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EXPONENTIAL STABILITY RESULT FOR A POROUS PROBLEM WITH FRACTIONAL TIME DELAYS

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ABSTRACT. In this work, we are concerned with a porous problem in a bounded one-dimensional domain under Dirichlet boundary conditions with fractional time delays and internal frictional dissipative terms. By a multiplier approach, an exponential stability result are obtained.

1. INTRODUCTION

The present paper is focused on the study of the stabilization of the porous system with delay terms and frictional dampings

$$(P)\begin{cases} \rho_1\varphi_{tt} - k\varphi_{xx} - \mu\psi_x + a_1\partial_t^{\alpha,\beta}\varphi\left(t-s\right) + \alpha_1\varphi_t = 0,\ x\in\Omega,\ t>0, \\ \rho_2\psi_{tt} - b\varphi_{xx} + \mu\varphi_x + m\psi + a_2\partial_t^{\alpha,\beta}\psi\left(t-s\right) + \alpha_2\psi_t = 0,\ x\in\Omega,\ t>0 \\ \varphi\left(x,0\right) = \varphi_0\left(x\right),\ \psi\left(x,0\right) = \psi_0\left(x\right), \quad x\in\Omega, \\ \varphi_t\left(x,0\right) = \varphi_1\left(x\right),\ \psi_t\left(x,0\right) = \psi_1\left(x\right), \quad x\in\Omega, \\ \varphi_t\left(x,t-s\right) = f_0\left(t-s\right), \quad x\in\Omega,\ t\in\left(0,s\right), \\ \psi_t\left(x,t-s\right) = g_0\left(t-s\right), \quad x\in\Omega,\ t\in\left(0,s\right), \\ \varphi\left(0,t\right) = \psi\left(0,t\right) = \varphi\left(L,t\right) = \psi\left(L,t\right), \quad t>0 \end{cases}$$

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where $\Omega = (0, L)$, the variable φ and ψ represent the displacement of a solid elastic material and volume fraction, respectively. Here $\rho_1, \rho_2, k, \mu,$ and m are the constitutive coefficients whose physical meaning is well known satisfying

$$\rho_1 > 0,\ \rho_2 > 0, b > 0,\ k > 0,\ \mu > 0,\ m > 0, \text{and } km > \mu^2.$$

The constant $s > 0$ is the time delay. The functions $\varphi_0, \varphi_1, \psi_0, \psi_1, f_0, g_0$ are the initial data belongs to a suitable space. The notation $\partial_t^{\alpha,\beta}$ stands the generalized Caputo's fractional derivative (see [5]) given by

$$\partial_t^{\alpha,\beta} u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\ e^{-\beta(t-s)}\ u_s(s)\ ds, \quad 0 < \alpha < 1,\ \beta > 0.$$

In the recent years, an increasing number of research have been discussed the stabilization of porous systems with several dissipative mechanisms and several results have been established. We recall a main result for this type of problem, it's shown that delays may destabilize a system that is uniformly asymptotically stable in the absence of delay, see [6] for more details.

It is important to emphasize here the particular case $k = \mu = m,$ the new system is known as the Timoshenko system where A. Adnane et.all [2] proved a uniformly exponential stability result by using frequency domain approach.

In absence of the delay terms, R. Quintanilla [16] considered the following problem

$$\left\{ \begin{array}{l} \rho_1 \varphi_{tt} - k\varphi_{xx} - \mu\psi_x = 0,\ x \in \Omega,\ t > 0, \\ \rho_2 \psi_{tt} - b\varphi_{xx} + \mu\varphi_x + m\psi + \tau\psi_t = 0,\ x \in \Omega,\ t > 0, \end{array} \right. \tag{1.1}$$

with some initial and boundary conditions. Employing Hurwitz theorem, he proved that dissipative terms is not enough to obtain exponential stability when the speed of propagation waves is different, otherwise if the speed of propagation of waves is equal, T.A. Apalara [3] studied the same system proving that the system is exponentially stable. By Adding the viscoelasticity term ($\gamma\varphi_{xxt}$) at the first equation of (1.1), A. Magana and R. Quintanilla [10] showed that the system is exponentially stable.

Note that the system (P)with fractional time delays can be looked as a porous system with memory terms acting only on time interval $(0, t-s)$. Regarding porous systems with memory term, B.Feng and T.A Apalara [7] considered problem (1.1) with memory term, that is

$$
\begin{cases} \rho_1\varphi_{tt} - k\varphi_{xx} - \mu\psi_x = 0, \ x \in \Omega, \ t > 0, \\ \rho_2\psi_{tt} - b\varphi_{xx} + \mu\varphi_x + m\psi + \int_0^1 g(t-s)\psi_{xx}\,ds = 0, \ x \in \Omega, \ t > 0. \end{cases} \tag{1.2}
$$

By assuming minimal conditions on the relaxation function, the authors established an optimal explicit and general energy decay results. For various other damping mechanisms introduced and more results on porous system, we refer reader to [1, 4, 8, 9, 11–13, 15, 17, 18] and the references therein.

Our goal in this paper is investigate the effect of presence of fractional time delays and frictional dampings on the asymptotic behavior of solutions of the system (P). We establish an exponential decay result under appropriate assumptions by using the multiplier method.

The plan of this paper is as follows. In section 2, we present some assumption, the augmented problem (P') and lemmas needed for this study. Section 3 is devoted to the proof of well-posedness result by using the semi-group method. In section 4, we prove decay result by using the multiplier method and appropriate Lyapunov functional.

2. Preliminaries

This section is concerned with the reformulation of the problem (P) into augmented system. For that, we need the following claims.

