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EXISTENCE OF SOLUTIONS OF THE TEN VELOCITY SPATIAL DISCRETE MODEL OF GAS

Tinfili Natta[1] and Amah Séna d'Almeida

ABSTRACT. We use the fractional step approximation method to prove the existence and the uniqueness of solutions of an initial boundary value problem for a spatial ten velocity discrete model in the study unsteady Couette flow.

1. INTRODUCTION

The development of industrial applications of gas flows in micro devices is rapid in recent years [5]. In these small systems the flows are in slip or transitional regimes and phenomena of rarefied gas flows such as velocity slip and temperature jump are observed [6, 10]. Due to the limitation of experimental conditions, the experiments are mainly limited to some simple structures [10] and the studies of micro gas flows still mainly rely on theoretical and computational methods. Therefore the investigation of gas dynamics problems such as Couette flows in slip and transitional regimes can give insight in the understanding of these kinds of flows. The study of these problems deserves the resolution of the Boltzmann equation as the Navier Stokes ones are not valid in such flow regimes. The Boltzmann equation is complex and several simplified models have been proposed.

[1]*corresponding author*

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The discrete velocity models whose velocity spaces are finite sets of vectors have its main features [1, 7]. In this paper the fractional step method is used to prove the existence and uniqueness and to find an approximating procedure for the solution of the ten spatial discrete velocity model C_1 [4] for the unsteady Couette flow. The first numerical resolution of the unsteady Couette flow in the scope of discrete kinetic theory [8] was carried out by means of the fractional step method with a four velocity planar discrete model. The fractional step method was used by Temam [12] to establish the existence and uniqueness of an initial boundary value problem for the two dimensional Carleman model [2] with zero dirichlet conditions on the boundaries and by Sultangazin [11] to obtain existence and uniqueness of the solution for a discrete model in one dimension. The aim of this paper is to extend the results to include the more physical boundary conditions of diffuse reflection and to discrete models having different speeds which are convenient in the treatement of flows involving thermal processes. In section 2 the discrete velocity model is presented and the mathematical problem is stated. The uniqueness of the solution is proved in section 3 and in section 4 follows the proof of the existence of an approximate solution and its convergence towards the exact solution.

2. Statement of the problem

The planar Couette problem of shear flow and heat transfer between parallel infinite and moving plates is interesting since it helps to understand the behavior of gas flow near a solid boundary. An exact solution of the steady problem for the ten velocity model C_1 is presented in [3, 4]. For the four velocities plane model of Broadwell, we have determined the exact analytic solution for the unsteady Couette problem [9].

The origin O of the orthonormal system of coordinates (x', y', z') of the physical space is chosen so that the plates are located at $y = \pm\frac{h}{2}$, $h > 0$. The velocities of the model C_1 are $\vec{u}_1 = c(-1, 1, 1)$, $\vec{u}_2 = c(1, 1, 1)$, $\vec{u}_3 = c(-1, -1, 1)$, $\vec{u}_4 = c(1, -1, 1)$, $\vec{u}_{9-i} = -\vec{u}_i$, $i \in \{1,\ 2,\ 3,\ 4\}$, and $\vec{u}_9 = -\vec{u}_{10} = c(0, 1, 0)$. The number density of particles of velocity $\vec{u}_i$ at the point $M(x', y', z')$ and at the time t' is denoted by $N_i(t', x', y', z'),\ i \in \{1,\ 2,\ 3,\ 4,\ 9,\ 10\}$.

The model has two different speeds and only linearly independent summational invariants [3, 4]. For sake of simplicity and by analogy with classical studies of the problem, we assume that the distribution of the velocities is symmetrical with respect to the $Ox'y'$ plane and we shall look for a solution of the unsteady problem under the assumption that the microscopic densities depend only on the time t' and the spatial variable y'. The number of unknown densities is reduced to six: the number densities $N_i(t', y')$, $i \in \{1,\ 2,\ 3,\ 4,\ 9,\ 10\}$. We choose the reference values t_0 for the time, c for the velocity, n_0 for the density and h for the transversal lenght and introduce the nondimensional quantities:

$$y = \frac{y'}{h}, \quad t = \frac{t'}{t_0}, \mathsf{Kn} = (sn_0h)^{-1}, \quad \mathsf{St} = \frac{h}{ct_0}, \quad n_i = \frac{N_i}{n_0}. \tag{2.1}$$

The initial boundary value problem in dimensionless variables for the unknown microscopic densities $n_i(t, y)$, $i \in \{1,\ 2,\ 3,\ 4,\ 9,\ 10\}$, takes the form:

$$(2.2)\quad \begin{cases} \mathsf{St}\frac{\partial n_1}{\partial t} + \frac{\partial n_1}{\partial y} = \frac{(\sqrt{2}+\sqrt{3})}{2Kn}(n_2n_3 - n_1n_4) + \frac{\sqrt{6}}{4Kn}(n_3n_9 - n_1n_{10}) & (2.2.1)\\ \mathsf{St}\frac{\partial n_2}{\partial t} + \frac{\partial n_2}{\partial y} = \frac{(\sqrt{2}+\sqrt{3})}{2Kn}(n_1n_4 - n_2n_3) + \frac{\sqrt{6}}{4Kn}(n_4n_9 - n_2n_{10}) & (2.2.2)\\ \mathsf{St}\frac{\partial n_3}{\partial t} - \frac{\partial n_3}{\partial y} = \frac{(\sqrt{2}+\sqrt{3})}{2Kn}(n_1n_4 - n_2n_3) + \frac{\sqrt{6}}{4Kn}(n_1n_{10} - n_3n_9) & (2.2.3)\\ \mathsf{St}\frac{\partial n_4}{\partial t} - \frac{\partial n_4}{\partial y} = \frac{(\sqrt{2}+\sqrt{3})}{2Kn}(n_2n_3 - n_1n_4) + \frac{\sqrt{6}}{4Kn}(n_2n_{10} - n_4n_9) & (2.2.4)\\ \mathsf{St}\frac{\partial n_9}{\partial t} + \frac{\partial n_9}{\partial t} = \frac{\sqrt{6}}{2Kn}[(n_1+n_2)n_{10} - (n_3+n_4)n_9] & (2.2.5)\\ \mathsf{St}\frac{\partial n_{10}}{\partial t} - \frac{\partial n_{10}}{\partial y} = \frac{\sqrt{6}}{2Kn}[(n_3+n_4)n_9 - (n_1+n_2)n_{10}] & (2.2.6)\\ n_i(0,y) = n_{0i}(y), \quad i \in \{1,\ 2,\ 3,\ 4,\ 9,\ 10\} & (2.2.7)\\ n_i\left(t, -\frac{h}{2}\right) = n_{iw}^{-}\lambda^{-}(t), \quad i \in \{1,\ 2,\ 9\} & (2.2.8)\\ n_i\left(t, +\frac{1}{2}\right) = n_{iw}^{+}\lambda^{+}(t), \quad i \in \{3,\ 4,\ 10\} & (2.2.9)\\ 2\left[n_1\left(t,-\frac{1}{2}\right) + n_2\left(t,-\frac{1}{2}\right) - n_3\left(t,-\frac{1}{2}\right) - n_4\left(t,-\frac{1}{2}\right)\right] & \\ \qquad +n_9\left(t,-\frac{1}{2}\right) - n_{10}\left(t,-\frac{1}{2}\right) = 0 & (2.2.10)\\ 2\left[n_1\left(t,+\frac{1}{2}\right) + n_2\left(t,+\frac{1}{2}\right) - n_3\left(t,+\frac{1}{2}\right) - n_4\left(t,+\frac{1}{2}\right)\right] & \\ \qquad +n_9\left(t,+\frac{1}{2}\right) - n_{10}\left(t,+\frac{1}{2}\right) = 0 & (2.2.11) \end{cases}$$

where St is the Strouhal number and Kn the Knudsen number. The boundary conditions (2.2.8) and (2.2.9) express the diffuse reflection of the gas particles with arbitrary accommodation and the relations (2.2.10) and (2.2.11) the impermeability of the plates. The initial densities n_i^0, $i \in \{1,\ 2,\ 3,\ 4,\ 9,\ 10\}$, and the accommodation coefficients $\lambda^{\pm}$ are continuous non negatives functions of y and t respectively. The quantities n_{iw}^{-}, $i \in \{1,\ 2,\ 9\}$ and n_{iw}^{+}, $i \in \{3,\ 4,\ 10\}$ are the microscopic densities of the fictituous discret gas in maxwellian equilibrium with the plates respectively at $y = -\frac{1}{2}$ and $y = \frac{1}{2}$ they are non negative functions of t.

