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ASYMPTOTIC NORMALITY OF L1-APPROACH A KERNEL ESTIMATOR OF CONDITIONAL CUMULATIVE DISTRIBUTION FUNCTION IN THE FUNCTIONAL SINGLE INDEX MODEL

Abdelkader Bouadjemi[1], Maroua Mebarki, and Abdelkrim Bouadjemi

ABSTRACT. The main of this paper is to treat the estimation of conditional distribution function for functional data. We defined the L1 norm estimator. Under some assumption in functional data analysis the asymptotic normality of estimator is established.

1. INTRODUCTION

In the digital age a whole variety of new scientific and social technical challenges have arisen, one of them being the study of functional data the problem of nonparametric estimation of this type of data has become a major concern and constitutes a current line of research that follows a craze from the community of statisticians, hence the extent of recent publications.

This frank success potentially results from the theoretical development and the multiple possibilities of application (imagery, econometrics, recognition of form,...). This dynamic around data of a functional nature has been popularized by the monograph by Ramsay (2002) which draws up an almost exhaustive

[1]*corresponding author*
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assessment of statistical methods for functional variables mainly from the practical point of view. We should also note that Ferraty and Vie (2006) have made a significant contribution to the study of non-parametric statistical methods which deal with various non-parametric estimation problems involving functional random variables.

In this context, the literature encompassing the study of the conditional distribution function has grown considerably in recent decades. It is necessary to specify that the first results obtained in the non-parametric framework are those of Ferraty et al (2006). One of the first papers in this perspective is that of Ezzahie and Ould said (2008a, 2008b) who studied asymptotic normality in both cases (i.i.d and α-mixing). Uniform convergence results of this estimator have been established by Laksaci and Maref (2009). For the robustness part, we refer to Azzedine et al 2008 and Attouch et al 2009. A pioneering work that was highlighted by Lakasaci et al (2009,2011) developing an alternative robust estimator method. For the recursive approach of our function we refer to Bouadjemi 2014, 2017 and Benziad et al. 20116. For recent advances in the field see Bouadjemi 2021 and Bouanani 2020.

In recent years the single-index model had been used to reduce the dimensionality of the explanatory variable while retaining the benefits of nonparametric smoothing. Moreover, these ideas had first been extended to the functional framework by Ferraty and al 2003 for the problem of functional regression. Ait Saidi et al. 2008 proposed to estimate the unique unknown functional index via the cross-validation technique. The estimation of the conditional density function of a real response variable Y given a Hilbertian explanatory variable has been studied by Attaoui 2011. Ling and LI 2014 extended these last results to the mixed case. Recently, Attaoui and Ling 2016 studied the nonparametric estimation of the conditional cumulative distribution function for the single functional index model. In this perspective, our paper aims to establish the asymptotic normality of the distribution function studied by Laksaci 2009 in the i.i.d case when the unique functional index.

The rest of the paper is organized as follows section 2. Introduced the L1-approach a kernel estimator of conditional cumulative distribution function in the functional single index model. Some assymptions and the main asymptotic results are established in section 3. The detailed proof of our main results are presented in appendix.

2. The functional model and its etimation

Let $\{(X_i, Y_i), 1 \leqslant i \leqslant n\}$ be n random variables independent and identically distributed as the random pair (X, Y) with values in $H \times \mathrm{R}$, where H is a separable real hilbert space with scalar product $< .,. >$ and its norm $\|.\| =< .,. >^{1/2}$, $\theta = \theta(t)$ is a functional single index valued in a sepzrable hilbert space H, we suppose that the conditional probability distribution of Y given $< X, \theta >=< x, \theta >$ exists and is given by:

$$\forall y \in \mathbb{R}, \quad F(\theta, y, x) = \mathbb{P}(Y \leqslant y| < X, \theta >=< x, \theta >).$$

The functional index θ acts a filter permitting the extraction of the part of X explaining the response Y, and plays an important role in a such model. We consider the semi-metric d_θ associated to the single $\theta \in H$ defined by $\forall x_1, x_2 \in H, d_\theta(x_1, x_2) := | < x_1 - x_2, \theta > |$.

In the FDA setup, there are several ways for estimate $F(\theta, y, x)$ by L1- norm approach method proposed by Laksaci and al (2009). In following, we denote by $F(\theta, y, x)$ the conditional distribution function of Y given $< x, \theta >$ and we define the kernel estimator for single index structure $\hat{\mathrm{F}}(\theta, y, x)\ of\ F(\theta, y, x)$ by:

$$\frac{\sum_{j=1}^{n} 1_{\{Y_i \leq y\}} K(h_k^{-1} d_\theta(x, X_i))}{\sum_{j=1}^{n} K(h_k^{-1} d_\theta(x, X_i))},$$

where K is a kernel, $\nVdash_A$ denote the indicator function on the set A and $h_k := h_{k,n}$ is a sequence of positif real numbers which goes to zero as n tends to infinity.

Notice that in this contribution we combine L1-norm approach and kernel estimator for single index structure.

3. Assumption and Main results

Throughout this paper, when no confusion is possible, we will denote by C o $C_{\theta,x}$ Some strictly positive generic constants.

We denote by $B(\chi, h_k) := \{\chi \in \mathcal{H} : | < \chi - x, \theta > | \leqslant h_k\}$, the ball centered at x, with radius h_k. Let $\mathcal{N}_x$ be a fixed neighborhood of $x \in \mathcal{H}$; S_R with a fixed copact subset of $\mathcal{R}$.