Lemma 2.1. *Let η be the function:*

$$
\eta(\xi) := |\xi|^{\frac{2\alpha-1}{2}}, \quad \xi \in \mathbb{R}, \ 0 < \alpha < 1.
$$

Then, the relationship between the "input" U and "output" O of the system

$$
\begin{cases} \phi_t(x,\xi,t) + (\xi^2+\beta)\phi(x,\xi,t) - U(x,t)\eta(\xi) = 0, \\ \qquad\qquad \xi \in \mathbb{R}, \ t > 0, \ \beta > 0, \\ \phi(x,\xi,0) = 0 \\ O(t) := (\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty}\phi(x,\xi,t)\eta(\xi)\,d\xi \end{cases} \tag{2.1}
$$

is given by

$$
O := I^{1-\alpha,\beta}U,
$$

where

$$I^{\alpha,\beta}u\left(t\right):=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-s\right)^{\alpha-1}\ e^{-\beta(t-s)}u\left(s\right)\ ds.$$

Lemma 2.2. *[2] if* $\lambda\in D_{\beta}=\mathbb{C}\setminus\left]-\infty,-\beta\right[$ *then*

$$\int_{-\infty}^{+\infty}\frac{\eta^{2}\left(\xi\right)}{\lambda+\beta+\xi^{2}}\,d\xi=\frac{\pi}{\sin\left(\alpha\pi\right)}\left(\lambda+\beta\right)^{\alpha-1}.$$

We suppose that the constants a_i, α_i, satisfy the following assumption

$$a_i\beta^{\alpha-1}<\alpha_i \quad \text{for } i=1,2. \tag{2.2}$$

Thought this work C denotes a generic positive constant that may change line to line.

Now, we introduce, as in [14], the new variables

$$z_1\left(x,\rho,t\right)=\varphi_t\left(x,t-s\rho\right),\quad x\in\Omega,\ \rho\in\left(0,1\right),\ t\in\mathbb{R}_+, \tag{2.3}$$

$$z_2\left(x,\rho,t\right)=\psi_t\left(x,t-s\rho\right),\quad x\in\Omega,\ \rho\in\left(0,1\right),\ t\in\mathbb{R}_+. \tag{2.4}$$

Consequently, we have for $i=1,2$

$$z_{it}\left(x,\rho,t\right)=\frac{-1}{s}z_{i\rho}\left(x,\rho,t\right),\quad x\in\Omega,\ \rho\in\left(0,1\right),\ t\in\mathbb{R}_+. \tag{2.5}$$

Denoting $z_{it}=\dfrac{\partial}{\partial t}z_i,\ z_{i\rho}=\dfrac{\partial}{\partial\rho}z_i$ for $i=1,2$. Then, by using (2.5),(2.4) and lemma 2.1, the problem (P) is equivalent to

(2.6)

$$(P')\begin{cases}\rho_1\varphi_{tt}-k\varphi_{xx}-\mu\psi_x+b_1\phi_1\star\eta+\alpha_1\varphi_t=0,\ (x,t)\in\Omega\times\mathbb{R}_\star^+,\\ \rho_2\psi_{tt}-b\psi_{xx}+\mu\varphi_x+m\psi+b_2\phi_2\star\eta+\alpha_2\psi_t=0,\ (x,t)\in\Omega\times\mathbb{R}_\star^+,\\ \phi_{it}+\left(\xi^2+\beta\right)\phi_i-z_i\left(x,1,t\right)\eta\left(\xi\right)=0,\ \text{for } i=1,2,\ (x,\xi,t)\in\Omega\times\mathbb{R}\times\mathbb{R}_\star^+,\\ sz_{it}\left(x,\rho,t\right)+z_{i\rho}\left(x,\rho,t\right)=0 \text{ for } i=1,2, (x,\xi,t)\in\Omega\times(0,1)\times\mathbb{R}_\star^+\\ \varphi\left(L,t\right)=\varphi\left(0,t\right)=\psi\left(L,t\right)=\psi\left(0,t\right)=0,\ t\in\mathbb{R}_\star^+,\\ z_1\left(x,0,t\right)=\varphi_t\left(x,t\right),\ (x,t)\in\Omega\times\mathbb{R}_\star^+,\\ z_2\left(x,0,t\right)=\psi_t\left(x,t\right),\ (x,t)\in\Omega\times\mathbb{R}_\star^+,\\ \varphi\left(x,0\right)=\varphi_0,\ \varphi_t\left(x,0\right)=\varphi_1,\ x\in\Omega,\\ \psi\left(x,0\right)=\psi_0,\ \psi_t\left(x,0\right)=\psi_1,\ x\in\Omega,\\ z_1\left(x,\rho,0\right)=f_0\left(x,-s\rho\right),\ (x,\rho)\in\Omega\times\left(0,1\right),\\ z_2\left(x,\rho,0\right)=g_0\left(x,-s\rho\right), (x,\rho)\in\Omega\times\left(0,1\right),\\ \phi_i\left(x,\xi,0\right)=0,\ \text{for } i=1,2,\ (x,\xi)\in\Omega\times\mathbb{R}_\star^+,\end{cases}$$

where $\phi_i \star \eta = \displaystyle\int_{-\infty}^{+\infty} \phi(x,\xi,t)\,\eta(\xi)\ d\xi\ dx$, and $b_i = (\pi)^{-1}\sin(\alpha\pi)\,a_i$, for $i = 1, 2$. Now, let us introduce the energy associated to solution of (P')

$$
\begin{aligned}
&E(t) \\
&= \frac{1}{2}\left[\rho_1\|\varphi_t\|^2 + k\|\varphi_x\|^2 + \rho_2\|\psi_t\|^2 + b\|\psi_x\|^2 + 2\mu\int_\Omega \varphi_x\psi\ dx + m\|\psi\|^2\right] \qquad (2.7)\\
&+\sum_{i=1}^{2}\frac{b_i}{2}\int_\Omega\int_{-\infty}^{+\infty}|\phi_i(x,\xi,t)|^2\ d\xi\ dx + \sum_{i=1}^{2} v_i s\int_\Omega\int_0^1 |z_i(x,\rho,t)|^2\ d\rho\ dx,
\end{aligned}
$$

where v_i is a positive constant verifying

$$
A_0 b_i < v_i < \alpha_i - b_i A_0,\ i = 1, 2,\ \text{with } A_0 = \int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{\xi^2+\beta}\ d\xi. \qquad (2.8)
$$