Let $\Omega = \left]-\frac{1}{2},\ \frac{1}{2}\right[$, be an interval of $\mathbb{R}$ and y a point of Ω. The Sobolev space $H^1(\Omega)$ defined by $H^1(\Omega) = \left\{u \mid u \in L^2(\Omega),\ \frac{\mathrm{d}u}{\mathrm{d}y} \in L^2(\Omega)\right\}$, is a Hilbert space for the scalar product $\langle\langle u, v\rangle\rangle = \langle u, v\rangle + \left\langle \frac{\mathrm{d}u}{\mathrm{d}y}, \frac{\mathrm{d}v}{\mathrm{d}y}\right\rangle$, where $\langle u, v\rangle = \int_\Omega u(y)v(y)\mathrm{d}y$ and $\|u\| = \sqrt{\langle u, u\rangle}$ defines the norm in $L^2(\Omega)$.

The remainder of the paper is devoted to the proof of the following result:

Theorem 2.1. *Let* $\mathrm{n}^0 = \{n_{01}, n_{02}, n_{03}, n_{04}, n_{09}, n_{010}\}$ *such that* $n_{0i}(y) \in H^1(\Omega) \cap L^\infty(\Omega)$ *and* $n_{0i}(y) \geq 0$ *a.e.. Then there exists a unique solution* $\mathrm{n} = \{n_1,\ n_2,\ n_3,\ n_4,\ n_9,\ n_{10}\}$ *of the problem* (2.2) *which satisfies:*

$$n_i(t,y) \in L^\infty\left([0,\ \mathsf{T}];\ H^1(\Omega)\right) \cap L^\infty([0,\ \mathsf{T}] \times \Omega) \quad \textit{and} \quad n_i(t,y) \geq 0 \ \textit{a.e.}. \tag{2.3}$$

3. Uniqueness of the solution of (2.2)

Let $\mathrm{n} = (n_1,\ n_2,\ n_3,\ n_4,\ n_9,\ n_{10})$ and $\mathrm{f} = (f_1,\ f_2,\ f_3,\ f_4,\ f_9,\ f_{10})$ be two solutions of the problem (2.2) and put $\mathrm{g} = \mathrm{n} - \mathrm{f} = (g_1,\ g_2,\ g_3,\ g_4,\ g_9,\ g_{10})$. Then we have the problem (3.1)-(3.2), with $\alpha = \frac{(\sqrt{2}+\sqrt{3})}{\mathsf{KnSt}}$ and $\beta = \frac{\sqrt{6}}{2\mathsf{KnSt}}$:

$$(3.1)\quad \left\{\begin{aligned}
\frac{\partial g_1}{\partial t} + \frac{1}{\mathsf{St}}\frac{\partial g_1}{\partial y} &= \alpha\left(n_2n_3 - n_1n_4 - f_2f_3 + f_1f_4\right)\\
&\quad +\beta\left(n_3n_9 - n_1n_{10} - f_3f_9 + f_1f_{10}\right) && (3.1.1)\\
\frac{\partial g_2}{\partial t} + \frac{1}{\mathsf{St}}\frac{\partial g_2}{\partial y} &= \alpha\left(n_1n_4 - n_2n_3 - f_1f_4 + f_2f_3\right)\\
&\quad +\beta\left(n_4n_9 - n_2n_{10} - f_4f_9 + f_2f_{10}\right) && (3.1.2)\\
\frac{\partial g_3}{\partial t} - \frac{1}{\mathsf{St}}\frac{\partial g_3}{\partial y} &= \alpha\left(n_1n_4 - n_2n_3 - f_1f_4 + f_2f_3\right)\\
&\quad +\beta\left(n_1n_{10} - n_3n_9 - f_1f_{10} + f_3f_9\right) && (3.1.3)\\
\frac{\partial g_4}{\partial t} - \frac{1}{\mathsf{St}}\frac{\partial g_4}{\partial y} &= \alpha\left(n_2n_3 - n_1n_4 - f_2f_3 + f_1f_4\right)\\
&\quad +\beta\left(n_2n_{10} - n_4n_9 - f_2f_{10} + f_4f_9\right) && (3.1.4)\\
\frac{\partial g_9}{\partial t} + \frac{1}{\mathsf{St}}\frac{\partial g_9}{\partial y} &= 2\beta\left[(n_1 + n_2)n_{10} - (n_3 + n_4)n_9\right.\\
&\quad \left.-(f_1 + f_2)f_{10} + (f_3 + f_4)f_9\right] && (3.1.5)\\
\frac{\partial g_{10}}{\partial t} - \frac{1}{\mathsf{St}}\frac{\partial g_{10}}{\partial y} &= 2\beta\left[(n_3 + n_4)n_9 - (n_1 + n_2)n_{10}\right.\\
&\quad \left.-(f_3 + f_4)f_9 + (f_1 + f_2)f_{10}\right] && (3.1.6)
\end{aligned}\right.$$

(3.2)
$$\begin{cases} g_i(0,y)=0, \quad i\in\{1,\ 2,\ 3,\ 4,\ 9,\ 10\} & (3.2.1)\\ g_i\left(t,-\frac{1}{2}\right)=0, \quad i\in\{1,\ 2,\ 9\} & (3.2.2)\\ g_i\left(t,+\frac{1}{2}\right)=0, \quad i\in\{3,\ 4,\ 10\} & (3.2.3)\\ 2\left[g_1\left(t,-\frac{1}{2}\right)+g_2\left(t,-\frac{1}{2}\right)-g_3\left(t,-\frac{1}{2}\right)-g_4\left(t,-\frac{1}{2}\right)\right] & \\ \qquad +g_9\left(t,-\frac{1}{2}\right)-g_{10}\left(t,-\frac{1}{2}\right)=0 & (3.2.4)\\ 2\left[g_1\left(t,+\frac{1}{2}\right)+g_2\left(t,+\frac{1}{2}\right)-g_3\left(t,+\frac{1}{2}\right)-g_4\left(t,+\frac{1}{2}\right)\right] & \\ \qquad +g_9\left(t,+\frac{1}{2}\right)-g_{10}\left(t,+\frac{1}{2}\right)=0 & (3.2.5)\end{cases}$$

As

$$\begin{aligned} n_1n_4-f_1f_4&=\frac{1}{2}\left[(n_1-f_1)(n_4+f_4)+(n_1+f_1)(n_4-f_4)\right]\\ n_2n_3-f_2f_3&=\frac{1}{2}\left[(n_2-f_2)(n_3+f_3)+(n_2+f_2)(n_3-f_3)\right]\\ n_3n_9-f_3f_9&=\frac{1}{2}\left[(n_3-f_3)(n_9+f_9)+(n_3+f_3)(n_9-f_9)\right]\\ n_1n_{10}-f_1f_{10}&=\frac{1}{2}\left[(n_1-f_1)(n_{10}+f_{10})+(n_1+f_1)(n_{10}-f_{10})\right]\end{aligned}$$

we can rewrite (3.1.1) as follows:

(3.3)
$$\begin{aligned} 2\frac{\partial g_1}{\partial t}+\frac{2}{\mathsf{St}}\frac{\partial g_1}{\partial y}&=-\alpha\left[(n_4+f_4)g_1+(n_1+f_1)g_4+(n_3+f_3)g_2+(n_2+f_2)g_3\right]\\ &\quad-\beta\left[(n_{10}+f_{10})g_1+(n_1+f_1)g_{10}+(n_9+f_9)g_3+(n_3+f_3)g_9\right].\end{aligned}$$

By multiplying (3.3) by g_1 and integrating on Ω, we obtain:

(3.4)
$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\|g_1\right\|^2+\frac{1}{\mathsf{St}}\int_{-\frac{1}{2}}^{\frac{1}{2}}2g_1\frac{\partial g_1}{\partial y}\mathrm{d}y+\alpha\int_{-\frac{1}{2}}^{\frac{1}{2}}(n_4+f_4)g_1^2\mathrm{d}y+\beta\int_{-\frac{1}{2}}^{\frac{1}{2}}(n_{10}+f_{10})g_1^2\mathrm{d}y\\ &\qquad=\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[-\alpha(n_1+f_1)g_4g_1+\alpha(n_3+f_3)g_2g_1+\alpha(n_2+f_2)g_3g_1\right.\\ &\qquad\left.-\beta(n_1+f_1)g_{10}g_1+\beta(n_9+f_9)g_3g_1+\beta(n_3+f_3)g_9g_1\right]\mathrm{d}y.\end{aligned}$$

Since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}2g_1(t,y)\frac{\partial g_1(t,y)}{\partial y}\mathrm{d}y=\left[g_1(t,y)^2\right]_{-\frac{1}{2}}^{\frac{1}{2}}=[g_1(t,1/2)]^2-\underbrace{[g_1(t,-1/2)]^2}_{0}=[g_1(t,1/2)]^2$$

then we have $\frac{2}{\mathsf{St}}\int_{-\frac{1}{2}}^{\frac{1}{2}} g_1 \frac{\partial g_1}{\partial y}\mathrm{d}y \geq 0$. So,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|g_1\right\|^2 \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} &[-\alpha(n_1+f_1)g_4g_1 + \alpha(n_3+f_3)g_2g_1 + \alpha(n_2+f_2)g_3g_1 \\ &-\beta(n_1+f_1)g_{10}g_1 + \beta(n_9+f_9)g_3g_1 + \beta(n_3+f_3)g_9g_1]\,\mathrm{d}y.\end{aligned} \tag{3.5}$$

