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VARIATIONAL ANALYSIS OF A VISCOELASTIC FRICTIONAL CONTACT WITH LONG-TERM MEMORY BODY WITH THERMAL EFFECTS

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ABSTRACT. In this article we study a mathematical model which describes the quasi-static process of contact between a piezoelectric body with long-term memory and an obstacle. The contact is modeled with a normal conformity condition and a version of Coulom's law. The evolution of temperature is described by a first kind evolution equation. The problem is formulated as a system of scalable elliptical variational inequalities for displacement, and a variational equality for electrical stress. We prove the existence of a unique weak solution to the problem. The proof is based on arguments from time-dependent variational inequalities, differential equations and fixed point.

1. INTRODUCTION

Due to the importance of contact processes in structural and mechanical systems, considerable progress has been made recently in mathematical modeling and analysis and numerical simulations and, therefore, the technical literature on this subject is quite abundant [1–3] . Contact and friction phenomena are increasingly taken into account in industrial issues. Engineering of wheel-in-rail contact,

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in the modeling of prosthetic medical devices, mechanical assemblies, formulation, etc. This requires having robust, reliable and precise resolution and analysis tools. The mechanical contact presents the most difficult nonlinearities to take into account. There are different contact resolution methods. The main idea is to link abstract results and models for different body types in order to apply these results in practice. We present a variant formulation of the problem for which we prove the existence and the uniqueness of the solution with respect to the data and the parameters. In most systems of continuum mechanics, there are situations in which a deformable body comes into contact with other bodies or with a rigid or deformable foundation.

The problem of the contact is primarily to know how the efforts are applied on a structure and how these structures react when they undergo these efforts. Mechanical contact problems are mainly found in fields as varied as aeronautics, mechanics, civil engineering and medicine. Taking into account the behavior of continuous media includes elasticity, plasticity, etc. Given the importance of the phenomenon, considerable efforts have been devoted to modelling. The general mathematical theory of contact mechanics emerged. He is interested in the mathematical structures which are at the origin of the problems of contact with different laws of behavior, see [1,2].

We study a new constitutive law called thermo-viscoelastic material with long-term memory given by

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{B}\varepsilon(u(t)) + \int \mathcal{G}(t-s, \varepsilon(u(s)))\, ds + \zeta^{*}\nabla\varphi(t) - \mathcal{M}\theta(t), \tag{1.1}$$

$$\mathbf{D}(t) = \zeta\varepsilon(u(t)) - \mathbf{B}\nabla\varphi(t) - \mathcal{P}\theta(t), \tag{1.2}$$

Here $\mathcal{A}$ and $\mathcal{B}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{G}$, $\mathrm{E}(\varphi) = -\nabla\varphi$, $\zeta = (e_{i,j,k})$, $\mathcal{M}$, B, $\mathcal{P}$ are respectively relaxation operator, electric field, piezoelectric, thermal expansion, electric permittivity pyroelectric tensors, and ζ is transpose of ζ. Note also that when $\zeta = 0$ and $D = 0$, (1.1)-(1.2) becomes the Kelvin-Voigt thermo-viscoelastic with long memory constitutive relation used in [4]. Moreover, when $\mathcal{M} = 0$ and $\mathcal{P} = 0$, the relations (1.1)-(1.2) becomes the Kelvin-Voigt electro-viscoelastic with long memory constitutive relation used in [5].

We use an evolution of the temperature field obtained from the conservation of energy and defined with the following differential equation

$$\dot{\theta}\,(t) - \mathrm{Div}\,(\mathcal{K}(\theta\,(t))\,) = \psi(\mathcal{M}\theta\,(t)\,,\dot{u}\,(t) + \mathrm{q}_{th}) \tag{1.3}$$

Here θ is the temperature, $\mathcal{K}$ denotes the thermal conductivity tenser, $\mathcal{M}$ the thermal expansion tensor, q_{th} is the density of volume heat sources and ψ is a nonlinear function, assumed here depends on thermal expansion tensor and the velocity.

The Coulomb friction is one of the most useful friction laws and known from the literature. This law has two basic ingredients mamely the concept of friction threshold and its dependence on the normal stress.

The normal compliance law is a contact law allowing penetration into the foundation, considered to be deformable. Various versions of the normal compliance law were recently presented in the literature [6], [7], [8], [9].

In this paper, we use the thermal and the mechanical contacts and we neglect the electrical contact for some reason used in materials such as car battery there is no electric field between the battery and the sheet metal.

2. The model

The physical setting is the following. An thermo-electro-viscoelastic body with long term memory occupies a bounded domain $\Omega \ \subset \mathbb{R}^d$ $(d = 2, 3)$ with outer Lipschitz surface Γ. This boundary is divided into three open disjoint Γ_1, Γ_2 and Γ_3, on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$. Let T> 0 and let $[0, T]$ be the time interval of interest. The body is subjected to the action of body forces of desity f_0, a volume electric charges of density q_0 and heat source of constant strength q_{th}. The body is clamped on $\Gamma_1\times\ [0, T]$, so the displacement field vanishes there. A surface traction of density f_2 act on $\Gamma_2\times\ [0, T]$. We also assume that the electrical potential vanishes on $\Gamma_a\times\ [0, T]$ and a surface electric charge of density q_b is prescribed on $\Gamma_b\times\ [0, T]$. Moreover, we suppose that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2)\times\ [0, T]$. Moreover, we suppose that the body forces and tractions vary slowly in time, and therefore the accelerations in the systemmay be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approach to the process.

The classical formulation of the mechanical problem is as follows.

Problem $\mathcal{P}$: Find the diplacement field u : $\Omega \times [0,T] \to \mathbb{R}^d$, the stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$, the electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, the electric displacement field D: $\Omega \times [0,T] \to \mathbb{R}^d$ and the temperature $\theta : \Omega \times [0,T] \to \mathbb{R}$ such that

(2.1)
$$\sigma(t) = \mathcal{A}\left(\varepsilon\left(\dot{\mathbf{u}}(t)\right)\right) + \mathcal{B}\varepsilon(\mathbf{u}(t)) + \int \mathcal{G}(t-s, \varepsilon(\mathbf{u}(s)))\, \mathrm{ds} + \zeta^* \nabla\varphi(t) - \mathcal{M}\theta(t) \quad \text{in } \Omega \times [0,T]$$

(2.2)
$$\mathbf{D}(t) = \zeta\varepsilon(\mathbf{u}(t)) - \mathbf{B}\nabla\varphi(t) - \mathcal{P}\theta(t) \quad \text{in } \Omega \times [0,T]$$

(2.3)
$$\dot{\theta}(t) - Div\left(k(\theta(t))\right) = \psi(\mathcal{M}\theta(t), \dot{u}(t) + q_{th}) \quad \text{in } \Omega \times [0,T]$$

(2.4)
$$Div\sigma + \mathbf{f}_0 = 0 \quad \text{in } \Omega \times [0,T]$$

(2.5)
$$Div\mathbf{D} = \mathrm{q}_0 \quad \text{in } \Omega \times [0,T]$$

(2.6)
$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times [0,T]$$

(2.7)
$$\sigma\nu = \mathrm{f}_2 \quad \text{on } \Gamma_2 \times [0,T]$$

(2.8)
$$\begin{cases} \sigma_\nu = -\alpha \mid \dot{\mathbf{u}}_\nu \mid, \mid \sigma_\nu \mid = -\mu\sigma_\nu \\ \sigma_\tau = -\lambda(\dot{\mathbf{u}}_\tau - \mathbf{v}^*), \ \text{ if } \lambda \geqslant 0 \end{cases} \text{on } \Gamma_3 \times [0,T]$$

(2.9)
$$-\mathrm{k}_{i,j}\theta_i n_j = k_e(\theta(t) - \theta_F) \quad \text{on } \Gamma_3 \times [0,T]$$

(2.10)
$$\mathbf{D}.\nu = \Psi(u_\nu - g)\Phi_l(\varphi - \varphi_0) \quad \text{on } \Gamma_3 \times [0,T]$$

(2.11)
$$\theta = 0 \quad \text{on } (\Gamma_1 \cup \Gamma_2) \times [0,T]$$

(2.12)
$$\varphi = 0 \quad \text{on } \Gamma_a \times [0,T]$$

(2.13)
$$\mathbf{D}.\nu = q_b \quad \text{on } \Gamma_b \times [0,T]$$

(2.14)
$$\mathbf{u}(0) = \mathbf{u}_0 \quad \theta(0) = \theta_0 \quad \text{in } \Omega$$

We now describe problem (2.1)-(2.14) and provide explanation of equations and the boundary conditions.

Equations (2.1) and (2.2) represent the thermo-electro-viscoelastic constitutive law with long term memory, the evolution of the temperature field is governed by differential equation given by the relation (2.3) where ψ is the mechanical source of the temperature growth, assumed to be rather general function of the strains. Next equations (2.4) and (2.5) are the steady equations for the stress and electric-displacement field, conditions (2.6) and (2.7) are the displacement and traction boundary conditions. Equation (2.11) means that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2)\times\ [0,T]$ which implies that there is only an electro-mechanical effect on $(\Gamma_1 \cup \Gamma_2)$. Next, (2.12) and (2.13) represent the electric boundary conditions for the electrical potential on Γ_a and the electric charges on Γ_b, respectively. Equation (2.14) represents the initial displacement field and the initial damage field where u_0 is the initial displacement, and θ_0 is the initial temperature.