Remark 2.1. *Under hypothesis $mk > \mu^2$, the energy E defined in (2.7) is positive.*

Lemma 2.3. *[2] For $z_i \in L^2(\Omega)$ and $\xi\phi_i \in L^2(\Omega\times(-\infty,+\infty))$ for $i = 1, 2$ we have*

$$
\begin{aligned}
\left|\int_\Omega z_i(x,\rho,t)\int_{-\infty}^{+\infty}\eta(\xi)\,\phi_i(x,\xi,t)\ d\xi\ dx\right| &\le A_0\int_\Omega |z_i(x,\rho,t)|^2\ dx \\
&+\frac{1}{4}\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi_i(x,\xi,t)|^2\ d\xi\ dx,
\end{aligned}
$$

where $A_0 := \displaystyle\int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{\xi^2+\beta}\ d\xi.$

Lemma 2.4. *Assume that (2.2) holds, then the energy functional defined by (2.7) satisfies*

$$
\begin{aligned}
\frac{dE(t)}{dt} &\le -C\sum_{i=1}^{2}\int_\Omega\left(|z_i(x,1,t)|^2 + |z_i(x,0,t)|^2\right)\ dx \\
&-\sum_{i=1}^{2}\frac{b_i}{2}\int_0^L\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi_i(x,\xi,t)|^2\ d\xi\ dx \le 0,
\end{aligned}
$$

for a positive constant C and $b_i = (\pi)^{-1}\sin(\alpha\pi)\,a_i$ with $i = 1, 2$.

Proof. Multiplying the first equation of (P') by φ_t, integration over $(0, L)$, by integration by parts with boundary conditions to get

$$\begin{aligned}&\frac{d}{dt}\left[\frac{\rho_1}{2}\|\varphi_t\|^2+\frac{k}{2}\|\varphi_x\|^2\right]+\mu\int_0^L\psi\varphi_{xt}\,dx\\ &+b_1\int_0^L\phi_L\star\eta\,\varphi_t\,dx+\alpha_1\|\varphi_t\|^2=0.\end{aligned}\tag{2.9}$$

Multiplying the second equation of (P') by ψ_t, integration over $(0,L)$, by integration by parts and boundary condition to find

$$\begin{aligned}&\frac{d}{dt}\left[\frac{\rho_2}{2}\|\psi_t\|^2+\frac{b}{2}\|\psi_x\|^2+\frac{m}{2}\|\psi\|^2\right]+\mu\int_0^L\varphi_x\psi_t\,dx\\ &+b_2\int_0^L\phi_2\star\eta\,\psi_t\,dx+\alpha_2\|\psi_t\|^2=0.\end{aligned}\tag{2.10}$$

Then, summing (2.9) and (2.10), we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{\rho_1}{2}\|\varphi_t\|^2+\frac{k}{2}\|\varphi_x\|^2+\frac{\rho_2}{2}\|\psi_t\|^2+\frac{b}{2}\|\psi_x\|^2+\frac{m}{2}\|\psi\|^2+\mu\int_0^L\psi\varphi_x\,dx\right]\\ &+b_1\int_0^L\phi_1\star\eta\,\varphi_t\,dx+b_2\int_0^L\phi_2\star\eta\,\psi_t\,dx+\alpha_1\|\varphi_t\|^2+\alpha_2\|\psi_t\|^2=0.\end{aligned}\tag{2.11}$$

Multiplying the equation j of (P') by $b_i\phi_i$ with $(i,j)=(1,3)$, respectively $(i,j)=(2,5)$ and integration over $(0,L)\times\mathbb{R}$, then we yield

$$\begin{aligned}&b_i\int_0^L\int_{-\infty}^{+\infty}\phi_{it}\phi_i\,d\xi\,dx+b_i\int_0^L\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi_i(x,\xi,t)|^2\;d\xi\,dx\\ &-b_i\int_0^L z_i(x,1,t)\int_{-\infty}^{+\infty}\eta(\xi)\,\phi_i(x,\xi,t)\;d\xi\,dx=0,\end{aligned}$$

which we give

$$\begin{aligned}&\frac{d}{dt}\left[\frac{b_i}{2}\int_0^L\int_{-\infty}^{+\infty}|\phi_i(x,\xi,t)|^2\;d\xi\,dx\right]\\ &+b_i\int_0^L\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi_i(x,\xi,t)|^2\;d\xi\,dx\\ &-b_i\int_0^L z_i(x,1,t)\int_{-\infty}^{+\infty}\eta(\xi)\,\phi_i(x,\xi,t)\;d\xi\,dx=0,\end{aligned}\tag{2.12}$$

with $i = 1, 2$. Multiplying the equation number j of (P') by $2v_i z_i$ with $(i, j) = (1, 4)$, respectively $(i, j) = (2, 6)$ and integrating over $(0, L) \times (0, 1)$, we get

$$\frac{d}{dt}\left\{ s v_i \int_0^L \int_0^1 |z_i(x, \rho, t)|^2 \; d\rho \; dx \right\} + v_i \int_0^L \left[|z_i(x, 1, t)|^2 - |z_i(x, 0, t)|^2 \right] \; dx = 0, \tag{2.13}$$

for $i = 1, 2$. By summing (2.11), (2.12) and (2.13) and using $\varphi_t(x, t) = z_1(x, 0, t)$ and $\psi(x, t) = z_2(x, 0, t)$, we arrive at

$$\begin{aligned}
\frac{dE(t)}{dt} &= -\sum_{i=1}^{2} (\alpha_i - v_i) \int_0^L |z_i(x, 0, t)|^2 \; dt \\
&- \sum_{i=1}^{2} b_i \int_0^L z_i(x, 0, t) \int_{-\infty}^{+\infty} \phi_i(x, \xi, t)\, \eta(\xi) \; d\xi \; dx \\
&- \sum_{i=1}^{2} b_i \int_0^L \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) |\phi_i(\xi)|^2 \; d\xi \; dx \\
&+ \sum_{i=1}^{2} b_i \int_0^L z_i(x, 1, t) \int_{-\infty}^{+\infty} \eta(\xi)\, \phi_i(x, \xi, t) \; d\xi \; dx - \sum_{i=1}^{2} v_i \int_0^L |z_i(x, 1, t)|^2 \; dx.
\end{aligned}$$