If μ is a positive number satisfying $\|f_i\| < \mu$ and $\|n_i\| < \mu$, then

$$\begin{aligned}&|-\alpha(n_1+f_1)g_4g_1 + \alpha(n_3+f_3)g_2g_1 + \alpha(n_2+f_2)g_3g_1 \\ &-\beta(n_1+f_1)g_{10}g_1 + \beta(n_9+f_9)g_3g_1 + \beta(n_3+f_3)g_9g_1| \\ &\leq 2\alpha\mu|g_4||g_1| + 2\alpha\mu|g_2||g_1| + 2\alpha\mu|g_3||g_1| + 2\beta\mu|g_{10}||g_1| + 2\beta\mu|g_3||g_1| + 2\beta\mu|g_9||g_1|\end{aligned}$$

and

$$\begin{aligned}&\int_{-\frac{1}{2}}^{\frac{1}{2}} [-\alpha(n_1+f_1)g_4g_1 + \alpha(n_3+f_3)g_2g_1 + \alpha(n_2+f_2)g_3g_1 \\ &-\beta(n_1+f_1)g_{10}g_1 + \beta(n_9+f_9)g_3g_1 + \beta(n_3+f_3)g_9g_1]\,\mathrm{d}y \\ &\leq 2\alpha\mu\|g_4\|\|g_1\| + 2\alpha\mu\|g_2\|\|g_1\| + 2\alpha\mu\|g_3\|\|g_1\| \\ &+ 2\beta\mu\|g_{10}\|\|g_1\| + 2\beta\mu\|g_3\|\|g_1\| + 2\beta\mu\|g_9\|\|g_1\|.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|g_1\|^2 \leq\ & 2\alpha\mu\left(\|g_2\|\|g_1\| + \|g_3\|\|g_1\| + \|g_4\|\|g_1\|\right) \\ &+ 2\beta\mu\left(\|g_3\|\|g_1\| + \|g_9\|\|g_1\| + \|g_{10}\|\|g_1\|\right).\end{aligned} \tag{3.6}$$

Similarly we can show that:

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\left\|g_2\right\|^2 \leq 2\alpha\mu\left(\|g_1\|\|g_2\| + \|g_3\|\|g_2\| + \|g_4\|\|g_2\|\right) \\ \qquad\qquad + 2\beta\mu\left(\|g_4\|\|g_2\| + \|g_9\|\|g_2\| + \|g_{10}\|\|g_2\|\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\|g_3\|^2 \leq 2\alpha\left(\mu\|g_1\|\|g_3\| + \|g_2\|\|g_3\| + \|g_4\|\|g_3\|\right) \\ \qquad\qquad + 2\beta\mu\left(\|g_1\|\|g_3\| + \|g_9\|\|g_3\| + \|g_{10}\|\|g_3\|\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\|g_4\|^2 \leq 2\alpha\mu\left(\|g_1\|\|g_4\| + \|g_2\|\|g_4\| + \|g_3\|\|g_4\|\right) \\ \qquad\qquad + 2\beta\mu\left(\|g_2\|\|g_4\| + \|g_9\|\|g_4\| + \|g_{10}\|\|g_4\|\right)\end{cases}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\|g_9\|^2 \leq 4\beta\mu\left(\|g_1\|\|g_9\| + \|g_2\|\|g_9\| + \|g_3\|\|g_9\|\right. \\ \qquad\qquad \left. + \|g_4\|\,\|g_9\| + 2\|g_{10}\|\|g_9\|\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\|g_{10}\|^2 \leq 4\beta\mu\left(\|g_1\|\|g_{10}\| + \|g_2\|\|g_{10}\| + \|g_3\|\|g_{10}\|\right. \\ \qquad\qquad \left. + \|g_4\|\|g_{10}\| + 2\|g_9\|\|g_{10}\|\right). \end{cases}$$

As $2\,\|g_k\|\,\|g_l\| \leq \|g_k\|^2 + \|g_l\|^2$, we can deduce that:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{2\left(\|g_1\|^2 + \|g_2\|^2 + \|g_3\|^2 + \|g_4\|^2\right) + \|g_9\|^2 + \|g_{10}\|^2\right\} \\ &\quad \leq 12\mu(\alpha+\beta)\left\{\|g_1\|^2 + \|g_2\|^2 + \|g_3\|^2 + \|g_4\|^2\right\} + 24\mu\beta\left\{\|g_9\|^2 + \|g_{10}\|^2\right\} \\ &\quad \leq \max\left\{(\alpha+\beta), 4\beta\right\} \times 6\mu\left\{2\left(\|g_1\|^2 + \|g_2\|^2 + \|g_3\|^2 + \|g_4\|^2\right) + \|g_9\|^2 + \|g_{10}\|^2\right\} \\ &\quad \leq 12\mu(\alpha+\beta)\left\{2\left(\|g_1\|^2 + \|g_2\|^2 + \|g_3\|^2 + \|g_4\|^2\right) + \|g_9\|^2 + \|g_{10}\|^2\right\}. \end{aligned}$$

Since $\|g_i(0)\| = 0,\ i \in \{1,\ 2,\ 3,\ 4,\ 9,\ 10\}$, we conclude using Gronwall's lemma that

$$2\left(\|g_1(t)\|^2 + \|g_2(t)\|^2 + \|g_3(t)\|^2 + \|g_4(t)\|^2\right) + \|g_9(t)\|^2 + \|g_{10}(t)\|^2 = 0, \tag{3.7}$$

$0 \leq t \leq \mathsf{T}$ and $g_1 = g_2 = g_3 = g_4 = g_9 = g_{10} = 0$, a.e..

4. Existence of the solution of (2.2)

We shall build approximate solutions by means of the fractional step method to establish the existence of the solution of the problem ((2.2)). Given $\mathsf{M} \in \mathbb{N}$, we shall define the families of positive elements $n_i^{m+\frac{1}{2}}$ of $L^\infty(\Omega)$, $i \in \Lambda$, $j = 1,\ 2$ and $0 \leq m \leq \mathsf{M} - 1$, with $\Lambda = \{1,\ 2,\ 3,\ 4,\ 9,\ 10\}$.

The time interval is splitted into M equal subintervals of length $\tau = \frac{\mathsf{T}}{\mathsf{M}}$. To start we let $n_i^0(y) = n_{0i}(y)$ and we denote $n_i^m(y) = n_i(m\tau, y)$, $i \in \Lambda$, $0 \leq m \leq \mathsf{M} - 1$, $\forall y \in \Omega$. We assume that n_i^0 belongs to $L^\infty(\Omega)$ and is positive $\forall\ i \in \Lambda$. When the functions n_i^m, $i \in \Lambda$, which approximate the solution at time $t_m = m\tau$ are known (belonging to $L^\infty(\Omega)$ and positive), the approximating functions corresponding at time $t_{m+1} = (m+1)\tau$ are obtained in two steps. In the first step, the equation (2.2) is solved in the spatially homogenous case on the subinterval $I_m = [t_m, t_{m+1}]$ with the initial condition n_i^m. The solution of this initial value problem at $t_{m+1} =$

$(m+1)\tau$ is denoted by $n_i^{m+\frac{1}{2}}$. In the second step, (2.2) is solved in the free flow case on the same subinterval with the initial condition $n_i^{m+\frac{1}{2}}$. The solution of this initial boundary value problem at $t_{m+1} = (m+1)\tau$ is denoted by n_i^{m+1}.

Proceeding recursively, we obtain the two sequences of the functions n_i^m, $m = 1,\ 2,\ \ldots,\ \mathsf{M}$ and $n_i^{m+\frac{1}{2}}$, $m = 1,\ 2,\ \ldots,\ \mathsf{M}-1$, $i \in \Lambda$, defined in Ω.

We now give details of the computations of $n_i^{m+\frac{1}{2}}$ and n_i^{m+1} in the first and second steps respectively.