We turn to the contact conditions (2.8)-(2.10) describe the frictional thermo-mecanical contact on the potential contact surface Γ_3. The relation (2.8) describes a normal compliance conditions with the Coulomb's law of dry friction where p_ν is a prescribed function, and g represents the gap in direction ν. The difference $u_\nu - g$, when positive, represents the penetration of the surface asperities into those of the foundation. Moreover the last two inequalities in the relation (2.8) describe Coulomb's law of dry friction. The equation (2.9) represents an associated temperature boundary condition on contact surface, where k_e is a heat exchange coefficient between the body and the obstacle. and θ_F is the temperature of the foundation. Finally, the equation (2.10) shows that there are no electric charges on the contact surface. We not here for the condition (2.8) we choose the following version of the normal compliance

$$p_\nu = \mu p_\tau.$$

This choice can be found often in the literature, here μ is the coefficient of friction and p_τ is called tangential compliance.

3. VARIATIONAL FORMULATION

In order to obtain the variational formulation of the problem $\mathcal{P}$, we use the following notations and preliminaries.

3.1. **Notations and preliminaries.**

We present the notation we recall some preliminary material. For more details, we refer the reader to [10], [11] .We recall that the canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively are given by

$$\mathbf{u}.\mathbf{v} = u_i.\upsilon_i, \quad \| \mathbf{v} \| = \sqrt{\mathbf{v}.\mathbf{v}} \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^d,$$
$$\sigma.\tau = \sigma_{ij}.\tau_{ij}, \quad \| \tau \| = \sqrt{\tau.\,\tau} \quad \text{for all} \quad \sigma, \tau \in \mathbb{S}^d,$$

We introduce the spaces

$$H = L^2(\Omega)^d = \left\{ \mathbf{v} = (\upsilon_i) \ : \upsilon_i \in L^2(\Omega) \right\},$$
$$\mathcal{H} = \left\{ \tau = \tau_{ij} \ : \ \tau_{ij} = \tau_{ji} \in L^2(\Omega) \right\},$$
$$H^1(\Omega)^d = \left\{ \mathbf{v} = (\upsilon_i) \in H \ : \varepsilon(\mathbf{v}) \in \mathcal{H} \right\},$$
$$\mathcal{H}_1 = \left\{ \tau \in : Div\tau \in H \right\}.$$

Here ε: $H^1(\Omega)^d \to$ and $Div : \mathcal{H} \to H$ are the linearized deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathrm{v}) = (\varepsilon_{ij}(\mathrm{v})), \qquad \varepsilon_{ij}(\mathrm{v}) = \frac{1}{2}(\upsilon_{i,j} + \upsilon_{j,i}), \qquad \mathrm{Div}\tau = (\tau_{ij,j}).$$

The spaces H, $\mathcal{H}$, $H^1(\Omega)^d$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u}, \mathbf{v})_H = \int_\Omega u_i.\upsilon_i \, dx, \qquad (\sigma.\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}.\tau_{ij} \, dx,$$
$$(\mathbf{u}, \mathbf{v})_{H^1(\Omega)^d} = (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \ \varepsilon(\mathbf{v}))_{\mathcal{H}},$$
$$(\sigma.\tau)_{\mathcal{H}} = (\sigma.\tau) + (\ Div\sigma, \ Div\tau)_H.$$

We introduce the closed subspase of $H^1(\Omega)^d$ defined by

$$\mathrm{V} = \left\{ \mathbf{v} \ \in H^1(\Omega)^d \ : \mathbf{v} = \mathbf{0} \ \ on \ \Gamma_1 \right\},$$

we also introduce the spaces

$$W_e = \left\{ \phi \in H^1(\Omega)^d \ : \phi = 0 \ on \ \Gamma_a \right\},$$
$$\mathcal{W}_e = \left\{ \mathbf{D} = (\mathbf{D}_i) \ : \ \mathbf{D}_i \in L^2(\Omega), \ Div\mathbf{D} \in L^2(\Omega) \right\},$$
$$W_{th} = \left\{ \mathbf{w} \in H^1(\Omega) \ : \ \mathbf{w} = \mathbf{0} \ a.e \ on \ (\Gamma_1 \cup \Gamma_2) \right\}.$$

Since $meas\Gamma_a > 0$ and $meas\Gamma_1 > 0$, the Korn's and Friedrichs-Poincaré inequalities holds, thus,

$$\| \varepsilon(\mathbf{v}) \|_{\mathcal{H}} \geqslant C_0 \| \mathbf{v} \|_{H^1(\Omega)^d}, \quad \forall \mathbf{v} \in V, \tag{3.1}$$

$$\| \nabla\phi \|_{\mathcal{W}_e} \geqslant C_1 \| \phi \|_{H^1(\Omega)}, \quad \forall \phi \in W_e, \tag{3.2}$$

$$\| \nabla\mathbf{w} \|_{H} \geqslant C_2 \| \mathbf{w} \|_{H^1(\Omega)}, \quad \forall \mathbf{w} \in W_{th}, \tag{3.3}$$

where here and below $C_0,\ C_1$ and C_2 are positive constants that depend on the problem data but is independents of the solutions, the value of which may change from line to line.

On the spaces $V,\ W_e, \mathcal{W}_e$ and W_{th} , we define the following inner products

$$(\mathbf{u}, \mathbf{v})_V = (\sigma, \varepsilon(\mathbf{v})_{\mathcal{H}}, \ \forall \mathbf{u}, \mathbf{v} \in V), \tag{3.4}$$

$$(\varphi,\ \phi)_{W_e} = (\nabla\varphi,\ \nabla\phi)_{\mathcal{W}_e}, \ \forall\varphi,\ \phi \in W_e, \tag{3.5}$$

$$(\mathbf{w}, \mathbf{z})_{W_{th}} = (\nabla\mathbf{w},\ \nabla\mathbf{z})_H, \quad \forall \mathbf{w}, \mathbf{z} \in W_{th}, \tag{3.6}$$

where

$$(\varphi,\ \phi)_{W_e} = \int_\Omega \nabla\varphi.\ \nabla\phi\ dx,$$
$$(\mathbf{D},\ \mathbf{E})_{w_e} = \int_\Omega \mathbf{D}.\mathbf{E}\ dx\ + \int_\Omega Div\mathbf{D}.Div\mathbf{E}\ dx.$$

There exists a constants $C_0,\ C_1$ and C_2 are positive constants, such that

$$\| \mathbf{v} \|_{L^2(\Gamma_3)\ ^d} \leqslant C_0 \| \mathbf{v} \|_V, \quad \forall \mathbf{v} \in V, \tag{3.7}$$

$$\| \phi \|_{L^2(\Gamma_3)} \leqslant C_1 \| \phi \|_W, \quad \forall \phi \in W_e, \tag{3.8}$$

$$\| \mathbf{z} \|_{L^2(\Gamma_3)} \geqslant C_2 \| \mathbf{z} \|_{W_{th}}, \quad \forall \mathbf{z} \in W_{th}, \tag{3.9}$$

and we denote by υ_ν and υ_τ the normal and tangential components of υ on Γ given by

$$\upsilon_\nu = \mathbf{v}.\nu\ , \qquad \mathbf{v}_\tau = \mathbf{v} - \upsilon_\nu \nu. \tag{3.10}$$

Similarly, we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \qquad \mathrm{v}_\tau = \sigma\nu - \sigma_\nu \nu, \tag{3.11}$$

for all $\sigma \in \mathcal{H}_1, \theta \in H^1(\Omega)^d$ and $D \in \mathcal{W}$, the following three Green's formulas holds:

$$(\sigma,\ \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\sigma, \mathbf{v}\ )_H = \int_\Gamma \sigma_\nu \mathbf{v}\ da, \qquad \forall \mathbf{v} \in H^1(\Omega)^d, \tag{3.12}$$

$$(\theta,\ \nabla \mathbf{w})_H\ + (Div\theta, \mathbf{w})_{L^2(\Omega)} = \int_\Gamma \theta\nu.\mathbf{w}\ da, \qquad \forall \mathbf{w} \in H^1(\Omega), \tag{3.13}$$

$$(\mathbf{D},\ \nabla\phi)_H\ + (Div\mathbf{D}, \phi)_{L^2(\Omega)} = \int_\Gamma \mathbf{D}\nu.\phi\ da, \qquad \forall\ \phi \in H^1(\Omega), \tag{3.14}$$

where

$$Div\theta = \theta_{i,i}\ , \qquad \mathrm{DivD} = (\mathrm{D}_{i,i}).$$

We recall the following Theorem

Theorem 3.1. *Let $V \subset H \subset V'$ be a Gelfand triple. Assume that $A : V \to V'$ is a hemicontinuous and monotone operator that satisfies*

$$(A\mathbf{v},\ \mathbf{v})_{V' \times V}\ \geqslant\ \omega \|\mathbf{v}\|_V^2\ + \varsigma, \quad \forall \mathbf{v} \in V, \tag{3.15}$$

$$\|A\mathbf{v}\|_{V'}^2\ \leqslant C(\|\mathbf{v}\|_V\ + 1), \quad \forall \mathbf{v} \in V. \tag{3.16}$$

For some constants $\omega > 0$, $C>0$ and $\varsigma \in \mathbb{R}$ then, given $u_0 \in H$ and $f \in L^2(0,\ T,\ V')$ there exist an unique function $u \in L^2(0,\ T,\ V') \cap C(0,\ T,\ H)$ satisfies

$$\mathbf{u} \in L^2(0,\ T,\ V') \cap\ \mathcal{C}(0,\ T,\ H),\ \dot{\mathbf{u}} \in L^2(0,\ T,\ V'),$$

$$\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = f\ (t) \quad a.e\ t \in (0,\ T),$$

$$\mathbf{u}(0) = \mathbf{u}_0.$$

The proof of this abstract result may be found in [12].