Thanks to lemma 2.3 with (10) and putting

$$C = \min_{i=1,2} (v_i - A_0 b_i, \alpha_i - v_i - b_i A_0) > 0, \;\; i = 1, 2,$$

hence we obtain

$$\begin{aligned}
\frac{dE(t)}{dt} &\leq -\sum_{i=1}^{2} (\alpha_i - v_i - b_i A_0) \int_0^L |z_i(x, 0, t)|^2 \; dx \\
&\quad - \sum_{i=1}^{2} (v_i - A_0 b_i) \int_0^L |z_i(x, 1, t)|^2 \; dx \\
&\quad - \sum_{i=1}^{2} \frac{b_i}{2} \int_0^L \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) |\phi_i(x, \xi, t)|^2 \; d\xi \; dx \\
&\leq -C \sum_{i=1}^{2} \int_0^L \{ |z_i(x, 0, t)|^2 + |z_i(x, 1, t)|^2 \} \, dx \\
&\quad - \sum_{i=1}^{2} \frac{b_i}{2} \int_0^L \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) |\phi_i(x, \xi, t)|^2 \; d\xi \; dx \\
&\leq 0.
\end{aligned}$$

□

3. Well-possdness

In this section, we prove the existence and uniqueness of global solution of (P') by using the semi group theory. Let us set $u = \varphi_t$ and $v = \psi_t$ and $U = (\varphi, u, \psi, v, \phi_1, \phi_2, z_1, z_2)^T$, then (P') can be rewrite as follows:

$$\left\{ \begin{array}{l} U_t(t) = AU(t), \\ U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, 0, 0, f_0(-\rho s), g_0(-\rho s))^T \end{array} \right. , \tag{3.1}$$

where the operator A is defined by

$$\begin{pmatrix} u \\ \dfrac{k}{\rho_1}\varphi_{xx} + \dfrac{\mu}{\rho_1}\psi_x - \dfrac{b_1}{\rho_1}\phi_1 \star \eta - \dfrac{\alpha_1}{\rho_1}u \\ v \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{\mu}{\rho_2}\varphi_x - \dfrac{m}{\rho_2}\psi - \dfrac{b_2}{\rho_2}\phi_2 \star \eta - \dfrac{\alpha_2}{\rho_2}v \\ -(\xi^2 + \beta)\phi_1 + z_1(x,1)\eta(\xi) \\ -(\xi^2 + \beta)\phi_2 + z_2(x,1)\eta(\xi) \\ -\dfrac{1}{s}z_{1\rho}(x,\rho) \\ -\dfrac{1}{s}z_{2\rho}(x,\rho) \end{pmatrix}$$

with

$$\phi_1 \star \eta = \int_{-\infty}^{+\infty} \phi_i(x,\xi)\eta(\xi)\ d\xi,\ i = 1,2,$$

and the domain

$$D(A) = \left\{ \begin{array}{l} U \in \mathcal{H}:\ (\varphi,\psi) \in (H^2(\Omega))^2,\ (u,v) \in (H_0^1(\Omega))^2, \\ z_{i\rho} \in L^2(\Omega \times (0,1)) \text{ for } i = 1,2, \\ u = z_1(\cdot,0),\ v = z_2(\cdot,0), \\ \xi\phi_i \in L^2(\Omega \times (\infty,+\infty)),\ \text{for } i = 1,2, \\ (\xi^2 + \beta)\phi_i - z_i(x,1)\eta(\xi) \in L^2(\Omega \times (-\infty,+\infty)),\ \text{for } i = 1,2 \end{array} \right\},$$

where the space is defined by

$$\mathcal{H} := \left(H_0^1(\Omega)\right)^2 \times \left(L_0^2(\Omega)\right)^2 \times \left(L^2(\Omega \times (\infty, +\infty))\right)^2 \times \left(L^2(\Omega \times (0,1))\right)^2$$

equipped with the inner product

$$\begin{aligned}
\langle U, \bar{U} \rangle &= \rho_2 \int_\Omega v\bar{v}\, dx + b \int_\Omega \psi_x \bar{\psi}_x\, dx \\
&+\mu \left[\int_\Omega \bar{\psi}\varphi_x\, dx + \int_\Omega \psi \bar{\varphi}_x\, dx \right] + m \int_\Omega \psi\bar{\psi}\, dx \\
&+k \int_\Omega \varphi_x \bar{\varphi}_x\, dx + \rho_1 \int_\Omega u\bar{u}\, dx \\
&+\sum_{i=1}^{2} b_i \int_\Omega \int_{-\infty}^{+\infty} \phi_i(x,\xi)\, \bar{\phi}_i(x,\xi)\; d\xi\, dx \\
&+2\sum_{i=1}^{2} v_i s \int_\Omega \int_0^1 z_i(x,\rho)\, \bar{z}_i(x,\rho)\; d\rho\, dx
\end{aligned} \tag{3.2}$$

for all $\bar{U} = \left(\bar{\varphi}, \bar{u}, \bar{\psi}, \bar{v}, \bar{\phi}_1, \bar{\phi}_2, \bar{z}_1, \bar{z}_2\right) \in \mathcal{H}$.