Our nonparametric model will be quite general in the sense that we will just need the following assumption:

(H1) $\mathbb{P}(X \in B_\theta(x, h_k)) = \phi_{\theta,x}(h) > 0$; and there exist a function $\beta_\theta(x)$

$$\forall t \in [0,1] \ \lim_{n\to\infty} \frac{\phi_x(ta_n)}{\phi_x(a_n)} = \beta_x(t).$$

(H2) For all $y \in \mathbb{R}$ $\forall(x_1, x_2) \in N_x^2$,

$$|F^{x_1}(t_1) - F^{x_2}(t_2)| \leq C \left(d(x_1,x_2)^{\beta_1} + |t_1 - t_2|^{\beta_2}\right),$$

with $C > 0, \beta_1 > 0,\ \beta_2 > 0$ and N_x is a fixed neighborhood of x.

(H3) The bandwidths (h_k, h_H) satisfy: $\lim_{n\to\infty} h_k = 0$ $\lim_{n\to\infty} h_H = 0$ and $\lim_{n\to\infty} n h_k^2 \phi_{\theta,x}(h_k) = 0$.

(H4) The kernel K from $\mathcal{R}$ into $\mathcal{R}^+$ is a differentiable function supported on $[0,1]$. Its derivative K' such that

$$\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds\right) > 0.$$

(H5) H has even bounded derivative verifies

$$\int_{\mathbb{R}} |t|^{\beta_2}\, H'(t)dt < \infty.$$

Comments on the assumptions. Note that these assumptions are not very restrict assumptions (H1) simply characterizes of the classical concentration property for the explanatory variable. Similar to the discussion by Ferraty and Vieu (2006), (H2) nd (H3) are the quite usual condition on the kernel function for nonparametric FDA. Assumption (H4) will play a major role in our results and it intervened to compute the exact constant terms involved in our assumption expansions see for instance Ferraty and Vieu (2007). Assumptions (H4 ii) are technical and permit to give an explicit assumption variance and will be used to remove the bias term in the asymptotic normality result.

3.1. **Main Results.** This part contains results on the asymptotic normality of the conditional cumulative distribution function in the single functional index midel.

Before announcing our main results, we introduce the quantity M, Which will appear in the variance terms:

$$M_j = K^j(1) - \int_{-1}^{1} (K^j(u))\varphi(u)du, \quad j = 1, 2.$$

Theorem 3.1. *Under Assumption (H1)-(H5), we have for any $x \in \mathcal{H}$,*

$$(n\phi_{\theta,x}(h_k))^{1/2} \left(\hat{F}(\theta, x) - F(<\theta, x>)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \text{ as } n \to \infty,$$

where $\sigma^2 = \dfrac{(F^x(y,\theta))^2 M_2}{M_1^2}$ *and* $\xrightarrow{\mathcal{D}}$ *means the convergence in distribution.*

Proof. For $i = 1, \ldots, n.$, we consider the quantities $k_i(\theta, x) := k(h_k{-}1(< x - \chi, \theta >))$, and let $\hat{F}_N(\theta, y, x)$ (resp.$\hat{F}_D(\theta, x)$ be defined as

- $\hat{F}_N(\theta, y, x) = \dfrac{1}{nE[k_1(\theta, x)]} \sum_{i=1}^{n} k_i(\theta, x) 1_{\{Y \leqslant y\}}$
- $\hat{F}_D(\theta, x) = \dfrac{1}{nE[k_1(\theta, x)]} \sum_{i=1}^{n} k_i(\theta, x)$.

This proof is based on the following decomposition

$$\hat{F}(\theta, x) - F(<\theta, x>) = \frac{Q(\theta, x) - B(\theta, x)(\hat{F}_D(\theta, x) - \mathbb{E}[\hat{F}_D(\theta, x)])}{\hat{F}_D(\theta, x)} + B(\theta, x).$$

Here

$$B(\theta, x) = \frac{\mathbb{E}[\hat{F}_N(\theta, y, x)] - F(<\theta, x>)\mathbb{E}[\hat{F}_D(\theta, x)]}{\mathbb{E}[\hat{F}_D(\theta, x)]}$$

and

$$Q(\theta, x) = \hat{F}_N(\theta, y, x) - \mathbb{E}[\hat{F}_N(\theta, y, x)] - F(<\theta, x>) \left[\hat{F}_D(\theta, x) - \mathbb{E}[\hat{F}_D(\theta, x)]\right].$$

Therefore, theorem 3.1 is obtained by the following some Lemmas. □

Lemma 3.1. *(Attaoui and Ling 2015) Under Assumption (H1)-(H5), we have for any $x \in \mathcal{H}$,*

$$\hat{F}_D(\theta, x) \xrightarrow{D} 1, \quad n \to \infty. \tag{3.1}$$

Lemma 3.2. *Under Assumption (H1)-(H5), we have for any $x \in \mathcal{H}$,*

$$B(\theta, x) = o(h_k^{\beta}), \quad n \to \infty. \tag{3.2}$$

Lemma 3.3. *Under Assumption (H1)-(H5), we have for any $x \in \mathcal{H}$,*

$$(n\phi_{\theta,x}(h_k))^{1/2} (Q(\theta, x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2), \quad n \to \infty. \tag{3.3}$$

Here $\sigma^2 = \frac{(F^x(y,\theta))^2 M_2}{M_1^2}$.

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Department of Mathematics,
Amine Elokkal El Hadj Moussa University of Tamanerasset
Tamanerasset,
Algeria.
Email address: bouadjemi@gmail.com

Department of Mathematics,
Amine Elokkal El Hadj Moussa University of Tamanerasset
Tamanerasset,
Algeria.
Email address: maroua.mebarki@yahoo.fr

Department of Computer Science
University of Relizane
Algeria.
Email address: abdelkrimbouadjemi@yahoo.fr