4.1. **First step: determination of the densities** $n_i^{m+\frac{1}{2}}$**.** The value $n_i^{m+\frac{1}{2}}$ of the microscopic densities at time $t = (m+1)\tau$ is the solution of the finite difference approximation scheme:

$$\mathsf{St}\frac{n_i^{m+\frac{1}{2}} - n_i^m}{\tau} = Q_i^{m+\frac{1}{2}}, \quad i \in \Lambda. \tag{4.1}$$

The n_i^m, $i \in \Lambda$ are known and the unique solution of equations (4.1) is given in explicit form by:

(4.2)

$$\begin{cases} n_1^{m+\frac{1}{2}} = \dfrac{1}{A}\left[n_1^m + \beta P\left(2n_1^m + 2n_2^m + n_9^m\right)\left(\alpha(P+Q) + \beta R\right)\right. \\ \qquad \left. + \alpha\left(n_1^m + n_2^m\right)P + 2\beta n_1^m(P+Q) + \beta\left(n_9^m P + n_1^m R\right)\right] \\ n_2^{m+\frac{1}{2}} = \dfrac{1}{A}\left[n_2^m + \beta Q\left(2n_1^m + 2n_2^m + n_9^m\right)\left(\alpha(P+Q) + \beta R\right)\right. \\ \qquad \left. + \alpha\left(n_1^m + n_2^m\right)Q + 2\beta n_2^m(P+Q) + \beta\left(n_9^m Q + n_2^m R\right)\right] \\ n_3^{m+\frac{1}{2}} = \dfrac{1}{A}\left[n_3^m + \beta P\left(2n_3^m + 2n_4^m + n_{10}^m\right)\left(\alpha(P+Q) + \beta R\right)\right. \\ \qquad \left. + \alpha\left(n_3^m + n_4^m\right)P + 2\beta n_3^m(P+Q) + \beta\left(n_{10}^m P + n_3^m R\right)\right] \\ n_4^{m+\frac{1}{2}} = \dfrac{1}{A}\left[n_4^m + \beta Q\left(2n_3^m + 2n_4^m + n_{10}^m\right)\left(\alpha(P+Q) + \beta R\right)\right. \\ \qquad \left. + \alpha\left(n_3^m + n_4^m\right)Q + 2\beta n_4^m(P+Q) + \beta\left(n_{10}^m Q + n_4^m R\right)\right] \\ n_9^{m+\frac{1}{2}} = \dfrac{n_9^m + \beta R\left(2n_1^m + 2n_2^m + n_9^m\right)}{1 + \beta(2P + 2Q + R)}, \quad n_{10}^{m+\frac{1}{2}} = \dfrac{n_{10}^m + \beta R\left(2n_3^m + 2n_4^m + n_{10}^m\right)}{1 + \beta(2P + 2Q + R)}, \end{cases}$$

with

(4.3) $$\begin{cases} \alpha = \frac{\tau\left(\sqrt{2}+\sqrt{3}\right)}{\mathsf{KnSt}}, \qquad \beta = \frac{\tau\sqrt{6}}{2\mathsf{KnSt}}, \\ P = n_1^m + n_3^m, \qquad Q = n_2^m + n_4^m, \qquad R = n_9^m + n_{10}^m, \\ A = (1 + \beta(2P + 2Q + R))\,(1 + \alpha(P + Q) + \beta R)\,. \end{cases}$$

As $n_i^m \geq 0$ a.e. and $n_i^m \in L^\infty(\Omega)\ \forall\, i \in \Lambda$, the expressions (4.2) show obviously that:

(4.4) $$n_i^{m+\frac{1}{2}} \geq 0 \text{ a.e.} \quad \text{and } n_i^{m+\frac{1}{2}} \in L^\infty(\Omega)\ \forall\, i \in \Lambda.$$

Moreover,

(4.5) $$\begin{aligned} &2\left(n_1^{m+\frac{1}{2}} + n_2^{m+\frac{1}{2}} + n_3^{m+\frac{1}{2}} + n_4^{m+\frac{1}{2}}\right) + n_9^{m+\frac{1}{2}} + n_{10}^{m+\frac{1}{2}} \\ =&2\left(n_1^m + n_2^m + n_3^m + n_4^m\right) + n_9^m + n_{10}^m, \end{aligned}$$

which means that the number of particles is conserved locally during the collisions.

4.2. **Second step: determination of the densities n_i^{m+1}.** We are interested in the endpoint value of the solution of the free flow equations in the subinterval I_m, so for $t = (m+1)\tau$ we solve the system of equations:

(4.6) $$\mathsf{St}\frac{n_i^{m+1} - n_i^{m+\frac{1}{2}}}{\tau} + v_i \frac{\partial n_i^{m+1}}{\partial y} = 0, \quad i \in \Lambda.$$

with the boundary conditions (2.2.8)-(2.2.11).

The system (4.6) is a system of first order differential equations that can be rewritten in the form:

(4.7) $$\frac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y} + \frac{\mathsf{St}}{\tau v_i} n_i^{m+1} = \frac{\mathsf{St}}{\tau v_i} n_i^{m+\frac{1}{2}}, \quad i \in \Lambda.$$

The boundary value problem (4.7)-(2.2.8)-(2.2.11) can be splitted into the two following boundary value problems:

(4.8) $$\begin{cases} \frac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y} + \frac{\mathsf{St}}{\tau v_i} n_i^{m+1} = \frac{\mathsf{St}}{\tau v_i} n_i^{m+\frac{1}{2}}, \quad i \in \{1,\ 2,\ 9\} & (4.8.1) \\ n_i^{m+1}\left(-\frac{1}{2}\right) = n_{iw}^{-} \lambda^{-(m+1)}, \quad i \in \{1,\ 2,\ 9\} & (4.8.2) \end{cases}$$

(4.9) $$\begin{cases} \frac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y} + \frac{\mathsf{St}}{\tau v_i} n_i^{m+1} = \frac{\mathsf{St}}{\tau v_i} n_i^{m+\frac{1}{2}}, \quad i \in \{3,\ 4,\ 10\} & (4.9.1) \\ n_i^{m+1}\left(+\frac{1}{2}\right) = n_{iw}^{+} \lambda^{+(m+1)}, \quad i \in \{3,\ 4,\ 10\} & (4.9.2) \end{cases},$$

where $\lambda^{\pm(m+1)} = \lambda^{\pm}((m+1)\tau)$ are the values of the accommodation coefficients at the time $(m+1)\tau$ at $y = \pm\frac{1}{2}$.

The problems (4.8) and (4.9) are linked by the impermeability boundary conditions (2.2.10) and (2.2.11) which are written at time $t = (m+1)\tau$ as:

$$\text{(4.10)} \qquad \begin{cases} 2\left[n_1^{m+1}\left(-\frac{1}{2}\right) + n_2^{m+1}\left(-\frac{1}{2}\right) - n_3^{m+1}\left(-\frac{1}{2}\right) - n_4^{m+1}\left(-\frac{1}{2}\right)\right] \\ \qquad +n_9^{m+1}\left(-\frac{1}{2}\right) - n_{10}^{m+1}\left(-\frac{1}{2}\right) = 0, \\ 2\left[n_1^{m+1}\left(+\frac{1}{2}\right) + n_2^{m+1}\left(+\frac{1}{2}\right) - n_3^{m+1}\left(+\frac{1}{2}\right) - n_4^{m+1}\left(+\frac{1}{2}\right)\right] \\ \qquad +n_9^{m+1}\left(+\frac{1}{2}\right) - n_{10}^{m+1}\left(+\frac{1}{2}\right) = 0. \end{cases}$$

The resolution of (4.8),(4.9) gives:

$$\text{(4.11)} \qquad \begin{cases} n_i^{m+1}(y) = \frac{\mathsf{St}}{\tau v_i} \int_{-\frac{1}{2}}^{y} \left[n_i^{m+\frac{1}{2}}(s) \exp\left(\frac{\mathsf{St}}{\tau v_i}(s-y)\right)\right] \mathrm{d}s \\ \qquad +C_i^{m+1} \exp\left(-\frac{\mathsf{St}}{\tau v_i} y\right), \ i \in \{1,\ 2,\ 9\}, \\ n_i^{m+1}(y) = -\frac{\mathsf{St}}{\tau v_i} \int_{y}^{\frac{1}{2}} \left[n_i^{m+\frac{1}{2}}(s) \exp\left(\frac{\mathsf{St}}{\tau v_i}(s-y)\right)\right] \mathrm{d}s \\ \qquad +C_i^{m+1} \exp\left(-\frac{\mathsf{St}}{\tau v_i} y\right), \ i \in \{3,\ 4,\ 10\}, \end{cases}$$

where the constant C_i^{m+1}, $i \in \Lambda$, are given by:

$$\begin{cases} C_1^{m+1} = \dfrac{1}{A_1}\left[\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(2n_1^{m+\frac{1}{2}} + 2n_2^{m+\frac{1}{2}} + n_9^{m+\frac{1}{2}}\right)(s) \times \exp\left(\frac{\mathsf{St}}{\tau}(s-1)\right) \mathrm{d}s \right. \\ \qquad \left. + \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(2n_3^{m+\frac{1}{2}} + 2n_4^{m+\frac{1}{2}} + n_{10}^{m+\frac{1}{2}}\right)(s) \times \exp\left(-\frac{\mathsf{St}}{\tau}s\right) \mathrm{d}s\right], \\ C_3^{m+1} = \dfrac{1}{A_2}\left[\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(2n_1^{m+\frac{1}{2}} + 2n_2^{m+\frac{1}{2}} + n_9^{m+\frac{1}{2}}\right)(s) \times \exp\left(\frac{\mathsf{St}}{\tau}s\right) \mathrm{d}s \right. \\ \qquad \left. + \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(2n_3^{m+\frac{1}{2}} + 2n_4^{m+\frac{1}{2}} + n_{10}^{m+\frac{1}{2}}\right)(s) \times \exp\left(-\frac{\mathsf{St}}{\tau}(s+1)\right) \mathrm{d}s\right], \\ C_2^{m+1} = \dfrac{1}{\alpha_{1,2}} C_1^{m+1},\ C_4^{m+1} = \dfrac{1}{\alpha_{3,4}} C_3^{m+1},\ C_9^{m+1} = \dfrac{1}{\alpha_{1,9}} C_1^{m+1},\ C_{10}^{m+1} = \dfrac{1}{\alpha_{3,10}} C_3^{m+1}, \end{cases}$$

with $\alpha_{1,2} = \frac{n_{1w}^-}{n_{2w}^-}$, $\alpha_{1,9} = \frac{n_{1w}^-}{n_{9w}^-}$, $\alpha_{3,4} = \frac{n_{3w}^+}{n_{4w}^+}$, $\alpha_{3,10} = \frac{n_{3w}^+}{n_{10w}^+}$ and $A_l = 2\tau\beta_l \sinh\left(\frac{\mathsf{St}}{\tau}\right)$, $l \in \{1,2\}$, where $\beta_1 = 2 + \frac{2}{\alpha_{1,2}} + \frac{1}{\alpha_{1,9}}$, $\beta_2 = 2 + \frac{2}{\alpha_{3,4}} + \frac{1}{\alpha_{3,10}}$.

It follows obviously from (4.4) that

$$n_i^{m+1} \geq 0 \text{ a.e. and } n_i^{m+1} \in L^\infty(\Omega)\ \forall\, i \in \Lambda. \tag{4.12}$$

4.3. **Proof of the constancy of the accommodation coefficients.** The a priori estimations we shall perform to prove the existence result depend on the boundary conditions, the determination of which rely on the knowledge of the accommodation coefficients. We prove here that they are constant for lack of their explicit determination. This result ensures that K_1 which is used in the sequel is constant.

The boundary conditions of diffuse reflection impose at $y = -\frac{1}{2}$, $\lambda_i^-(t) = \lambda^-(t)$, $i \in \{1,\ 2,\ 9\}$ and at $y = \frac{1}{2}$, $\lambda_i^+(t) = \lambda^+(t)$, $i \in \{3,\ 4,\ 10\}$. Moreover the normal components with respect to the plates of the velocities of the discrete model in consideration have the same absolute value $|v_i| = |v_j|$ with $i \in \Lambda$ and $j \in \Lambda$. Hence $\lambda^-(t) = n^-(t)$ and $\lambda^+(t) = n^+(t)$ where $n^-(t) = n\left(t, -\frac{1}{2}\right)$, $n^+(t) = n\left(t, +\frac{1}{2}\right)$ and $n = 2(n_1 + n_2 + n_3 + n_4) + n_9 + n_{10}$ [3,4]. Therefore $\lambda^\pm$ is constant if and only if $n^\pm$ is constant.

We deduce from (4.1) by adding up upon $i \in \{1,\ 2,\ 9\}$ and $i \in \{3,\ 4,\ 10\}$ respectively:

$$2\left(\frac{n_1^{m+\frac{1}{2}} - n_1^m}{\tau} + \frac{n_2^{m+\frac{1}{2}} - n_2^m}{\tau}\right) + \frac{n_9^{m+\frac{1}{2}} - n_9^m}{\tau} = 0$$

and

$$2\left(\frac{n_3^{m+\frac{1}{2}} - n_3^m}{\tau} + \frac{n_4^{m+\frac{1}{2}} - n_4^m}{\tau}\right) + \frac{n_{10}^{m+\frac{1}{2}} - n_{10}^m}{\tau} = 0.$$

Hence

$$2\left(n_1^{m+\frac{1}{2}} + n_2^{m+\frac{1}{2}}\right) + n_9^{m+\frac{1}{2}} = 2\left(n_1^m + n_2^m\right) + n_9^m, \tag{4.13}$$

$$2\left(n_3^{m+\frac{1}{2}} + n_4^{m+\frac{1}{2}}\right) + n_{10}^{m+\frac{1}{2}} = 2\left(n_3^m + n_4^m\right) + n_{10}^m. \tag{4.14}$$

The sum (4.13)+(4.14), yields $n^{m+\frac{1}{2}} = n^m$, $\forall\, m$.

Letting $n_- = 2(n_1 + n_2) + n_9$ and $n_+ = 2(n_3 + n_4) + n_{10}$ we infer from equations (4.13) and (4.14) that $n_-^{m+\frac{1}{2}} = n_-^m$ and $n_+^{m+\frac{1}{2}} = n_+^m$; thus in the interval $[m\tau,\ (m+1)\tau[$ we have:

$$\frac{\partial n_-^{m+\frac{1}{2}}}{\partial t} = \frac{\partial n_-^m}{\partial t} = 0 \quad \text{and} \quad \frac{\partial n_+^{m+\frac{1}{2}}}{\partial t} = \frac{\partial n_+^m}{\partial t} = 0. \tag{4.15}$$

On the one hand we obtain from (4.6) by addition:

$$\text{(4.16)} \qquad \mathsf{St}\frac{n_-^{m+1} - n_-^{m+\frac{1}{2}}}{\tau} + \frac{\partial}{\partial y}\left(n_-^{m+1}\right) = 0,$$

$$\text{(4.17)} \qquad \mathsf{St}\frac{n_+^{m+1} - n_+^{m+\frac{1}{2}}}{\tau} - \frac{\partial}{\partial y}\left(n_+^{m+1}\right) = 0.$$

On the other hand, the projection of the conservation equations of the mass and the momentum on the y axis yields

$$\mathsf{St}\frac{\partial}{\partial t}\left(n_-^{m+\frac{1}{2}} + n_+^{m+\frac{1}{2}}\right) + \frac{\partial}{\partial y}\left(n_-^{m+\frac{1}{2}} - n_+^{m+\frac{1}{2}}\right) = 0$$

and

$$\mathsf{St}\frac{\partial}{\partial t}\left(n_-^{m+\frac{1}{2}} - n_+^{m+\frac{1}{2}}\right) + \frac{\partial}{\partial y}\left(n_-^{m+\frac{1}{2}} + n_+^{m+\frac{1}{2}}\right) = 0;$$

hence we deduce, taking into account (4.15), that $n_-^{m+\frac{1}{2}}$ and $n_+^{m+\frac{1}{2}}$ do not depend on y.

The integration of the equations (4.16) and (4.17) gives therefore

$$\text{(4.18)} \qquad n_-^{m+1} = \lambda_1 \exp\left(-\frac{\mathsf{St}}{\tau}y\right) + n_-^{m+\frac{1}{2}} \quad \text{and} \quad n_+^{m+1} = \lambda_2 \exp\left(\frac{\mathsf{St}}{\tau}y\right) + n_+^{m+\frac{1}{2}}.$$

We get from the impermeability conditions $(2.2.10)$ and $(2.2.11)$ written at $t = (m+1)\tau$ the relations:

$$\text{(4.19)} \qquad \left(n_-^{m+1} - n_+^{m+1}\right)\left(-\tfrac{1}{2}\right) = \left(n_-^{m+1} - n_+^{m+1}\right)\left(+\tfrac{1}{2}\right) = 0,$$

which are equivalent to the system:

$$\text{(4.20)} \qquad \begin{cases} \lambda_1 \exp\left(+\frac{1}{2}\frac{\mathsf{St}}{\tau}\right) - \lambda_2 \exp\left(-\frac{1}{2}\frac{\mathsf{St}}{\tau}\right) = 0, \\ \lambda_1 \exp\left(-\frac{1}{2}\frac{\mathsf{St}}{\tau}\right) - \lambda_2 \exp\left(+\frac{1}{2}\frac{\mathsf{St}}{\tau}\right) = 0. \end{cases}$$

The unique solution of the system (4.20) is $\lambda_1 = \lambda_2 = 0$, that is $n_-^{m+1} = n_-^{m+\frac{1}{2}}$ and $n_+^{m+1} = n_+^{m+\frac{1}{2}}$, $\forall m$. Hence we get by addition $n^{m+1} = n^{m+\frac{1}{2}}$, $\forall m$. Thus $n^{m+1} = n^m$, $\forall m$ and the total density is constant at every time step. As it is a continuous function of the time it is constant on $[0,\ \mathsf{T}]$. Consequently λ^- and λ^+ are constants on $[0,\ \mathsf{T}]$.