Theorem 3.1. *Assume that (2.2) holds. Then, any $U_0 \in \mathcal{H}$ the problem (3.1) has unique weak solution*

$$U \in C\left((0,\infty), \mathcal{H}\right).$$

Moreover,if $U_0 \in D(A)$, then we have

$$U \in C\left([0,\infty), D(A)\right) \cap C^1\left([0,\infty), \mathcal{H}\right).$$

Proof. We prove that A is a maximal dissipative operator. For this, we first show that A is dissipative. We remark from lemma 2.4 and (3.1) that

$$E'(t) = \frac{1}{2}\frac{d}{dt}\|U\|^2 = \langle U_t, U\rangle_{\mathcal{H}} = \langle AU, U\rangle_{\mathcal{H}} \leq 0.$$

Thus, A is dissipative.

In second step, we prove the surjectively of $I - A$. Indeed, let

$$F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$$

looking for $U \in D(A)$ such that

$$(I - A)\, U = F,$$

this implies

$$
\left\{
\begin{array}{l}
\varphi - u = f_1\left(x\right), \\
\left(1 + \dfrac{\alpha_1}{\rho_2}\right) u - \dfrac{k}{\rho_1}\varphi_{xx} - \dfrac{\mu}{\rho_1}\psi_x + \dfrac{b_1}{\rho_1}\phi_1 \star \eta = f_2\left(x\right), \\
\psi - v = f_3\left(x\right), \\
v\left(1 + \dfrac{\alpha_2}{\rho_2}\right) - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{\mu}{\rho_2}\varphi_x + \dfrac{m}{\rho_2}\psi \\
+\dfrac{b_2}{\rho_2}\phi_2 \star \eta = f_4\left(x\right), \\
\left(1 + \xi^2 + \beta\right)\phi_1 - z_1\left(x,1\right)\eta\left(\xi\right) = f_5\left(x,\xi\right), \\
\left(1 + \xi^2 + \beta\right)\phi_2 - z_2\left(x,1\right)\eta\left(\xi\right) = f_6\left(x,\xi\right), \\
z_1 + \dfrac{1}{s}z_{1\rho} = f_7\left(x,\rho\right), \\
z_2 + \dfrac{1}{s}z_{2\rho} = f_8\left(x,\rho\right).
\end{array}
\right. \tag{3.3}
$$

We suppose $(\varphi,\psi) \in \left(H_0^1\left(\Omega\right)\right)^2$, then the first and second equation in (3.3) give

$$
\left\{
\begin{array}{l}
u = \varphi - f_1, \\
v = \psi - f_3.
\end{array}
\right. \tag{3.4}
$$

On other hand, the solution of $(3.3)_7$ and $(3.3)_8$ with taking into consideration that $z_1\left(x,0\right) = u = \varphi - f_1,\ z_2\left(x,0\right) = v = \psi - f_3$ are given by

$$
\begin{aligned}
z_1\left(x,\rho\right) &= e^{-s\rho}\left[\varphi\left(x\right) - f_1\left(x\right)\right] \\
&\quad + s\ e^{-s\rho}\int_0^{\rho} e^{s\tau} f_7\left(x,\tau\right)\ d\tau \in L^2\left(\Omega \times (0,1)\right),
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
z_2\left(x,\rho\right) &= e^{-s\rho}\left[\psi\left(x\right) - f_3\left(x\right)\right] \\
&\quad + s\ e^{-s\rho}\int_0^{\rho} e^{s\tau} f_8\left(x,\tau\right)\ d\tau \in L^2\left(\Omega\left(0,1\right)\right).
\end{aligned} \tag{3.6}
$$

Noting from $(3.3)_5$ and $(3.3)_6$ that

$$
\phi_1 = \frac{f_5 + z_1\left(x,1\right)\eta\left(\xi\right)}{1 + \xi^2 + \beta}, \tag{3.7}
$$

$$
\phi_2 = \frac{f_6 + z_2\left(x,1\right)\eta\left(\xi\right)}{1 + \xi^2 + \beta}. \tag{3.8}
$$

Thanks to (3.7) and (3.8), we deduce that

$$(\xi\phi_1, \xi\phi_2) \in L^2(\Omega \times (-\infty, +\infty)).$$

Now, substituting (20) and (3.6) for $\rho = 1$ in (22) and (3.8) with (3.4), then relations$(3.3)_{2,4}$ become

$$\begin{aligned} &\left(1+\frac{e^{-s}}{\rho_1}b_{11}+\frac{\alpha_1}{\rho_1}\right)\varphi-\frac{k}{\rho_1}\varphi_{xx}-\frac{\mu}{\rho_1}\psi_x \\ &= f_2+\left(1+\frac{e^{-s}}{\rho_1}b_{11}+\frac{\alpha_1}{\rho_1}\right)f_1-\frac{b_1}{\rho_1}\left(\int_{-\infty}^{+\infty}\frac{\eta(\xi)}{1+\xi^2+\beta}\,d\xi\right)f_5 \\ &-\frac{b_1}{\rho_1}s\,e^{-s}\left(\int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{1+\xi^2+\beta}\,d\xi\right)\int_0^1 e^{s\tau}f_7(x,\tau)\;d, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\left(1+\frac{e^{-s}}{\rho_2}b_{22}+\frac{\alpha_2}{\rho_2}\right)\psi-\frac{b}{\rho_2}\psi_{xx}+\frac{\mu}{\rho_2}\varphi_x+\frac{m}{\rho_2}\psi \\ &= f_4+\left(\frac{\alpha_2}{\rho_2}+\frac{e^{-s}}{\rho_2}b_{22}+1\right)f_3-\frac{b_2}{\rho_2}\left(\int_{-\infty}^{+\infty}\frac{\eta(\xi)}{1+\xi^2+\beta}\,d\xi\right)f_6 \\ &-\frac{b_2}{\rho_2}s\,e^{-s}\left(\int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{1+\xi^2+\beta}\,d\xi\right)\int_0^1 e^{s\tau}f_8(x,\tau)\;d\tau, \end{aligned} \tag{3.10}$$

where

$$b_{ii} = b_i\int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{1+\xi^2+\beta}\,d\xi,\ \text{for } i=1,2.$$