We denote by $\mathcal{C}(0,\ T;\ X)$ and $\mathcal{C}^1(0,\ T;\ X)$ the space of continuous and continuously differentiable functions from $[0,\ T]$ to X , respectively, with the norms

$$\|\mathbf{f}\|_{\mathcal{C}(0,\ T;\ X)} = \max_{t\in[0,\ T]} \|\mathbf{f}(t)\|_{\mathcal{C}(0,\ T;\ X)},$$

$$\|\mathbf{f}\|_{\mathcal{C}^1(0,\ T;\ X)} = \max_{t\in[0,\ T]} \|\mathbf{f}(t)\|_X + \max_{t\in[0,\ T]} \|\dot{\mathbf{f}}(t)\|_X.$$

If X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1 \times X_2}$.

3.2. **Assumptions on the problem's data.** The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$(3.17)\quad \begin{cases} \text{(a) There exists } L_{\mathcal{A}} > 0 \text{ such that,} \\ \qquad \|\mathcal{A}(\mathrm{x}, \varepsilon_1) - \mathcal{A}(\mathrm{x}, \varepsilon_2)\| \leqslant L_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\| \\ \text{for all } \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^{\mathrm{d}}, \text{a.e. } \mathrm{x} \in \Omega. \\ \text{(b) There exists } \mathrm{m}_{\mathcal{A}} > 0 \text{ such that} \\ \qquad (\mathcal{A}(\mathrm{x}, \varepsilon_1) - \mathcal{A}(\mathrm{x}, \varepsilon_2))/(\varepsilon_1 - \varepsilon_2) \geqslant \mathrm{m}_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|^2 \\ \text{for all } \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^{\mathrm{d}}, \text{ a.e. } \mathrm{x} \in \Omega. \\ \text{(c) The mapping } \mathrm{x} \mapsto \mathrm{A}(\mathrm{x}, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \text{for any } \varepsilon \in \mathbb{S}^d. \\ \text{(d) The mapping } \mathrm{x} \mapsto \mathrm{A}(\mathrm{x}, 0) \text{ belong to } \mathcal{H} \end{cases}$$

The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$(3.18)\quad \begin{cases} \text{(a) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \qquad \|\mathcal{B}(\mathrm{x}, \varepsilon_1) - \mathcal{B}(\mathrm{x}, \varepsilon_2)\| \leqslant L_{\mathcal{B}} \|\varepsilon_1 - \varepsilon_2\| \\ \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^{\mathrm{d}}, \text{ a.e. } \mathrm{x} \in \Omega. \\ \text{(b) The mapping } \mathrm{x} \mapsto \mathcal{B}(\mathrm{x}, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \text{for any } \varepsilon \in \mathbb{S}^{\mathrm{d}}. \\ \text{(c) The mapping } \mathrm{x} \mapsto \mathcal{B}(\mathrm{x}, 0) \text{ belong to } \mathcal{H}. \end{cases}$$

The relation operator $\mathcal{G} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$(3.19)\quad \begin{cases} (a) \text{ There exists a cons tan t } L_{\mathcal{G}} \succ 0 \text{ such that} \\ \quad \|\mathcal{G}(\mathrm{x}, \mathrm{t}_1,\ \varepsilon_1) - \mathcal{G}(\mathrm{x}, \mathrm{t}_2, \varepsilon_2)\| \leqslant L_{\mathcal{G}} \|\varepsilon_1 - \varepsilon_2\| \text{ for all } \mathrm{t}_1,\ \mathrm{t}_2 \in (0,\ \mathrm{T}), \\ \text{for all } \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^{\mathrm{d}}, \text{ a.e. } \mathrm{x} \in \Omega. \\ (b) \text{ The mapping } \mathrm{x} \mapsto \mathcal{G}(\mathrm{x}, \mathrm{t}, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \text{for any } \mathrm{t} \in (0, \mathrm{T}), \text{ for any } \varepsilon \in \mathbb{S}^d. \\ (c) \text{ The mapping } \mathrm{x} \mapsto \mathcal{G}(\mathrm{x}, \mathrm{t}, \varepsilon) \text{ is continuous, in } (0, \mathrm{T}), \\ \text{for any } \varepsilon \in \mathbb{S}^d, \text{ a.e. } \mathrm{x} \in \Omega. \\ (d) \text{ The mapping } \mathrm{x} \mapsto \mathcal{G}(\mathrm{x}, 0, 0) \in \mathcal{H}. \end{cases}$$

The piezoelectric operator $\varepsilon\ : \Omega \times \mathbb{S}^d \ \to \mathbb{S}^d$

$$(3.20)\qquad \begin{cases} (a)\ \varepsilon(\mathrm{x},\mathrm{t}) = (\mathrm{e}_{ijk}\tau_{jk})\ ,\forall\tau = (\tau_{jk}) \in \mathbb{S}^d\ ,\ \text{a.e. } \mathrm{x} \in \Omega. \\ (b)\ \mathrm{e}_{ijk}\tau_{jk} = \mathrm{e}_{ikj}\tau_{jk}\ \in L^\infty(\Omega)\ ,\ 1 \leqslant i,j,k \leqslant d. \end{cases}$$

The thermal expansion operator $\mathcal{M} : \Omega \times \mathbb{R} \to \mathbb{R}$

$$(3.21)\qquad \begin{cases} (a)\ \text{There exists a cons tan t } L_{\mathcal{M}} \succ 0 \quad \text{such that} \\ \qquad \|\mathcal{M}(\mathrm{x},\theta_1) - \mathcal{M}(\mathrm{x},\theta_2)\| \ \leqslant L_{\mathcal{M}}\ \|\theta_1 - \theta_2\|\ \ \text{for all } \theta_1,\ \theta_1 \in \mathbb{R}\ . \\ (b)\ \text{Themapping } \mathrm{x} \mapsto \mathcal{M}\ (\mathrm{x},\theta) \text{ is Lebesgue measurable on } \Omega \\ \ \text{for any } \theta \in \mathbb{R}. \\ \text{(c) The mapping } \mathrm{x} \mapsto \mathcal{M}\ (\mathrm{x},0)\ \in \mathcal{H}. \end{cases}$$

The nonlinear constitutive function $\psi\ : \Omega \times \mathbb{R} \times \mathrm{V} \longrightarrow \mathbb{R}$ satisfies

$$(3.22)\qquad \begin{cases} (a)\ \text{There exists a cons tan t } L_\psi > 0 \quad \text{such that} \\ \quad \|\psi(\mathrm{x},\mathcal{M}\theta_1,\mathbf{v}_1) - \psi(\mathrm{x},\mathcal{M}\theta_2,\mathrm{v}_2)\| \leqslant L_\psi(\|\mathcal{M}\theta_1 - \mathcal{M}\theta_2\| + \|\mathbf{v}_1 - \mathrm{v}_2\| \\ \text{for all } \theta_1,\ \theta_1 \in \mathbb{R},\ \text{for all } \mathbf{v}_1,\mathbf{v}_2 \in V,\ \text{a.e. } \mathrm{x} \in \Omega. \\ (b)\ \text{The mapping } x \mapsto \psi(\mathrm{x},\ \mathcal{M}\theta,\ \mathbf{v}) \text{ is Lebesgue measurable on } \Omega \\ \text{for any } \theta \in \mathbb{R}, \text{ for any } \mathbf{v}\ \in V. \\ (c)\ \text{The mapping } \mathrm{x} \mapsto \psi\ (\mathrm{x},0,0)\ \in L^2(\Omega). \end{cases}$$

The electric permittivity operator $\mathbf{B} = (\mathbf{B}_{ij}) : \Omega \times \mathbb{R}^d \ \to \mathbb{R}^d$ satisfies

$$(3.23)\qquad \begin{cases} (a)\ \mathbf{B}(\ x, E\ ) = (\mathbf{B}_{ij}(x)E_j) \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{a.e. } \mathrm{x} \in \Omega. \\ (b)\ \mathbf{B}_{ij} = \mathbf{B}_{ji} \in L^\infty(\Omega).\ 1 \leqslant i,\ j \leqslant d. \\ (c)\ \textit{There exists}\ \text{ a cons tan t } \mathrm{M}_{\mathbf{B}} > 0 \quad \text{such that } \mathbf{B}E.E \geqslant \mathrm{M}_{\mathbf{B}} \mid E \mid^2 \\ \text{for all } E = (E_i) \in \mathbb{R}^d, \text{a.e in} \in \Omega. \end{cases}$$

The pyroelectric operator $\mathcal{P} : \Omega \times \mathbb{R} \to \mathbb{R}$

$$(3.24)\qquad \begin{cases} (a)\ \text{There exists a cons tan t } L_{\mathcal{P}} > 0 \quad \text{such that} \\ \quad \|\mathcal{P}(\mathrm{x},\theta_1) - \mathcal{P}(\mathrm{x},\theta_2)\| \leqslant L_{\mathcal{P}}\ \|\theta_1 - \theta_2\| \text{ for all } \theta_1,\theta_2 \in \mathbb{R},\ \text{a.e. } \mathrm{x} \in \Omega. \\ (b)\ m_{ij} = mji \in L^\infty(\Omega).\ 1 \leqslant i,\ j \leqslant d. \\ (c)\ \text{The mapping } \mathrm{x}\ \longmapsto \mathcal{P}\ (\mathrm{x},0) \text{ belongs to } \mathcal{W}_e. \end{cases}$$