4.4. A priori estimates (I).

Lemma 4.1. *We have*

$$2\left(\left\|n_1^{m+\frac{j}{2}}\right\|_{L^\infty(\Omega)}+\left\|n_2^{m+\frac{j}{2}}\right\|_{L^\infty(\Omega)}+\left\|n_3^{m+\frac{j}{2}}\right\|_{L^\infty(\Omega)}+\left\|n_4^{m+\frac{j}{2}}\right\|_{L^\infty(\Omega)}\right)+\left\|n_9^{m+\frac{j}{2}}\right\|_{L^\infty(\Omega)}+\left\|n_{10}^{m+\frac{j}{2}}\right\|_{L^\infty(\Omega)}\leq K_0,$$

where K_0 *is a constant,* $0\leq m\leq \mathsf{M}-1$ *and* $j\in\{1,\ 2\}$.

Proof. Let $\lambda_i^{m+\frac{j}{2}}=\left\|n_i^{m+\frac{j}{2}}\right\|_{L^\infty(\Omega)}, \quad 0\leq m\leq \mathsf{M}-1, \quad i\in\Lambda, \quad j\in\{1,\ 2\}$. We have:

$$\begin{aligned}
\lambda_i^{m+1} &= \left\|n_i^{m+1}\right\|_{L^\infty(\Omega)}, \quad i\in\Lambda\\
&\leq \begin{cases} \sup\limits_{y\in[-\frac{1}{2},\ \frac{1}{2}]}\left|\dfrac{\mathsf{St}}{\tau v_i}\displaystyle\int_{-\frac{1}{2}}^{y}\left[n_i^{m+\frac{1}{2}}(s)\exp\left(\dfrac{\mathsf{St}}{\tau v_i}(s-y)\right)\right]\mathrm{d}s \right.\\ \qquad\qquad \left. +C_i^{m+1}\exp\left(-\frac{\mathsf{St}}{\tau v_i}y\right)\right|, \quad i\in\{1,\ 2,\ 9\}\\ \sup\limits_{y\in[-\frac{1}{2},\ \frac{1}{2}]}\left|-\dfrac{\mathsf{St}}{\tau v_i}\displaystyle\int_{y}^{\frac{1}{2}}\left[n_i^{m+\frac{1}{2}}(s)\exp\left(\dfrac{\mathsf{St}}{\tau v_i}(s-y)\right)\right]\mathrm{d}s \right.\\ \qquad\qquad \left. +C_i^{m+1}\exp\left(-\frac{\mathsf{St}}{\tau v_i}y\right)\right|, \quad i\in\{3,\ 4,\ 10\}\end{cases}\\
&\leq \begin{cases} \lambda_i^{m+\frac{1}{2}}\sup\limits_{y\in[-\frac{1}{2},\ \frac{1}{2}]}\displaystyle\int_{-\frac{1}{2}}^{y}\dfrac{\mathsf{St}}{\tau}\exp\left(\dfrac{\mathsf{St}}{\tau v_i}(s-y)\right)\mathrm{d}s\\ \qquad\qquad +\sup\limits_{y\in[-\frac{1}{2},\ \frac{1}{2}]}C_i^{m+1}\exp\left(-\dfrac{\mathsf{St}}{\tau v_i}y\right), \quad i\in\{1,\ 2,\ 9\}\\ \lambda_i^{m+\frac{1}{2}}\sup\limits_{y\in[-\frac{1}{2},\ \frac{1}{2}]}\displaystyle\int_{y}^{\frac{1}{2}}\dfrac{\mathsf{St}}{\tau}\exp\left(\dfrac{\mathsf{St}}{\tau v_i}(s-y)\right)\mathrm{d}s\\ \qquad\qquad +\sup\limits_{y\in[-\frac{1}{2},\ \frac{1}{2}]}C_i^{m+1}\exp\left(-\dfrac{\mathsf{St}}{\tau v_i}y\right), \quad i\in\{3,\ 4,\ 10\}\end{cases} \qquad (4.21)\\
&\leq \begin{cases}\lambda_i^{m+\frac{1}{2}}+C_i^{m+1}\exp\left(\frac{\mathsf{St}}{\tau}\frac{1}{2}\right), \quad i\in\{1,\ 2,\ 9\}\\ \lambda_i^{m+\frac{1}{2}}+C_i^{m+1}\exp\left(\frac{\mathsf{St}}{\tau}\frac{1}{2}\right), \quad i\in\{3,\ 4,\ 10\}\end{cases}
\end{aligned}$$

We also have

$$\begin{aligned}&2\left(n_1^m+n_2^m+n_3^m+n_4^m\right)+n_9^m+n_{10}^m\\ &\leq 2\left(\lambda_1^m+\lambda_2^m+\lambda_3^m+\lambda_4^m\right)+\lambda_9^m+\lambda_{10}^m \quad \text{a.e.,}\end{aligned} \qquad (4.22)$$

and from (4.5) we can write

$$(4.23)\qquad \begin{aligned} &2\left(\lambda_1^{m+\frac{1}{2}}+\lambda_2^{m+\frac{1}{2}}+\lambda_3^{m+\frac{1}{2}}+\lambda_4^{m+\frac{1}{2}}\right)+\lambda_9^{m+\frac{1}{2}}+\lambda_{10}^{m+\frac{1}{2}}\\ &\le 2\left(\lambda_1^m+\lambda_2^m+\lambda_3^m+\lambda_4^m\right)+\lambda_9^m+\lambda_{10}^m. \end{aligned}$$

Finally, using (4.21) and (4.4), we obtain:

$$\begin{aligned} \lambda^{m+1} &\le 2\left(\lambda_1^{m+\frac{1}{2}}+\lambda_2^{m+\frac{1}{2}}+\lambda_3^{m+\frac{1}{2}}+\lambda_4^{m+\frac{1}{2}}\right)+\lambda_9^{m+\frac{1}{2}}+\lambda_{10}^{m+\frac{1}{2}}\\ &\quad+\left[2\left(C_1^{m+1}+C_2^{m+1}+C_3^{m+1}+C_4^{m+1}\right)+C_9^{m+1}+C_{10}^{m+1}\right]\exp\left(\frac{\mathsf{St}}{\tau}\frac{1}{2}\right)\\ &\le 2\left(\lambda_1^m+\lambda_2^m+\lambda_3^m+\lambda_4^m\right)+\lambda_9^m+\lambda_{10}^m+\left(2n_{1w}^-+2n_{2w}^-+n_{9w}^-\right)\lambda^{-(m+1)}\\ &\quad+\left(2n_{3w}^++2n_{4w}^++n_{10w}^+\right)\lambda^{+(m+1)}\\ &\le 2\left(\|n_1^0\|_{L^\infty(\Omega)}+\|n_2^0\|_{L^\infty(\Omega)}+\|n_3^0\|_{L^\infty(\Omega)}+\|n_4^0\|_{L^\infty(\Omega)}\right)\\ &\quad+\|n_9^0\|_{L^\infty(\Omega)}+\|n_{10}^0\|_{L^\infty(\Omega)}\\ &\quad+\left(2n_{1w}^-+2n_{2w}^-+n_{9w}^-\right)\lambda^{-(m+1)}+\left(2n_{3w}^++2n_{4w}^++n_{10w}^+\right)\lambda^{+(m+1)}=K_0, \end{aligned}$$

where $\lambda^{m+1}=2\left(\lambda_1^{m+1}+\lambda_2^{m+1}+\lambda_3^{m+1}+\lambda_4^{m+1}\right)+\lambda_9^{m+1}+\lambda_{10}^{m+1}$.

Thus, $2\left(\lambda_1^{m+1}+\lambda_2^{m+1}+\lambda_3^{m+1}+\lambda_4^{m+1}\right)+\lambda_9^{m+1}+\lambda_{10}^{m+1}\le K_0$, and the result follows. □

4.5. **A priori estimates (II).** We give here a priori estimates on the derivatives of the $n_i^{m+\frac{1}{2}}$, $i\in\Lambda$.

Lemma 4.2. *If τ is small enough $\left(\tau<\frac{\mathsf{KnSt}}{12\left(2\left(\sqrt{2}+\sqrt{3}\right)+\sqrt{6}\right)K_0}=\tau_0\right)$ then the norms $\left\|\frac{\mathrm{d}n_i^{m+\frac{j}{2}}}{\mathrm{d}t}\right\|$ with $j\in\{1,\ 2\}$ and $i\in\Lambda$, are increased by a constant $\forall\ \tau<\tau_0$ and $\forall\ m$.*

Proof.