Our goal now is to prove that the solutions (φ, ψ) of the system (3.9) and (3.10) belongs to $(H^2(\Omega))^2$. Indeed, let $(\bar{\varphi}, \bar{\psi}) \in (H_0^1(\Omega))^2$, Multiplying the equation (3.9) and (3.10) by $\rho_1\bar{\varphi}$ and $\rho_2\bar{\psi}$ respectively, by integration by parts, then summing the resulting equations to arrive at

$$M\left(\varphi, \psi; \bar{\varphi}, \bar{\psi}\right) = L\left(\bar{\varphi}, \bar{\psi}\right), \tag{3.11}$$

where the bilinear from $M : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is given by

$$\begin{aligned} M\left(\varphi,\psi;\bar{\varphi},\bar{\psi}\right) &= \left(\rho_1+\alpha_1+e^{-s}b_{11}\right)\int_\Omega \varphi\bar{\varphi}\,dx+\left(\rho_2+\alpha_2+e^{-s}b_{22}\right)\int_\Omega \psi\bar{\psi}\,dx \\ &+k\int_\Omega \varphi_x\bar{\varphi}_x\,dx+b\int_\Omega \psi_x\bar{\psi}_x\,dx+\mu\left[\int_\Omega\left(\psi\bar{\varphi}_x+\varphi_x\bar{\psi}\right)\;dx\right]+m\int_\Omega \psi\bar{\psi}\,dx, \end{aligned}$$

and linear from $L : \ H_0^1(\Omega) \to \mathbb{R}$ by

$$
\begin{aligned}
L\left(\bar{\varphi},\bar{\psi}\right) := {} & \left(\alpha_1+\rho_1+e^{-s}b_{11}\right)\int_\Omega f_1\bar{\varphi}\,dx \\
& +\left(\alpha_2+\rho_2+e^{-s}b_{22}\right)\int_\Omega f_3\bar{\psi}\,dx+\rho_1\int_\Omega f_2\bar{\varphi}\,dx \\
& +\rho_2\int_\Omega f_4\bar{\psi}\,dx-b_1\left(\int_{-\infty}^{+\infty}\frac{\eta(\xi)}{1+\xi^2+\beta}\right)\int_\Omega f_5\bar{\varphi}\,dx \\
& -b_2\left(\int_{-\infty}^{+\infty}\frac{\eta(\xi)}{1+\xi^2+\beta}\right)\int_\Omega f_6\bar{\psi}\,dx \\
& -b_1 s\,e^{-s}\left(\int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{1+\xi^2+\beta}\,d\xi\right)\int_\Omega\int_0^1 e^{s\tau}f_7(x,\tau)\,d\tau\,dx \\
& -b_2 s\,e^{-s}\left(\int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{1+\xi^2+\beta}\,d\xi\right)\int_\Omega\int_0^1 e^{s\tau}f_8(x,\tau)\,d\tau\,dx,
\end{aligned}
$$

with simple and straightforward calculation, it follows that M is coercive and continuous operator and L is continuous hence, virtue to the Lax-Milgram theorem, the problem (3.11) has a unique solution $(\varphi,\psi)\in\left(H_0^1(\Omega)\right)^2$. Due to the classical elliptic regularity, it result that $(\varphi,\psi)\in\left(H^2(\Omega)\right)^2$. Finally, it remains only to show that

$$
\left(\xi^2+\beta\right)\phi-z_i(x,1)\,\eta(\xi)\in L^2\left(\Omega\times(-\infty,+\infty)\right)\ \text{for } i=1,2.
$$

Returning to $(3.3)_5$ and $(3.3)_6$ we have

$$
\begin{aligned}
\left(\xi^2+\beta\right)\phi_1-z_1(x,1)\,\eta(\xi)=f_5-\phi_1\in L^2\left(\Omega\times(-\infty,+\infty)\right), \\
\left(\xi^2+\beta\right)\phi_2-z_2(x,1)\,\eta(\xi)=f_6-\phi_2\in L^2\left(\Omega\times(-\infty,+\infty)\right).
\end{aligned}
$$

Hence, we conclude that $U\in D(A)$, so, the operator $I-A$ is surjective. □

4. Decay exponential

In this section, we prove the decay result using the multiplier method. For this purpose, we need to some lemmas and functionals. The first, we introduce the following functions

$$
\begin{aligned}
k_1(t) := {} & \rho_1\int_\Omega \varphi_t\varphi\,dx+\rho_2\int_\Omega\psi\psi_t\,dx \\
& +\sum_{i=1}^{2}\frac{b_i}{2}\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)\left|M_i(x,\xi,t)\right|^2\,d\xi\,dx,
\end{aligned}
\tag{4.1}
$$

where

$$M_i(x,\xi,t) = \int_0^t \phi_i(x,\xi,\tau)\ d\tau - \frac{s\eta(\xi)}{\xi^2+\beta}\int_0^1 f_0^i(x,-\rho s)\ d\rho + \frac{\varphi_0^i \eta(\xi)}{\xi^2+\beta}$$

with

$$f_0^i(x,-\rho s) = \begin{cases} f_0(x,-\rho s), & i=1 \\ g_0(x,-\rho s), & i=2 \end{cases}$$

and

$$\varphi_0^i(x) = \begin{cases} \varphi_0(x), & i=1 \\ \psi_0(x), & i=2. \end{cases}$$

Denoting from now on

$$\varphi^i = \begin{cases} \varphi, & i=1 \\ \psi, & i=2. \end{cases}$$

Lemma 4.1. *(see [2]) Let $(\varphi,\phi_1,z_1,\psi,\phi_2,z_2)$ be a regular solution of problem (P'), then we have for i=1,2*

$$\begin{aligned}
&\int_\Omega\int_{-\infty}^{+\infty} \left(\xi^2+\beta\right) M_i(x,\xi,t)\ \phi_i(x,\xi,t)\ d\xi\ dx \\
&-\int_\Omega\int_{-\infty}^{+\infty} \varphi^i(x,t)\,\phi_i(x,\xi,t)\,\eta(\xi)\ d\xi\ dx \\
&= -s\int_\Omega\int_0^1 z_i(x,\rho,t)\int_{-\infty}^{+\infty}\eta(\xi)\,\phi_i(x,\xi,t)\ d\xi\ d\rho\ dx \\
&-\int_\Omega\int_{-\infty}^{+\infty} |\phi_i(x,\xi,t)|^2\ d\xi\ dx.
\end{aligned}$$