The thermal conductivity operator $\mathcal{K} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$

$$(3.25)\quad \begin{cases} (a)\ \text{There exists a constant } L_{\mathcal{K}} \succ 0 \ \text{ such that} \\ \quad \|\mathcal{K}(\mathrm{x},\theta_1) - \mathcal{K}(\mathrm{x},\theta_2)\| \leqslant L_{\mathcal{K}} \|\mathrm{r}_1 - \mathrm{r}_2\| \ \text{ for all } \mathrm{r}_1,\ \mathrm{r}_2 \in \mathbb{R}, \text{a.e.}, \mathrm{x} \in \Omega. \\ (b)\ m_{ij} = mji \in \mathrm{L}^\infty(\Omega).\ 1 \leqslant i,j \leqslant d. \\ (c)\ \text{The mapping } \mathrm{x} \longmapsto S\ (\mathrm{x},0,0) \text{ belongs to } \mathrm{L}^2(\Omega). \end{cases}$$

The normal compliance function $p_\nu : \Omega \times \mathbb{R} \to \mathbb{R}_+$.

$$(3.26)\quad \begin{cases} (a)\ \text{There exists a constant } L_\nu > 0 \ \text{ such that} \\ \quad \|p_\nu(\mathrm{x},\mathrm{r}_1) - p_\nu(\mathrm{x},\mathrm{r}_2)\| \leqslant L_\nu \|\mathrm{r}_1 - \mathrm{r}_2\| \ \text{ for all } \mathrm{r}_1,\ \mathrm{r}_2 \in \mathbb{R},\ \text{a.e. } \mathrm{x} \in \Omega. \\ (b)\ \text{The mapping } \mathrm{x} \mapsto p_\nu(\mathrm{x},\ \mathrm{r}) \text{ is Lebesgue measurable on } \Gamma_3, \\ \text{for any } r \in \mathbb{R}. \\ (c)\ \text{The mapping } \mathrm{x} \mapsto p_\nu(\mathrm{x},\ 0) \text{ belongs to } L^2(\Gamma_3). \end{cases}$$

We also suppose that the body forces and surfaces tractions have the regularity

$$(3.27)\qquad \mathbf{f}_0 \in \mathcal{C}(0,\ T;\ H), \qquad \mathbf{f}_2 \in \mathcal{C}(0,\ T;\ L^2(\Gamma_2)^d),$$

$$(3.28)\qquad q_0 \in \mathcal{C}(0,\ T; L^2(\Omega)\ ), \qquad q_2 \in \mathcal{C}(0,\ T;\ L^2(\Gamma_\mathrm{b})),$$

$$(3.29)\qquad q_2\ (t) = 0 \quad \text{on } \Gamma_3,\ \forall t \in [0,T]\,.$$

The functions g and μ have the following properties:

$$(3.30)\qquad g \in L^2(\Gamma_3),\ \mathrm{g}(\mathrm{x}) \geqslant 0,\ \text{ a.e. on } \Gamma_3$$

$$(3.31)\qquad \mu \in L^\infty(\Gamma_3),\ \mu(\mathrm{x}) > 0,\ \text{a.e. on } \Gamma_3$$

$$(3.32)\qquad \mathrm{u}_0 \in \mathrm{V}$$

and the initial temperature field satisfies

$$(3.33)\quad \theta_0 \in \mathrm{W}_{th}\,,\ \theta_F \in L^2(0,\ \mathrm{T}, \mathrm{L}^2(\Gamma_3)),\ k_e \in L^\infty(\Omega,\ \mathbb{R}_+),\ \mathrm{q}_{th} \in L^2(0,\ \mathrm{T}, \mathrm{W}'_{\mathrm{th}}).$$

Using the above notation and Green's formulas given by (3.12)-(3.14), we obtain the variatonal formulation of the mechanical problem (2.1)-(2.14), for all functions $\mathbf{v} \in V$, $\mathbf{w} \in \mathrm{W}_{th}$, $\phi \in \mathrm{W}_e$ and a.e t$\in (0,\ T)$ given as follows.

3.3. Pr**oblem.** $\mathcal{P}_V$. Find the displacement field $\mathbf{u} : [0,\ T] \to V$, the stress field $\sigma : [0,\ T] \to \mathcal{H}_1$, the electric potential $\varphi : [0,\ T] \to W$, the electric displacement field **D:**$[0,\ T] \to H$ and the temperature $\theta : [0,\ T] \to V$ such that

(3.34)
$$\sigma(t) = \mathcal{A}\left(\varepsilon\left(\dot{\mathbf{u}}(t)\right)\right) + \mathcal{B}\varepsilon(\mathbf{u}(t)) + \int \mathcal{G}(t-s, \varepsilon(\mathbf{u}(s)))\, ds + \varepsilon^{*}\nabla\varphi(t) - \mathcal{M}\theta(t),$$

(3.35)
$$(\sigma(t),\ \varepsilon(\mathbf{v}-\dot{\mathbf{u}}(t))_{\mathcal{H}} + j((\mathbf{u}(t), \mathbf{v})) - j\left(\mathbf{u}(t), \dot{\mathbf{u}}(t)\right) \geqslant (\mathbf{f}(t),\ \mathbf{v}-\dot{\mathbf{u}}(t))_{V},$$

(3.36)
$$\mathbf{D}(t) = \varepsilon\varepsilon(\mathbf{u}(t)) - \mathbf{B}\nabla\varphi(t) - \mathcal{P}\theta(t),$$

(3.37)
$$(\mathbf{D}(t),\ \nabla\phi)_{H} = -(q_e(t), \phi)_{W},$$

(3.38)
$$(\dot{\theta}(t), \mathbf{w})_{\mathrm{W}_{th}\times\mathrm{W}'_{\mathrm{th}}} + (\mathcal{K}(\nabla\varphi(t)), \nabla\mathbf{w}) = (j_{th}(\theta, \mathbf{w}) + \psi(\mathcal{M}\theta, \dot{\mathbf{u}}(t)) + (q_{th}, \mathbf{w})_{\mathrm{W}'_{\mathrm{th}}\times\mathrm{W}_{th}},$$

(3.39)
$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \theta(0) = \theta_0, \qquad \text{in } \Omega.$$

Here $j : V \times V \times L^2(\Gamma_3) \to \mathbb{R}$, $\mathbf{f} : [0,\ T] \to V$, $q_e : [0,\ T] \to W$, j_{th} :$\mathrm{W}_{th}\times\mathrm{W}_{th} \to \mathbb{R}$ are respectively, defined by

(3.40)
$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \alpha \parallel u_\nu \parallel (\mu \parallel \mathbf{v}_\tau - \mathbf{v}^{*} \parallel) + \boldsymbol{v}_\nu)da,$$

(3.41)
$$\mathbf{f}(t, \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t).\mathbf{v}\ dx + \int_{\Gamma_2} \mathbf{f}_2(t).\mathbf{v}\ da,$$

(3.42)
$$(q_e(t), \phi))_W = \int_\Omega q_0(t).\phi\ dx - \int_{\Gamma_b} q_2(t).\phi\ da,$$

(3.43)
$$j_{th} = -\int_{\Gamma_3} k_e(\theta.\nu - \theta_F.\nu)\mathbf{w}\ da.$$

We define the mapping $h : V \times W_e \to W_e$ by

$$(h(\mathbf{u}, \varphi), \phi)_W = \int_{\Gamma_3} \Psi((u_\nu - g)\Phi_l(\varphi - \varphi_0)\xi\ da,$$

for all $\mathbf{v} \in V$,

$$(\mathbf{B}\nabla\varphi, \nabla\phi)_H - (\varepsilon\varepsilon(\mathbf{u}, \nabla\phi)_H + (h(\mathbf{u}, \varphi), \phi)_{W_e} = (q(t), \phi)_{We},$$

for all $u,\ v\ \in\ V,\ \theta,\ w \in$ W$_{th}$ and $\phi\ \in$ W$_e$ and $t\ \in [0,\ T]$. We note that the definitions of f and q_e are based on the Riesz representation theorem. Moreover, the conditions (3.29) and (3.30) imply that

(3.44) $$\mathrm{f}\ \in \mathcal{C}(0\ ,\mathrm{T};\ \mathrm{V}),\qquad \mathrm{q}_e{\in}\mathcal{C}(0\ ,\mathrm{T};\ \mathrm{W}_e).$$

4. Existence and uniqueness of a solution

Now, we propose our existence and uniqueness result.