• On the one hand we differentiate (4.7) with respect to y, multiply by $\frac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y}$ and integrate on Ω. We obtain:

$$(4.24)\qquad \begin{cases} \left\|\dfrac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y}\right\|^2\le\left\langle\dfrac{\mathrm{d}n_i^{m+\frac{1}{2}}}{\mathrm{d}y},\ \dfrac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y}\right\rangle+\dfrac{\tau}{2\mathsf{St}}\left(\dfrac{\mathrm{d}n_i^{m+1}\left(-\frac{1}{2}\right)}{\mathrm{d}y}\right)^2, i\in\{1,2,9\},\\ \left\|\dfrac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y}\right\|^2\le\left\langle\dfrac{\mathrm{d}n_i^{m+\frac{1}{2}}}{\mathrm{d}y},\ \dfrac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y}\right\rangle+\dfrac{\tau}{2\mathsf{St}}\left(\dfrac{\mathrm{d}n_i^{m+1}\left(+\frac{1}{2}\right)}{\mathrm{d}y}\right)^2, i\in\{3,4,10\}. \end{cases}$$

Using the inequality $\|u\|^2 + \|v\|^2 \geq 2\langle u, v\rangle$ the system (4.24) gives:

$$\begin{cases} \left\|\dfrac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y}\right\|^2 \leq \left\|\dfrac{\mathrm{d}n_i^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \dfrac{\tau}{\mathsf{St}}\left(\dfrac{\mathrm{d}n_i^{m+1}\left(-\frac{1}{2}\right)}{\mathrm{d}y}\right)^2, i \in \{1,\ 2,\ 9\}, \\ \left\|\dfrac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y}\right\|^2 \leq \left\|\dfrac{\mathrm{d}n_i^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \dfrac{\tau}{\mathsf{St}}\left(\dfrac{\mathrm{d}n_i^{m+1}\left(+\frac{1}{2}\right)}{\mathrm{d}y}\right)^2, i \in \{3,\ 4,\ 10\}. \end{cases} \tag{4.25}$$

By taking $y = +\frac{1}{2}$ and $y = -\frac{1}{2}$ in (4.7), the system (4.25) becomes:

$$\begin{cases} \left\|\dfrac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y}\right\|^2 \leq \left\|\dfrac{\mathrm{d}n_i^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \dfrac{\mathsf{St}}{\tau}\left(n_i^{m+\frac{1}{2}}\left(-\dfrac{1}{2}\right) - n_i^{m+1}\left(-\dfrac{1}{2}\right)\right)^2, i \in \{1,2,9\} \\ \left\|\dfrac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y}\right\|^2 \leq \left\|\dfrac{\mathrm{d}n_i^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \dfrac{\mathsf{St}}{\tau}\left(n_i^{m+\frac{1}{2}}\left(+\dfrac{1}{2}\right) - n_i^{m+1}\left(+\dfrac{1}{2}\right)\right)^2, i \in \{3,4,10\} \end{cases} \tag{4.26}$$

Since λ^+ and λ^- are constant (see section 4.3), we deduce from (2.2.8) and (2.2.9)that $n_i\left(t, -\frac{1}{2}\right)$, $i \in \{1,\ 2,\ 9\}$ and $n_i\left(t, +\frac{1}{2}\right)$, $i \in \{3,\ 4,\ 10\}$ are constant. Thus, we use the system (4.2) and **Lemma 4.1** to obtain:

$$\begin{cases} \left\|\dfrac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y}\right\|^2 \leq \left\|\dfrac{\mathrm{d}n_i^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \dfrac{\mathsf{St}}{\tau}\left(2\alpha K_0^2 + 8\beta K_0^2\right)^2, i \in \{1,\ 2,\ 3,\ 4\}, \\ \left\|\dfrac{\mathrm{d}n_i^{m+1}}{\mathrm{d}y}\right\|^2 \leq \left\|\dfrac{\mathrm{d}n_i^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \dfrac{\mathsf{St}}{\tau}\left(2\beta K_0^2\right)^2, i \in \{9,\ 10\}. \end{cases} \tag{4.27}$$

Summing up each side of the inequalities (4.27), we get:

$$\begin{aligned} &\left\|\tfrac{\mathrm{d}n_1^{m+1}}{\mathrm{d}y}\right\|^2 + \left\|\tfrac{\mathrm{d}n_2^{m+1}}{\mathrm{d}y}\right\|^2 + \left\|\tfrac{\mathrm{d}n_3^{m+1}}{\mathrm{d}y}\right\|^2 + \left\|\tfrac{\mathrm{d}n_4^{m+1}}{\mathrm{d}y}\right\|^2 + \left\|\tfrac{\mathrm{d}n_9^{m+1}}{\mathrm{d}y}\right\|^2 + \left\|\tfrac{\mathrm{d}n_{10}^{m+1}}{\mathrm{d}y}\right\|^2 \\ \leq &\left\|\tfrac{\mathrm{d}n_1^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \left\|\tfrac{\mathrm{d}n_2^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \left\|\tfrac{\mathrm{d}n_3^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \left\|\tfrac{\mathrm{d}n_4^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \left\|\tfrac{\mathrm{d}n_9^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 \\ + &\left\|\tfrac{\mathrm{d}n_{10}^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \tfrac{4\left(119+48\sqrt{2}+33\sqrt{3}\right)}{(\mathsf{Kn})^2\mathsf{St}}\tau K_0^4. \end{aligned} \tag{4.28}$$

We notice that when τ tends to 0, $\frac{4\left(119+48\sqrt{2}+33\sqrt{3}\right)}{(\mathsf{Kn})^2\mathsf{St}}\tau K_0^4$ tends to 0.

• On the other hand, according to (4.1) we have for $i = 1$:

$$\begin{aligned}\mathrm{D}n_1^{m+\frac{1}{2}} &= \alpha\left(n_2^{m+\frac{1}{2}}\mathrm{D}n_3^{m+\frac{1}{2}} + n_3^{m+\frac{1}{2}}\mathrm{D}n_2^{m+\frac{1}{2}} - n_1^{m+\frac{1}{2}}\mathrm{D}n_4^{m+\frac{1}{2}} - n_4^{m+\frac{1}{2}}\mathrm{D}n_1^{m+\frac{1}{2}}\right)\\ &\quad + \beta\left(n_3^{m+\frac{1}{2}}\mathrm{D}n_9^{m+\frac{1}{2}} + n_9^{m+\frac{1}{2}}\mathrm{D}n_3^{m+\frac{1}{2}} - n_1^{m+\frac{1}{2}}\mathrm{D}n_{10}^{m+\frac{1}{2}} - n_{10}^{m+\frac{1}{2}}\mathrm{D}n_1^{m+\frac{1}{2}}\right) + \mathrm{D}n_1^m.\end{aligned}$$

D is the operator $\frac{\mathrm{d}}{\mathrm{d}y}$, with K_0 the constant defined in **Lemma 4.1**.

Since $\max\{(\alpha+\beta),\ \alpha,\ \beta,\ 2\beta,\ 4\beta\} < 2(\alpha+\beta) = \epsilon$, we can deduce that:

$$\begin{aligned}\left\|\mathrm{D}n_1^{m+\frac{1}{2}}\right\| &\le 2(\alpha+\beta)K_0\left(\left\|\mathrm{D}n_1^{m+\frac{1}{2}}\right\| + \left\|\mathrm{D}n_2^{m+\frac{1}{2}}\right\| + \left\|\mathrm{D}n_3^{m+\frac{1}{2}}\right\|\right.\\ &\quad \left. + \left\|\mathrm{D}n_4^{m+\frac{1}{2}}\right\| + \left\|\mathrm{D}n_9^{m+\frac{1}{2}}\right\| + \left\|\mathrm{D}n_{10}^{m+\frac{1}{2}}\right\|\right) + \left\|\mathrm{D}n_1^m\right\|\end{aligned} \tag{4.29}$$

Using the inégality $2XY \le X^2 + Y^2$ for all $X, Y \in \mathbb{R}$, we have:

$$\begin{aligned}\left\|\mathrm{D}n_1^{m+\frac{1}{2}}\right\|^2 &\le \left(24(\alpha+\beta)^2 K_0^2 + 2(\alpha+\beta)K_0\right)\left(\left\|\mathrm{D}n_1^{m+\frac{1}{2}}\right\|^2 + \left\|\mathrm{D}n_2^{m+\frac{1}{2}}\right\|^2\right.\\ &\quad \left. + \left\|\mathrm{D}n_3^{m+\frac{1}{2}}\right\|^2 + \left\|\mathrm{D}n_4^{m+\frac{1}{2}}\right\|^2 + \left\|\mathrm{D}n_9^{m+\frac{1}{2}}\right\|^2 + \left\|\mathrm{D}n_{10}^{m+\frac{1}{2}}\right\|^2\right)\\ &\quad + \left(1 + 12(\alpha+\beta)K_0\right)\left\|\mathrm{D}n_1^m\right\|^2.\end{aligned} \tag{4.30}$$

We assume that $24(\alpha+\beta)K_0 < 1$, that is to say $\tau < \frac{\mathsf{KnSt}}{12\left(2\left(\sqrt{2}+\sqrt{3}\right)+\sqrt{6}\right)K_0} = \tau_0$.