Lemma 4.2. *(see [2]) Let $(\varphi,\phi_1,z_1,\psi,\phi_2,z_2)$ be a regular solution of problem (P'), then we have for i=1,2*

$$\begin{aligned}
&\left|\int_\Omega\int_{-\infty}^{+\infty} \left(\xi^2+\beta\right)|M_i(x,\xi,t)|^2\ d\xi\ dx\right| \\
&\le 3s^2A_0\int_\Omega\int_0^1 |z_i(x,\rho,t)|^2\ d\rho\ dx \\
&+3A_0C_\star^2\|\varphi_x^i\|^2 + \frac{3}{\beta}\int_\Omega\int_{-\infty}^{+\infty}|\phi_i(x,\xi,t)|^2\,d\xi dx.
\end{aligned}$$

Lemma 4.3. *Assume (2.2) hold. The functional k_1 defined in (4.1) satisfies*

$$\begin{aligned} k_1'(t) \leq & -\frac{k}{2}\|\varphi_x\|^2 - \frac{b}{2}\|\psi_x\|^2 - \frac{b}{4C_{\star\star}}\|\psi\|^2 \\ & - \sum_{i=1}^{2} b_i \int_\Omega \int_{-\infty}^{+\infty} |\phi_i(x,\xi,t)|^2 \, d\xi \, dx + C\left(\|\varphi_t\|^2 + \|\psi_t\|^2\right) \\ & + s^2 \sum_{i=1}^{2} v_i \int_\Omega \int_0^1 |z_i(x,\rho,t)|^2 \, d\rho \, dx \\ & + \sum_{i=1}^{2} \frac{b_i}{4} \int_\Omega \int_{\infty}^{+\infty} \left(\xi^2+\beta\right) |\phi_i(x,\xi,t)|^2 \, d\xi \, dx, \end{aligned}$$

where $\lambda = k - \frac{\mu^2}{m}$ and $C_{\star\star}$ is the Poincaré constant.

Proof. Differentiating k_1 respect to t, using the first and second equation of (P'), by integration by parts and we take the boundary conditions into account to yield

$$\begin{aligned} k_1'(t) = & -k\|\varphi_x\|^2 - b\|\psi_x\|^2 - 2\mu \int_\Omega \psi\varphi_x \, dx - m\|\psi\|^2 \\ & -b_1 \int_\Omega \varphi \left(\int_{-\infty}^{+\infty} \phi_1(x,\xi,t)\,\eta(\xi) \, d\xi \right) dx \\ & -b_2 \int_\Omega \psi \left(\int_{-\infty}^{+\infty} \phi_2(x,\xi,t)\,\eta(\xi) \, d\xi \right) dx \\ & -\alpha_1 \int_\Omega \varphi\varphi_t \, dx - \alpha_2 \int_\Omega \psi\psi_t \, dx \\ & +\rho_1\|\varphi_t\|^2 + \rho_2\|\psi\|^2 \\ & +\sum_{i=1}^{2} b_i \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2+\beta\right) M_i(x,\xi,t)\,\phi_i(x,\xi,t)\, d\xi dx, \end{aligned} \tag{4.2}$$

by virtue of lemma 4.1, so (28) becomes

$$\begin{aligned} k_1'(t) = & -\lambda\|\varphi_x\|^2 - b\|\psi_x\|^2 - \frac{1}{m}\|\mu\varphi + m\psi\|^2 \\ & -\alpha_1 \int_\Omega \varphi\varphi_t \, dx - \alpha_2 \int_\Omega \psi\psi_t \, dx + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 \\ & -s \sum_{i=1}^{2} b_i \int_\Omega \int_0^1 z_i(x,\rho,t) \int_{-\infty}^{+\infty} \eta\phi_i(x,\xi,t) \, d\xi \, d\rho \, dx \\ & -\sum_{i=1}^{2} b_i \int_\Omega \int_{-\infty}^{+\infty} |\phi_i(x,\xi,t)|^2 \, d\xi \, dx. \end{aligned} \tag{4.3}$$

Now, we will estimate the terms of the right-hand side of (4.3) as follows. Since $b_I A_0 < v_i$ for $i = 1, 2,$ and integrating the inequality in lemma 2.3 over $(0,1)$ respect to ρ to arrive at

$$\begin{aligned}
& - s\sum_{i=1}^{2} b_i \int_\Omega \int_0^1 z_i\left(x,\rho,t\right) \int_{-\infty}^{+\infty} \eta\left(\xi\right) \phi_i\left(x,\xi,t\right)\ d\xi\ d\rho\ dx \\
& \leq s^2 \sum_{i=1}^{2} v_i \int_\Omega \int_0^1 \left|z_i\left(x,\rho,t\right)\right|^2\ d\rho\ dx \qquad (4.4)\\
& + \sum_{i=1}^{2} \frac{b_i}{4} \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2+\beta\right) \left|\phi_i\left(x,\xi,t\right)\right|^2\ d\xi\ dx.
\end{aligned}$$

Using the Young's and Poincaré inequalities to find

$$\begin{aligned}
-\alpha_1 \int_\Omega \varphi\varphi_t\ dx - \alpha_2 \int_\Omega \psi\psi_t\ dx \leq{}& \alpha_1\delta_1 C_* \|\varphi_x\|^2 + \alpha_2\delta_2 C_{**}\|\psi_x\|^2 \\
& + \frac{\alpha_1 C_*}{4\delta_1}\|\varphi_t\|^2 + \frac{\alpha_2}{4\delta_2}\|\psi_t\|^2.
\end{aligned}$$