Theorem 4.1. *Assume that* (3.19)-(3.35) *hold. Then there exists a constant* α_0 *which depends only on* $\Omega, \Gamma_1,\ \Gamma_3$ and $\mathcal{A}$ *such that if*

(4.1) $$L_\nu(1+ \parallel \mu \parallel_{L^\infty(\ \Gamma_3)}) < \alpha_0,$$

where $\alpha_0\ = \dfrac{m_{\mathcal{A}}}{C_0^{\acute{e}}}$ *such that* $m_{\mathcal{A}}$ *is defined in* (3.19) *and* C_0 *defined by* (3.7)*. Then there exists a unique solution* $\{\mathbf{u},\ \sigma, \theta,\ \varphi,\ \mathbf{D}\}$ *to problem* $\mathcal{P}_V$*. Moreover, the solution satisfies*

(4.2) $$\mathbf{u} \in \mathcal{C}^1(0\ ,\mathrm{T};\ \mathrm{V}),$$

(4.3) $$\sigma \in\ \mathcal{C}(0\ ,\mathrm{T}; \mathcal{H}_1),$$

(4.4) $$\theta \in\ L^2(0,\ \mathrm{T}, \mathrm{W_{th}}) {\cap}\, \mathcal{C}(0\ ,\mathrm{T}; \mathrm{L}^2(\Omega)\ ),$$

(4.5) $$\varphi \in\ \mathcal{C}(0\ ,\mathrm{T};\ \mathrm{W}_e),$$

(4.6) $$\mathbf{D} \in \mathcal{C}(0\ ,\mathrm{T};\ \mathcal{W}_e).$$

The proof of Theorem 4.1 is carried in several steps. It is based on results of evolutionary variational inequalities, ordinary differential equations and fixed point arguments.

To prove the theorem we consider the following three auxiliary problems for given $\eta \in \mathcal{C}(0\ ,\mathrm{T};\ \mathrm{V}),\ \chi \in\ L^2(0,\ \mathrm{T}, \mathrm{W}'_{\mathrm{th}})\ ,\ \lambda \in\ \mathcal{C}(0\ ,\mathrm{T};\ \mathrm{W}_e)$ we consider the following three auxiliary problems.

4.1. Problem $\mathcal{PV}_\eta$. Find a displacement field $\mathbf{u}_\eta :\ [0\,,\mathrm{T}] \to \mathrm{V}$ and a stress field $\sigma_\eta :\ [0\,,\mathrm{T}] \to \mathcal{H}$ such that

$$\sigma_\eta\ (t) = \mathcal{A}\left(\varepsilon\left(\dot{u_\eta}\,(t)\right)\right) + \mathcal{B}\varepsilon\left(u_\eta\,(t)\right), \tag{4.7}$$

$$\begin{aligned}&\left(\sigma_\eta\ (t)\,, \varepsilon\left(\mathbf{v}-\dot{\mathbf{u}_\eta}\,(t)\right)\right)_{\mathcal{H}} + j\left(\mathbf{u}_\eta\,(t)\,,\mathbf{v}\right) - j\left(\mathbf{u}_\eta\,(t)\,, \dot{\mathbf{u}}_\eta\,(t)\right)\\ &\geqslant \left(\mathrm{f}\,(t)\,,\ \mathbf{v}-\dot{\mathbf{u}_\eta}\,(t)\right)_{\mathbf{V}},\end{aligned} \tag{4.8}$$

$$\mathbf{u}_\eta\ (0) = \mathbf{u}_0,\ in\ \Omega, \tag{4.9}$$

for all $\mathbf{u}_\eta, \mathbf{v} \in \mathrm{V}$ and $t \in \mathcal{C}^1(0\,,\mathrm{T})$.

4.2. Problem $\mathcal{PV}_\chi$. Find the temperature $\theta_\chi : [0\,,\mathrm{T}] \to \mathrm{W_{th}}$ which is solution of the variational problem

$$\begin{aligned}&\left(\theta_\chi\,(t)\,,\ \mathbf{w}\right)_{\mathrm{W}'_{\mathrm{th}}\times \mathrm{W_{th}}} + \left(\mathcal{K}(\nabla\theta\,(t))\,,\ \nabla\mathbf{w}\right)_{\mathrm{W}'_{\mathrm{th}}\times \mathrm{W_{th}}}\\ &= \left(\chi\,(t) + q_{th}\,(t)\,,\ \mathbf{w}\right)_{\mathrm{W}'_{\mathrm{th}}\times \mathrm{W_{th}}},\end{aligned} \tag{4.10}$$

$$\theta_\chi\,(0) = \theta_0, \qquad \text{in}\ \ \Omega, \tag{4.11}$$

for all $\theta_\chi, \mathbf{w} \in \mathrm{W_{th}}$, a.e.t $\in(0\,,\mathrm{T})$,

4.3. Problem $\mathcal{PV}_\lambda$. Find an electrical potential φ_λ: $[0\,,\mathrm{T}] \to \mathrm{W}_e$, $\mathbf{D}_\lambda : [0\,,\mathrm{T}] \to \mathcal{W}_e$ such that

$$\mathbf{D}_\lambda(t) = \mathbf{B}\nabla\varphi_\lambda(t) - \mathcal{E}\varepsilon(\mathbf{u}_\eta\,(t)) - \mathcal{P}\theta, \tag{4.12}$$

$$(\mathbf{B}\nabla\varphi_\lambda(t), \nabla\phi)_H - (\mathcal{E}\varepsilon(\mathbf{u}_\eta\,(t)\,, \nabla\phi)_H = (\lambda(t), \phi)_{\mathrm{W}_e}, \tag{4.13}$$

for all $\varphi_\lambda, \phi \in \mathrm{W}_e$, $t \in(0\,,\mathrm{T})$.

We begin with an auxiliary result on the priorities of the functional $j :\ V \times V \to \mathbb{R}$ and $j_{th} : \mathrm{W_{th}} \times \mathrm{W_{th}} \to \mathbb{R}$ defined by (3.42) and (3.43), respectively.

Lemma 4.1. *Under the hypotheses (3.19)-(3.35), functionals j and j_{th} satisfy*

$$j(\mathbf{u}, .)\ \textit{is convex and lawer semicontinuous on}\ V, \tag{4.14}$$

(4.15) $$j(\mathbf{u}_1,\ \mathbf{v}_2) +\ j(\mathbf{u}_2,\ \mathbf{v}_1) - j(\mathbf{u}_1,\ \mathbf{v}_1) - j(\mathbf{u}_2,\ \mathbf{v}_2)$$
$$\leqslant\ C_0^2\ (\|\ \mu\ \|_{L^\infty(\ \Gamma_3)}\ +1)\ \|\ \mathbf{u}_1 - \mathbf{u}_2\ \|_V\|\ \mathbf{v}_1 - \mathbf{v}_2\ \|_V,$$

for all $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{v}_1,\ \mathbf{v}_2 \in V$,

(4.16) $$\|\ (\theta_1,\ \mathbf{w}) -\ j_{th}\ (\theta_2,\ \mathbf{w})\ \|_{L^2(\ \Gamma_3)} \leqslant C_{j_{th}}\ \|\ (\theta_1(t) -\ \theta_2(t)\|_{\mathrm{W_{th}}},$$

for all $\theta_1,\ \theta_2, \mathbf{w}\ \in \mathrm{W_{th}}$.

Proof. We use condition (3.33) and inequality (3.7) to see that the functional j defined by (3.42) is a seminorm on V and moreover,

$$\|\ j(u_1, \mathbf{v}) + j(u_2, \mathbf{v})\ \| \leqslant C_0^2\ \ \|\ \alpha\ \|_{L^\infty(\ \Gamma_3)}\ (\|\ \mu\ \|_{L^\infty(\ \Gamma_3)}\ +1)\ \|\ \mathbf{u}_1 - \mathbf{u}_2\ \|_V\|\ \mathbf{v}_1 - \mathbf{v}_2\ \|_V\ .$$

Thus, the seminorm j is continuous on V and, therefor, (4.14) holds. From the definition of the functional j given by (3.42), we have

(4.17) $$j(\mathbf{u}_1, \mathbf{v}_2) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_1, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2)$$
$$= \int_{\Gamma_3} (\alpha\ \|\ \mathbf{u}_{1\nu} - \alpha\ \|\ \mathbf{u}_{2\nu})(\mu\ \|\ \mathbf{v}_{2\tau} - \mathbf{v}^*\ \| - \mu\ \|\ \mathbf{v}_{1\tau} - \mathbf{v}^*\ \|) + \boldsymbol{v}_{2\nu} - \boldsymbol{v}_{1\nu})da$$
$$\leqslant\ C_0^2\ \|\ \alpha\ \|_{L^\infty(\Gamma_3)}\ (\|\ \mu\ \|_{L^\infty(\Gamma_3)}\ +1)\ \|\ \mathbf{u}_1 - \mathbf{u}_2\ \|_V\|\ \mathbf{v}_1 - \mathbf{v}_2\ \|_V\ .$$

Next we use the following majoration: For the functional j_{th} defined by (3.43)

$$j_{th}(\theta, \mathbf{w}\ ) = -\int_{\Gamma_3} k_e(\theta.\nu - \theta_F.\nu)\mathbf{w}\ da,\ for\ all\ \theta,\ \theta_F,\ \mathbf{w} \in \mathrm{W_{th}}.$$

Thus by the assumption (3.35) and majoration (4.17), we get

$$\|\ j_{th}(\theta_1, \mathbf{w}\ ) - j_{th}(\theta_2, \mathbf{w}\ )\ \|_{L^2(\ \Gamma_3)} \leqslant \|\ k_e\ \|_{L^\infty(\ \Gamma_3)}\|\ \theta_1(t) - \theta_2(t)\ \|_{L^2(\ \Gamma_3)},$$

$$\|\ j_{th}(\theta_1, \mathbf{w}\ ) - j_{th}(\theta_2, \mathbf{w}\ )\ \|_{L^2(\ \Gamma_3)} \leqslant C_1\ \|\ k_e\ \|_{L^\infty(\ \Gamma_3)}\|\ \theta_1(t) - \theta_2(t)\ \|_{\mathrm{W_{th}}},$$

thus we can write

$$\|\ j_{th}(\theta_1, \mathbf{w}\ ) - j_{th}(\theta_2, \mathbf{w}\ )\ \|_{L^2(\ \Gamma_3)} \leqslant C_{j_{th}}\ \|\ \theta_1(t) - \theta_2(t)\ \|_{\mathrm{W_{th}}},\ for\ \theta_1, \theta_2 \in \mathrm{W}_{th}.$$ □

We have the following result for problem $\mathcal{PV}_\eta$.

