259299754
$f_6(x) = x^3 + \log(1+x)$	$3.29048930341427\mathrm{e} - 1009$

TABLE 2. Comparison of numerical results for different derivative free methods

Methods	LM1	TOM1	KTM
IT	4	4	4
$\|f_n(x)\|$	$1.7012\text{e}{-}1008$	$-1.7012\text{e}-1008$	$1.7012\text{e}{-}1008$
$\|x_n - x_{n-1}\|$	$1.15651\text{e}{-}384$	$6.53836\text{e}{-}479$	$1.28117\text{e}{-}482$
COC	8	8	8
$f_4(x) = a\tan(x)$,	$x_0 = -0.5$		
Methods	LM1 1	TOM1	KTM
IT	4	4	4
$\|f_n(x)\|$	$-4.58804\text{e}-3256$	0	$-6.13231\text{e}-1403$
$\|x_n - x_{n-1}\|$	$3.93565\text{e}{-}450$	$4.52535\text{e}{-}328$	$1.35884\text{e}{-}395$
COC	8	11	11
$f_5(x) = x^2 - (1-x)^{25}$,	$x_0 = 0.35$		
Methods	LM1	TOM1	KTM
IT	5	5	5
$\|f_n(x)\|$	$-3.54417\text{e}-1009$	$7.97438\text{e}-1009$	$9.30345\text{e}-1009$
$\|x_n - x_{n-1}\|$	$3.34248\text{e}{-}312$	$4.14351\text{e}{-}663$	$2.28469\text{e}{-}734$
COC	8.00002	8	8
$f_6(x) = x^3 + \log(1+x)$, $x_0 = 0.25$			
Methods	LM1	TOM1	KTM
IT	4	4	
$\|f_n(x)\|$	$3.56272\text{e}-3026$	$-9.10251\text{e}-3029$	Division by zero
$\|x_n - x_{n-1}\|$	$2.93636\text{e}{-}309$	$8.46255\text{e}{-}389$	
COC	8.00001	8	

TABLE 3. Comparison of numerical results for different derivative free methods

$f_1(x) = \sin(3x) + x\cos(x)$,	$x_0 = 1$		
Methods	LM2	SKHM	SKHM2
IT	4	4	4
$\lvert f_n(x)\rvert$	$7.79717\mathrm{e}-1008$	$7.79717\mathrm{e}-1008$	$7.79717\mathrm{e}-1008$
$\lvert x_n - x_{n-1}\rvert$	$2.22754\mathrm{e}{-333}$	$1.2117\mathrm{e}-376$	$2.06062\mathrm{e}{-431}$
COC	8	8	8
$f_2(x) = \exp(\sin(x)) - x + 1$,	$x_0 = 2.3$		
Methods	LM2	SKHM	SKHM2
IT	4	4	4
$\lvert f_n(x)\rvert$	$-8.60172\mathrm{e}-572$	$5.67067\mathrm{e}-1008$	$5.67067\mathrm{e}-1008$
$\lvert x_n - x_{n-1}\rvert$	$2.49632\mathrm{e}{-496}$	$7.29951\mathrm{e}{-440}$	$3.24786\mathrm{e}{-490}$

6. CONCLUSION

In this paper, we have created of two class of optimal derivative-free methods of order eight. It is proven by convergence analysis that the new derivative-free methods maintain their order of efficiency index and convergence. One of the advantages of those three-step class of optimal derivative-free methods of order eight is that they are highly efficient when used with derivatives of high computational cost as those methods do not have to evaluate the functions derivatives. Another advantage of those new methods is that they give better approximation of exact root when compared to other methods.

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