We choose $\delta_1 = \dfrac{\lambda}{2C_\star \alpha_1}$ and $\delta_2 = \dfrac{b}{2\alpha_2 C_{\star\star}}$ consequently

$$-\alpha_1 \int_\Omega \varphi\varphi_t\ dx - \alpha_2 \int_\Omega \psi\psi_t\ dx \leq \frac{\lambda}{2}\|\varphi_x\|^2 + \frac{b}{2}\|\psi_x\|^2 + C\left(\|\varphi_t\|^2 + \|\psi_t\|^2\right). \qquad (4.5)$$

Inserting (4.4) and (4.5) in (4.3), then using the Poincaré inequality it follows that

$$\begin{aligned}
k_1'\left(t\right) \leq{}& -\frac{\lambda}{2}\|\varphi_x\|^2 - \frac{b}{4}\|\psi_x\|^2 - \frac{b}{4C_{\star\star}}\|\psi\|^2 \\
& + \sum_{i=1}^{2} \frac{b_i}{4} \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2+\beta\right) \left|\phi_i\left(x,\xi,t\right)\right|^2\ d\xi\ dx + C\|\varphi_t\|^2 + C\|\psi_t\|^2 \\
& + s^2 \sum_{i=1}^{2} v_i \int_\Omega \int_0^1 \left|z_i\left(x,\rho,t\right)\right|^2\ d\rho\ dx \\
& - \sum_{i=1}^{2} b_i \int_\Omega \int_{-\infty}^{+\infty} \left|\phi_i\left(x,\xi,t\right)\right|^2\ d\xi\ dx.
\end{aligned}$$

□

Next, we introduce the following functional

$$k_2\left(t\right) := s\sum_{i=1}^{2} \int_\Omega \int_0^1 e^{-s\rho} \left|z_i\left(x,\rho,t\right)\right|^2\ d\rho\ dx. \qquad (4.6)$$

Lemma 4.4. *the functional k_2 define in (4.6) satisfies*

$$k_2'(t) \leq -s\, e^{-s} \sum_{i=1}^{2} \int_{\Omega} \int_0^1 |z_i(x,\rho,t)|^2 \, d\rho\, dx + \|\varphi_t\|^2 + \|\psi_t\|^2. \tag{4.7}$$

Proof. We take derivative of k_2 respect to t and using $(2.6)_4$ and $(2.6)_6$, we arrive at

$$\begin{aligned} k_2'(t) &= \sum_{i=1}^{2} \int_{\Omega} |z_i(x,0,t)|^2 \, dx - \sum_{i=1}^{2} \int_{\Omega} e^{-s} |z_i(x,1,t)|^2 \, d\rho\, dx \\ &\quad -s \sum_{i=1}^{2} \int_{\Omega} \int_0^1 e^{-s\rho} |z_i(x,\rho,t)|^2 \, d\rho\, dx. \end{aligned}$$

We see that $z_i(x,0,t) = \varphi_t^i(x,t)$ and using the fact $e^{-s\rho} \geq e^{-s}$ therefore, we obtain (4.7). □

Lemma 4.5. *the functionals k_1 and k_2 defined in (4.1) and (4.6) satifies*

$$|k_1(t)| < CE(t) \text{ and } k_2(t) \leq CE(t)$$

for some positive constant C.

Proof. Using lemma 4.2 and Young's inequality we find easily that $|k_1(t)| < CE(t)$. Since $e^{-s\rho} < 1$, then we obtain

$$k_2(t) \leq s \sum_{i=1}^{2} \int_{\Omega} \int_0^1 |z_i(x,\rho,t)|^2 \, d\rho\, dx \leq CE(t).$$

□

It is ready to state and prove the main result for this we introduce the perturbed energy as follows.

$$L(t) := NE(t) + \varepsilon k_1(t) + k_2(t) \text{ for } \varepsilon > 0 \text{ and } N > 0.$$

Theorem 4.1. *Assume (2.2) holds and $U_0 \in \mathcal{H}$, then any solution of (P') satisfies*

$$E(t) \leq d_1\, e^{-d_2\, t}\ \forall t \geq 0,$$

for some positive constants d_1 and d_2 independent of t.

Proof. by using lemma 4.3 and lemma 4.4 we deduce that

$$\begin{aligned} L'(t) \leq & -(NC-\varepsilon C-1)\|\varphi_t\|^2-(NC-\varepsilon C-1)\|\psi_t\|^2 \\ & -\frac{\lambda\varepsilon}{2}\|\varphi_x\|^2-\frac{b\varepsilon}{4}\|\psi_x\|^2-\frac{b\varepsilon}{4C_{**}}\|\psi\|^2 \\ & -\varepsilon\sum_{i=1}^{2} b_i\int_{\Omega}\int_{-\infty}^{+\infty}|\phi_i(x,\xi,t)|^2\, d\xi\, dx \\ & -\sum_{i=1}^{2} s\left(e^{-s}-v_i s\varepsilon\right)\int_{\Omega}\int_0^1 |z_i(x,\rho,t)|^2\, d\rho\, dx. \end{aligned}$$

At this point, we choose ε small enough such that $e^{-s}-v_i s\,\varepsilon>0$ for $i=1,2$. Then, picking N large enough so as $N>\max\left(\dfrac{C\varepsilon+1}{C},\dfrac{\varepsilon}{2}\right)$. This implies that there exist constant d such that

$$L'(t)\leq -d\,E(t)\ \ \forall t\geq 0.$$

In the other hand, we remark virtue of lemma 4.5 that $L(t)$ and $E(t)$ are equivalent for all $t>0$. So there exist $d_1>0$ such that

$$L'(t)\leq -d_1 L(t)\ \ \forall t\geq 0, \tag{4.8}$$

therefore, by integration simple (4.8) over $(0,t)$, we deduce that

$$L(t)\leq L(0)\,e^{-d_1\,t}\ \forall t\geq 0.$$

Hence, we conclude that

$$E(t)\leq d_2 e^{-d_1\,t}\ \forall t\geq 0,$$

with $d_2>0$. □

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