Lemma 4.2. *Under the hypotheses* (3.19) -(3.35)*, for every* $\eta \in \mathcal{C}(0\ , \mathrm{T};\ \mathrm{V})$*, problem* $\mathcal{PV}_\eta$ *has a unique solution* $\{\mathbf{u}_\eta,\ \sigma_\eta\}$ *, such that*

(4.18) $$\mathbf{u}_\eta\ \in \mathcal{C}^1(0\ , \mathrm{T};\ \mathrm{V})\ ,\ \sigma_\eta \in \mathcal{C}^1(0\ , \mathrm{T};\ \mathcal{H}_1).$$

Moreover, if $\{\mathbf{u}_i,\ \sigma_i\}$ *is the solutions of the problem* $\mathcal{PV}_{\eta_i}$, *corresponding* $\eta = \eta_i \in \mathcal{C}(0\ ,\mathrm{T};\ \mathrm{V})$ *for* $i = 1, 2$, *then*

(4.19) $$\| \mathbf{u}_1(t) - \mathbf{u}_2(t) \|_V \leqslant C \int_0^T \| \eta_1(s) - \eta_2(s) \|_V \, ds.$$

Proof. We chose $\mathbf{v} = \dot{\mathbf{u}}(t) \pm \zeta$ in (4.8) , where $\zeta \in D\ (\Omega)^d$ is arbitrary, we find

$$(\sigma_\eta\ (t),\ \varepsilon(\phi)) = (\mathbf{f}(t),\ \phi)_V.$$

Using the definition (3.43) for $\mathbf{f}$, we deduce

(4.20) $$Div\sigma_\eta(t)\ + \mathbf{f}_0(t) = 0, \quad t \in (0,\ T).$$

With the regularity assumption (3.29) on $\mathbf{f}_0$ we see that $Div\sigma_\eta(t) \in H$. Therefore, $\sigma_\eta(t) \in \mathcal{H}_1$.

For all $\mathbf{u},\ \mathbf{v} \in V$ and $t \in [0,\ T]$,

(4.21) $$\| A\mathbf{u} - A\mathbf{v} \|_V \leqslant L_{\mathcal{A}} \| \mathbf{u} - \mathbf{v} \|_V,$$

which shows that $A : V \to V$ is Lipschitz continuous,

(4.22) $$(A\mathbf{u} - A\mathbf{v})_V \geqslant m_{\mathcal{A}} \| \mathbf{u} - \mathbf{v} \|_V^2,\ \forall \mathbf{u},\ \mathbf{v} \in V$$

and by (4.19) and (3.19) we obtain

(4.23) $$\| B\mathbf{u} - B\mathbf{v} \|_V \leqslant L_{\mathcal{B}} \| \mathbf{u} - \mathbf{v} \|_V,$$

if 4.1 is satisfied, since strongly monotone and Lipschitz continuous operator on V and B is Lipschitz continuous operator on V , $j(\mathbf{u}, .)$ satisfies the conditions (4.14) and (4.15), $\mathbf{u}_0$ satisfies the assumption (3.34) and we note that for any fixed $\eta \in \mathcal{C}(0\ ,\mathrm{T};\ \mathrm{V})$ we use the definitions (3.44) and (4.21) to show that $\mathbf{f}_\eta\mathcal{C}(0\ ,\mathrm{T};\ \mathrm{V})$. we deduce from classical results for evolutionary elliptic variational ineqalities (see for example [10] that there exists a unique function $\mathbf{u}_\eta \in \mathcal{C}(0\ ,\mathrm{T};\ \mathrm{V})$. Moreover, for $\mathbf{u}_i = \mathbf{u}_{\eta_i}$ solutions of the problem $\mathcal{PV}_{\eta_i}$ for i=1, 2, then we have

(4.24) $$\| \mathbf{u}_1(t) - \mathbf{u}_2(t) \|_V \leqslant C \int_0^t \| \dot{\mathbf{u}}_1\ (s) - \dot{\mathbf{u}}_2(s) \|_V \, ds$$

using (4.25) the inequality (4.26) becomes

(4.25) $$\| \mathbf{u}_1(t) - \mathbf{u}_2(t) \|_V \leqslant C \int_0^T \| \eta_1(s) - \eta_2(s) \|_V \, ds.$$

□

For the problem $\mathcal{PV}_\chi$ we have the following result.

Lemma 4.3. *Under the hypotheses* (3.19)-(3.35)*, for every* c*, problem* $\mathcal{PV}_\chi$ *has a unique weak solution such that*

(4.26) $$\theta_\chi \in L^2(0,\ T,\ \mathrm{W}_{th}) \cap C(0,\ T,\ \mathrm{L}^2(\Omega)),$$

Moreover, if θ_i *is the solution to problem* $\mathcal{PV}_{\chi_i}$*, corresponding* $\chi = \chi_i \in C(0,\ T,\ \mathrm{W}'_{th})$,

(4.27) $$\| \theta_1(t) - \theta_2(t) \|^2_{L^2(\Omega)} \leqslant C \int_0^T \| \chi_1(s) - \chi_2(s) \|^2_{\mathrm{W}'_{th}} ds.$$

Proof. The problem (4.10)-(4.11) may be written as

$$\dot{\theta}_\chi(t) + K\ \theta_\chi(t) = Q(t),$$

$$\theta_\chi(0) = \theta_0,$$

where, $K : \mathrm{W}_{th} \to \mathrm{W}'_{th}$ and $Q : [0,\ T] \to \mathrm{W}'_{th}$ are defined as

(4.28) $$(K_\tau, \mathbf{w})_{\mathrm{W}'_{th}\times \mathrm{W}_{th}} = \sum_{i,j=1}^{d} \int_\Omega k_{i,j} \frac{\partial \tau}{\partial x_j} \frac{\partial \mathbf{w}}{\partial x_i} dx + \int_{\Gamma_3} \tau.\mathbf{w}\ da,$$

(4.29) $$(Q, \mathbf{w})_{\mathrm{W}'_{th}\times \mathrm{W}_{th}} = (\chi(t) + q_{th}(t),\ \mathbf{w})_{\mathrm{W}'_{th}\times \mathrm{W}_{th}}.$$

It follows from the definition of the operator K, we obtain

(4.30) $$\| K\tau - K\mathbf{w} \|_{\mathrm{W}'_{th}} \leqslant L_\mathcal{K} \| \tau - \mathbf{w} \|_{\mathrm{W}'_{th}},$$

which shows that is $K : \mathrm{W}_{th} \longrightarrow \mathrm{W}'_{th}$ is continuous and by (4.32)-(3.27)(c), we obtain

(4.31) $$(K\tau - K\mathbf{w}, \tau - \mathbf{w})_{\mathrm{W}'_{th}\times \mathrm{W}_{th}} \geqslant m_\mathcal{K} \| \tau - \mathbf{w} \|^2_{\mathrm{W}_{th}}, \ \ \forall \tau,\ \mathbf{w} \in \mathrm{W}_{th},$$

K is a monotone operator. Choosing $\mathbf{w} = \mathbf{0}_{\mathrm{W}_{th}}$ in (4.33) we fined

$$\begin{aligned} ((K\tau, \tau)_{\mathrm{W}'_{th}\times \mathrm{W}_{th}} &\geqslant m_\mathcal{K} \| \tau \|^2_{\mathrm{W}_{th}} - \| K\mathbf{0}_{\mathrm{W}_{th}} \|_{\mathrm{W}'_{th}} \| \tau \|_{\mathrm{W}_{th}}) \\ &\geqslant \frac{1}{2} m_\mathcal{K} \| \tau \|^2_{\mathrm{W}_{th}} - \frac{1}{2m_\mathcal{K}} \| K\mathbf{0}_{\mathrm{W}_{th}} \|_{\mathrm{W}'_{th}}, \ \ \forall \tau \in \mathrm{W}_{th}. \end{aligned}$$

Thus, K satisfies condition (3.16) with $\omega = \frac{m_{\mathcal{K}}}{2}$ and $\zeta = -\frac{1}{2m_{\mathcal{K}}} \parallel K\mathbf{0}_{\mathrm{W}_{th}} \parallel_{\mathrm{W}'_{th}}$ and by (4.32) we deduce that

$$\parallel K\tau \parallel_{\mathrm{W}'_{th}} \leqslant L_{\mathcal{K}} \parallel \tau \parallel_{\mathrm{W}_{th}} + \parallel K\mathbf{0}_{\mathrm{W}_{th}} \parallel_{\mathrm{W}'_{th}}, \ \ \forall \tau \in \mathrm{W}_{th}.$$

This inequality implies that K satisfies condition (3.18). Moreover, for $\chi(t) \in L^2(0,\ T,\ \mathrm{W}_{th})$ and $q_{th}(t) \in \mathrm{L}^2(0,\ T,\ \mathrm{L}^2(\Omega))$ which implies $Q \in \mathrm{L}^2(0,\ T,\ \mathrm{W}'_{th})$ and $\theta_0 \in \mathrm{L}^2(\Omega)$. From theorem 3.2 there exists a unique function function $\theta_\chi \in L^2(0,\ T,\ \mathrm{W}_{th}) \ \cap \mathcal{C}(0,\ T,\ \mathrm{L}^2(\Omega))$ which satisfies the problem $\mathcal{PV}_\chi$. we take $\chi = \chi_1$ and $\chi = \chi_2$ in (4.10) , we deduce by choosing $\mathbf{w} = \theta_1(t) - \theta_2(t)$ as test function.

$$(\dot{\theta}_1(t) - \dot{\theta}_2(t),\ \theta_1(t) - (\theta_2(t))_{\mathrm{W}'_{th}\times\mathrm{W}_{th}} + K\ \theta_1(t) - K\ \theta_2(t),\ \theta_1(t) - \ \theta_2(t)\ )_{\mathrm{W}'_{th}\times\mathrm{W}_{th}}$$
$$=(\chi_1(t) - \chi_2(t),\ \theta_1(t) - \ \theta_2(t)\ )_{\mathrm{W}'_{th}\times\mathrm{W}_{th}}.$$

Then integrating the last property over $(0,t)$, using (3.15)-(4.32) and (4.33), we deduce (4.29). □

For the last problem $\mathcal{PV}_\lambda$ we have the following result.

Lemma 4.4. *Under the hypotheses* (3.19)-(3.35)*, for every* $\lambda \in \mathcal{C}(0,\ T,\ \mathrm{W}_e)$*, problem* $\mathcal{PV}_\lambda$ *has a unique solution* $\{\varphi_\lambda\ ,\ \mathbf{D}_\lambda\}$ *, such that*

$$\varphi_\lambda \ \in \mathcal{C}(0,\ T,\ \mathrm{W}_e),\ \mathbf{D}_\lambda \in \mathcal{C}(0,\ T,\ \mathcal{W}_e). \tag{4.32}$$

Moreover, if $\{\varphi_i\ ,\ \mathbf{D}_i\}$ *is the solutions to problem* $\mathcal{PV}_{\lambda i}$*, corresponding* $\lambda = \lambda_i \in \mathcal{C}(0,\ T,\ \mathrm{W}_e)$ *for* $i = 1, 2$*, then*

$$\parallel \varphi_1(t) - \varphi_2(t) \parallel_{\mathrm{W}_e} \leqslant C \parallel \mathbf{u}_1(t) - \mathbf{u}_2(t) \parallel_V + \parallel \lambda_1(t) - \lambda_2(t) \parallel_{\mathrm{W}_e}. \tag{4.33}$$

Proof. We define the operator $F : W \to W$ by

$$(F\varphi,\ \phi)_W = (\mathbf{B}\nabla\varphi(t), \nabla\phi)_H \ - (\zeta\varepsilon(\mathbf{u}_\eta(t), \nabla\phi)_W,\ \forall\varphi,\ \phi \in \mathrm{W}_e. \tag{4.34}$$

Let $\varphi_1,\ \varphi_2 \in W$. By (3.25), we find that

$$(F\varphi_1 - F\varphi_2,\ \phi_1 - \phi_2)_W \geqslant m_B \parallel \varphi_1 - \varphi_2 \parallel^2_{\mathrm{W}_e} \forall\varphi,\ \phi \in \mathrm{W}_e. \tag{4.35}$$

On the other hand , using the assumption (3.22)-(3.25) we find

$$(F\varphi_1 - F\varphi_2,\ \phi)_W \leqslant C_\zeta \parallel \varphi_1 - \varphi_2 \parallel^2_{\mathrm{W}_e} \parallel \phi \parallel_{\mathrm{W}_e} \ \ \forall\varphi,\ \phi \in \mathrm{W}_e\ ,$$

where C_ζ is a positive constant which depends on ζ. Thus

$$\| F\varphi_1 - F\varphi_2 \|_{W_e} \leqslant C_\zeta \| \varphi_1 - \varphi_2 \|_{W_e}, \tag{4.36}$$

and by (4.37)-(4.38) we obtain that $F(t)$ is a monotone and Lipchitz continuous operator on W_e and there exists a unique element $\varphi_\lambda \in W_e$ such that

$$F(t)\varphi_\lambda(t) = \lambda(t)\ \forall \varphi_\lambda \in W_e. \tag{4.37}$$

We obtain that $\varphi_\lambda(t)$ is a solution of $\mathcal{PV}_\lambda$. Let $\lambda_1,\ \lambda_2 \in \mathcal{C}(0,\ T,\ W_e)$. From (3.22)-(3.25) and (4.13) that

$$m_B\ \| \varphi_1 - \varphi_2 \|^2_{W_e} \leqslant C(\| \mathbf{u}_1(t) - \mathbf{u}_2(t) \|_V + \| \lambda_1(t) - \lambda_2(t) \|_{W_e}) \| \varphi_1 - \varphi_2 \|_{W_e},$$

which implies

$$\| \varphi_1 - \varphi_2 \|^2_{W_e} \leqslant C(\| \mathbf{u}_1(t) - \mathbf{u}_2(t) \|_V + \| \lambda_1(t) - \lambda_2(t) \|_{W_e}), \tag{4.38}$$

Where C is $\frac{C_\zeta}{m_B}$ and for every $\lambda \in \mathcal{C}(0,\ T,\ W_e)$ and $\mathbf{u}_\eta \in \mathcal{C}^1(0,\ T,\ V)$, the inequality (4.34) implies that $\varphi_\lambda \in \mathcal{C}(0,\ T,\ W_e)$. Then for $\lambda \in \mathcal{C}(0,\ T,\ W_e)$ the previousinequality and the regularity of q_e imply that $\varphi_\lambda \in \mathcal{C}(0,\ T,\ W_e)$. By (3.44) and definition of the divergence operator that

$$(Div\mathbf{D}_\lambda, \phi\ )_H = (q_e, \phi)_H \quad \forall \phi \in H^1(\Omega), \tag{4.39}$$

which shows that $\mathbf{D}_\lambda \in \mathcal{C}(0,\ T,\ \mathcal{W}_e)$.

Finally, as of these results and by the properties of the operators $\mathcal{G},\ \zeta,\ \mathcal{M}, \mathcal{P}$ and the function $\psi \in [0,\ T]$, we consider the element

$$\begin{aligned} \Lambda(\eta,\theta)(t) &= (\Lambda_1(\eta,\chi,\ \lambda)(t),\ \Lambda_2(\eta,\chi,\ \lambda)(t),\ \Lambda_3(\eta,\chi,\ \lambda)(t)) \\ &\in V \times L^2(W'_{th}) \times W_e \end{aligned} \tag{4.40}$$

defined by

$$(\Lambda_1(\eta,\chi,\ \lambda)(t) = \int_0^t \mathcal{G}(t-s,\ \varepsilon(\mathbf{u}_\eta(s)))ds + \zeta^*\nabla\varphi_\lambda(t) - \mathcal{M}\theta_\chi,\ \forall t \in [0,\ T]\,, \tag{4.41}$$

$$\Lambda_2(\eta,\chi,\ \lambda)(t) = \psi(\mathcal{M}\theta,\ \dot{\mathbf{u}}(t)) + j_{th}(\theta,\ \mathbf{w}),\ \forall t \in [0,\ T]\,, \tag{4.42}$$

$$\Lambda_3(\eta,\chi,\ \lambda)(t) = \mathcal{P}\theta_\lambda + q_e(t),\ \forall t \in [0,\ T]\,. \tag{4.43}$$

□

We have the following result.

Lemma 4.5. *Let* (4.1) *is satisfied. Then for* $(\eta,\chi,\ \lambda)\in\mathcal{C}(0,\ T,\ V)\times \mathrm{L}^2(\mathrm{W}'_{th})\times \mathrm{W}_e$, *the function* $\Lambda(\eta,\chi,\ \lambda):[0,\ T]\to V\times \mathrm{L}^2(\mathrm{W}'_{th})\times \mathrm{W}_e$ *is a continuous, and there is a unique element* $(\eta^*,\chi^*,\ \lambda^*)\in\mathcal{C}(0,\ T,\ V)\times \mathrm{L}^2(\mathrm{W}'_{th})\times \mathrm{W}_e$ *, such that* $\|\ \Lambda(\eta_1,\chi_1,\ \lambda_1)(t)-\Lambda(\eta_2,\chi_2,\ \lambda_2)(t)\ \|^2_{V\times \mathrm{L}^2(\mathrm{W}'_{th})\times \mathrm{W}_e}=(\eta^*,\chi^*,\ \lambda^*)$.

Proof. Let $(\eta,\chi,\ \lambda)\in\mathcal{C}(0,\ T,\ V)\times \mathrm{L}^2(\mathrm{W}'_{th})\times \mathrm{W}_e$ and $t_1,\ t_2,\ s_1,\ s_2\in[0,\ T]$. From (3.20)-(3.24) and (3.26), we have

$$\begin{aligned}&\|\ \Lambda(\eta_1,\chi_1,\ \lambda_1)(t)-\Lambda(\eta_2,\chi_2,\ \lambda_2)(t)\ \|_{V\times \mathrm{L}^2(\mathrm{W}'_{th})\times \mathrm{W}_e}\\ &\leqslant L_{\mathcal{G}}\int_0^t \|\ \mathbf{u}_1(s)-\mathbf{u}_2(s)\ \|_V\ ds\ +C_\zeta\ \|\ \varphi_1(t)-\varphi_2(t)\ \|_{\mathrm{W}}\\ &+L_\psi(L_{\mathcal{M}}\ \|\ \theta_1(t)-\ \theta_2(t)\ \|_{\mathrm{L}^2(\Omega)}+\|\ \dot{\mathbf{u}}_1\ (s)-\dot{\mathbf{u}}_2(s)\ \|_V)\\ &+\|\ j_{th}(\theta_1,\mathbf{w}_1\ )-j_{th}(\theta_2,\mathbf{w}_2\ )\ \|_{L^2(\ \Gamma_3)}+C_\zeta\ \|\ \dot{\mathbf{u}}_1\ (s)-\dot{\mathbf{u}}_2(s)\ \|_V\\ &+L_{\mathcal{P}}\ \|\ \theta_1(t)-\ \theta_2(t)\ \|_{\mathrm{L}^2(\Omega)}+\|\ q_{e1}(t)-q_{e2}(t)\ \|_W\ .\end{aligned}$$

We use (3.39) and (4.13) from (3.22)-(3.25) and (3.26) we obtain

$$\begin{aligned}&\|\ q_{e1}(t)-q_{e2}(t)\ \|_W\leqslant C_{\mathbf{B}}\ \|\ \varphi_1(t)-\varphi_2(t)\ \|_{\mathrm{W}}\\ &+C_\zeta\ \|\ \mathbf{u}_1(s)-\mathbf{u}_2(s)\ \|_V+L_{\mathcal{P}}\ \|\ \theta_1(t)-\ \theta_2(t)\ \|_{\mathrm{L}^2(\Omega)}\ .\end{aligned}$$

Inserting the last inequality in (4.45) and by (4.35)

$$\begin{aligned}&\|\ \Lambda(\eta_1,\chi_1,\ \lambda_1)(t)-\Lambda(\eta_2,\chi_2,\ \lambda_2)(t)\ \|_{V\times \mathrm{L}^2(\mathrm{W}'_{th})\times \mathrm{W}_e}\\ &\leqslant L_{\mathcal{G}}\int_0^t \|\ \mathbf{u}_1(s)-\mathbf{u}_2(s)\ \|_V\ ds+(C_\zeta+C_{\mathbf{B}})\ \|\ \lambda_1(t)-\lambda_2(t)\ \|_{\mathrm{W}_e}\\ &+(L_\psi L_{\mathcal{M}}+2L_{\mathcal{P}}+C_{j_{th}})\ \|\ \theta_1(t)-\ \theta_2(t)\ \|_{\mathrm{L}^2(\Omega)}\\ &+(L_\psi+C_\zeta)\ \|\ \dot{\mathbf{u}}_1\ (s)-\dot{\mathbf{u}}_2(s)\ \|_V+(2C_\zeta+C_{\mathbf{B}})\ \|\ \mathbf{u}_1(s)-\mathbf{u}_2(s)\ \|_V,\end{aligned}\tag{4.44}$$

we obtain by (4.26)

$$\begin{aligned}L_{\mathcal{G}}\int_0^t\ \|\ \mathbf{u}_1(s)-\mathbf{u}_2(s)\ \|_V\ ds+(L_\psi+C_\zeta)\ \|\ \dot{\mathbf{u}}_1\ (s)-\dot{\mathbf{u}}_2(s)\ \|_V\\ +(2C_\zeta+C_{\mathbf{B}})\ \|\ \mathbf{u}_1(s)-\mathbf{u}_2(s)\ \|_V\end{aligned}\tag{4.45}$$

$$\begin{aligned}
&\leqslant L_{\mathcal{G}} \int_0^t \int_0^s \| \dot{\mathbf{u}}_1 (r) - \dot{\mathbf{u}}_2(r) \|_V \, drds \\
&\quad +(L_\psi + 3C_\zeta + C_{\mathbf{B}}) \int_0^t \int_0^s \| \dot{\mathbf{u}}_1 (r) - \dot{\mathbf{u}}_2(r) \| \\
&\leqslant (L_{\mathcal{G}} + L_\psi + 3C_\zeta + C_{\mathbf{B}}) \int_0^t \int_0^s \| \dot{\mathbf{u}}_1 (r) - \dot{\mathbf{u}}_2(r) \|_V \, drds \\
&\leqslant C \int_0^t \int_0^s \| \dot{\mathbf{u}}_1 (r) - \dot{\mathbf{u}}_2(r) \|_V \, drds,
\end{aligned}$$

we have

$$\begin{aligned}
(4.46)\qquad &\int_0^t \int_0^s \| \dot{\mathbf{u}}_1 (r) - \dot{\mathbf{u}}_2(r) \|_V \, drds \\
&\leqslant C \int_0^t \int_0^s \| \mathbf{u}_1(r) - \mathbf{u}_2(r) \|_V + \| \eta_1(s) - \eta_2(s) \|_V \, drds.
\end{aligned}$$

The inequality (4.29) becomes

$$\begin{aligned}
\int_0^t \int_0^s \| \dot{\mathbf{u}}_1 (r) - \dot{\mathbf{u}}_2(r) \|_V \, drds &\leqslant C \int_0^t \int_0^s \int_0^r \| \dot{\mathbf{u}}_1 (z) - \dot{\mathbf{u}}_2(z) \|_V \, dzdrds \\
&\quad + C \int_0^t \int_0^s \| \eta_1(r) - \eta_2(r) \|_V \, drds.
\end{aligned}$$

From Gronwall's inequality

$$\begin{aligned}
\int_0^t \int_0^s \| \dot{\mathbf{u}}_1 (r) - \dot{\mathbf{u}}_2(r) \|_V \, drds &\leqslant C \int_0^t \int_0^s \| \eta_1(r) - \eta_2(r) \|_V \, drds \\
&\leqslant C \int_0^T \| \eta_1(s) - \eta_2(s) \|_V \, ds.
\end{aligned}$$

The equations (4.35)-(4.44)-(4.46) become

$$\begin{aligned}
(4.47)\qquad &\| \Lambda(\eta_1, \chi_1,\ \lambda_1)(t) - \Lambda(\eta_2, \chi_2,\ \lambda_2)(t) \|^2_{V \times \mathrm{L}^2(\mathrm{W}'_{th}) \times \mathrm{W}_e} \\
\leqslant\ \ & C \int_0^T \| \Lambda(\eta_1, \chi_1,\ \lambda_1)(t) - \Lambda(\eta_2, \chi_2,\ \lambda_2)(t) \|^2_{V \times \mathrm{L}^2(\mathrm{W}'_{th}) \times \mathrm{W}_e} .
\end{aligned}$$

Existence

Let $(\eta^*, \chi^*,\ \lambda^*) \in \mathcal{C}(0,\ T,\ V) \times \mathrm{L}^2(\mathrm{W}'_{th}) \times \mathrm{W}_e$, the fixed point of Λ defined by $(4.42) - (4.44)$ we denote

$$\mathbf{u}_* = \mathbf{u}_{\eta^*}\ ,\ \theta_* = \theta_{\chi^*}\ , \varphi_* = \varphi_\lambda.$$

$$\sigma_* = \mathcal{A}(\varepsilon(\mathbf{u}_*)) + \mathcal{B}(\varepsilon(\mathbf{u}_*)) + \int_0^t \mathcal{G}(t-s,\ \varepsilon(\mathbf{u}_*(s)))ds + \zeta^* \nabla \varphi_*(t) - \mathcal{M}\theta_*$$

$$\mathbf{D}_* = \zeta\varepsilon(\mathbf{u}_*) - \mathbf{B}\nabla\varphi_* - P\theta_*.$$

Let $\{\mathbf{u}_*\ ,\ \sigma_*\}$, θ_* and $\{\varphi_*,\ \mathbf{D}_*\}$ be the solution of the problems $\mathcal{PV}_{\eta^*}$, $\mathcal{PV}_{\chi^*}$ and $\mathcal{PV}_{\lambda^*}$ respectively , the equalities Λ_1 $(\eta^*, \chi^*,\ \lambda^*) = \eta^*,\ \Lambda_2$ $(\eta^*, \chi^*,\ \lambda^*) = \chi^*$ and Λ_3 $(\eta^*, \chi^*,\ \lambda^*) = \lambda^*$ combined with (4.42)-(4.44) show that (3.36)-(3.40) are satisfied. The regularity (4.2)-(4.6) follow from Lemmas 4.3-4.4 and 4.5.

Uniqueness

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.42)-(4.44) and the unique solution of problems$\mathcal{PV}_{\eta^*}$, $\mathcal{PV}_{\chi^*}$ and $\mathcal{PV}_{\lambda^*}$ which complete the proof. □

5. Conclusion

- We have presented a mathematical model which describes the quasi-static process of contact between a piezoelectric body with long-term memory and an obstacle.
- The problem was posed as a variational inequality for the displacements and a variational equality for the electric potential.
- The existence of a unique weak solution for the problem was established using arguments from the theory of evolutionary variational inequalities and a fixed point theorem.
- This work opens the way to the study of other problems with other conditions of conductive materials.

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