We establish inequalities analogous to (4.30) for $i = 2, 3, 4, 9, 10$ and by summing, we obtain the following inequality:

$$\begin{aligned}&\left\|\mathrm{D}n_1^{m+\frac{1}{2}}\right\|^2 + \left\|\mathrm{D}n_2^{m+\frac{1}{2}}\right\|^2 + \left\|\mathrm{D}n_3^{m+\frac{1}{2}}\right\|^2\\ &+ \left\|\mathrm{D}n_4^{m+\frac{1}{2}}\right\|^2 + \left\|\mathrm{D}n_9^{m+\frac{1}{2}}\right\|^2 + \left\|\mathrm{D}n_{10}^{m+\frac{1}{2}}\right\|^2\\ &\le \gamma\left(\left\|\mathrm{D}n_1^m\right\|^2 + \left\|\mathrm{D}n_2^m\right\|^2 + \left\|\mathrm{D}n_3^m\right\|^2 + \left\|\mathrm{D}n_4^m\right\|^2 + \left\|\mathrm{D}n_9^m\right\|^2 + \left\|\mathrm{D}n_{10}^m\right\|^2\right),\end{aligned} \tag{4.31}$$

where $\gamma = \frac{(1+12(\alpha+\beta)K_0)}{(1-18(\alpha+\beta)K_0)}$.

Thus:

$$\begin{aligned}&\left\|\frac{\mathrm{d}n_1^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \left\|\frac{\mathrm{d}n_2^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \left\|\frac{\mathrm{d}n_3^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2\\ &+ \left\|\frac{\mathrm{d}n_4^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \left\|\frac{\mathrm{d}n_9^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2 + \left\|\frac{\mathrm{d}n_{10}^{m+\frac{1}{2}}}{\mathrm{d}y}\right\|^2\end{aligned} \tag{4.32}$$

$$\leq \delta \left(\left\| \frac{\mathrm{d}n_1^m}{\mathrm{d}y} \right\|^2 + \left\| \frac{\mathrm{d}n_2^m}{\mathrm{d}y} \right\|^2 + \left\| \frac{\mathrm{d}n_3^m}{\mathrm{d}y} \right\|^2 + \left\| \frac{\mathrm{d}n_4^m}{\mathrm{d}y} \right\|^2 + \left\| \frac{\mathrm{d}n_9^m}{\mathrm{d}y} \right\|^2 + \left\| \frac{\mathrm{d}n_{10}^m}{\mathrm{d}y} \right\|^2 \right).$$

When τ tends to 0, $\delta = \frac{\mathsf{KnSt}+6\left(2\left(\sqrt{2}+\sqrt{3}\right)+\sqrt{6}\right)\tau K_0}{\mathsf{KnSt}-9\left(2\left(\sqrt{2}+\sqrt{3}\right)+\sqrt{6}\right)\tau K_0}$ tends to 1.

From (4.28) and (4.32), we conclude easily that the norms $\left\| \frac{\mathrm{d}n_i^{m+\frac{j}{2}}}{\mathrm{d}y} \right\|$, $j \in \{1,\ 2\}$ and $i \in \Lambda$, are increased $\forall\ \tau < \tau_0$ and $\forall\ m$. □

4.6. **A priori estimates (III).** We consider the functions $n_{i\tau}^j$ defined on $[0,\ \mathsf{T}[$ with values in $H^1(\Omega) \cap L^\infty(\Omega)$ such that

$$n_{i\tau}^j(t) = n_i^{m+\frac{j}{2}}, \tag{4.33}$$

for $t \in [m\tau,\ (m+1)\tau[$, $0 \leq m \leq \mathsf{M}-1$, $i \in \Lambda$ and $j \in \{1,\ 2\}$.

Let $\tilde{n}_{i\tau}$, $i \in \Lambda$, the applications of $[0,\ \mathsf{T}] \longrightarrow H^1(\Omega) \cap L^\infty(\Omega)$ linear over each interval $[m\tau,\ (m+1)\tau[$ such that

$$\tilde{n}_{i\tau}(m\tau) = n_i^m, \quad 0 \leq m \leq \mathsf{M}. \tag{4.34}$$

According to **Lemma 4.1** and (4.26) we have:

Lemma 4.3. *The functions $n_{i\tau}^j$ and $\tilde{n}_{i\tau}$, $i \in \Lambda$ and $j \in \{1,\ 2\}$, remain in bounded sets of $L^\infty\left([0,\ \mathsf{T}];\ H^1(\Omega)\right)$ and $L^\infty([0,\ \mathsf{T}] \times \Omega)$.*

We shall establish the other a priori estimates for these functions. The equality (4.6) is written:

$$n_{i\tau}^2(t,y) - n_{i\tau}^1(t,y) + \frac{\tau v_i}{\mathsf{St}} \frac{\partial}{\partial y}\left(n_{i\tau}^2(t,y)\right) = 0, \quad i \in \Lambda. \tag{4.35}$$

And according to **Lemma 4.2**, we have:

$$\left\| n_{i\tau}^2 - n_{i\tau}^1 \right\|_{L^\infty([0,\ \mathsf{T}];\ L^2(\Omega))} \leq \tau K_2, \quad i \in \Lambda, \tag{4.36}$$

where K_2 is a positive constant independent of τ. Adding the equalities (4.1) and (4.6) we find:

$$n_i^{m+1} - n_i^m + \frac{\tau v_i}{\mathsf{St}} \frac{\partial}{\partial y}\left(n_i^{m+1}\right) - \frac{\tau}{\mathsf{St}} Q_i^{m+\frac{1}{2}} = 0, \quad i \in \Lambda. \tag{4.37}$$

This implies that

$$\frac{n_i^{m+1} - n_i^m}{\tau} + \frac{v_i}{\mathsf{St}}\frac{\partial}{\partial y}\left(n_i^{m+1}\right) - \frac{1}{\mathsf{St}}Q_i^{m+\frac{1}{2}} = 0, \quad i \in \Lambda. \tag{4.38}$$

Using the notations (4.33) and (4.34) we obtain:

$$\frac{\partial}{\partial t}\left(\tilde{n}_{i\tau}\right) + \frac{v_i}{\mathsf{St}}\frac{\partial}{\partial}\left(n_{i\tau}^2\right) - \frac{1}{\mathsf{St}}Q_{i\tau}^1 = 0, \quad i \in \Lambda. \tag{4.39}$$

It follows from (4.39) and **Lemma 4.3** that $\frac{\mathrm{d}n_{i\tau}}{\mathrm{d}y}$, $i \in \Lambda$, remain in the bounded sets of $L^\infty\left([0,\ \mathsf{T}];\ L^2(\Omega)\right)$ when τ tends to 0. By definition of the functions $\tilde{n}_{i\tau}$ and $n_{i\tau}^2$, we verify that

$$\left\|\tilde{n}_{i\tau} - n_{i\tau}^2\right\|_{L^\infty([0,\ \mathsf{T}];\ L^2(\Omega))} \leq \sup_{0\leq m\leq \mathsf{M}-1}\left\|n_i^{m+1} - n_i^m\right\|, \quad i \in \Lambda. \tag{4.40}$$

With (4.37) and **Lemma 4.3**, we conclude that

$$\left\|\tilde{n}_{i\tau} - n_{i\tau}^2\right\|_{L^\infty([0,\ \mathsf{T}];\ L^2(\Omega))} \leq \tau K_3, \quad i \in \Lambda, \tag{4.41}$$

where K_3 is a positive constant independent of τ.

4.7. **Transition to the limit and approximation theorem.** According to **Lemma 4.3**, we can extract from the sequence τ a subsequence (also denoted τ for simplicity) such that:

$$n_{i\tau}^j \longrightarrow n_i^j, \quad j \in \{1,\ 2\}, \tag{4.42}$$

in $L^\infty\left([0,\ \mathsf{T}];\ H^1(\Omega)\right)$ weak-star and in $L^\infty([0,\ \mathsf{T}] \times \Omega)$ weak-star. From (4.36) we necessarily have:

$$n_i^1 = n_i^2, \quad \text{(note } n_i\text{)}, \quad i \in \Lambda. \tag{4.43}$$

According to **Lemma 4.1** and (4.36), the family $\tilde{n}_{i\tau}$, $\forall\ i \in \Lambda$, is an equicontinuous bounded family in $\mathcal{C}\left([0,\ \mathsf{T}];\ L^2(\Omega)\right)$. We can therefore choose the sequence extracted so that the sequence $\tilde{n}_{i\tau}$, $\forall\ i \in \Lambda$, is convergent in $\mathcal{C}\left([0,\ \mathsf{T}];\ L^2(\Omega)\right)$. According to (4.41), (4.42) and (4.43) the limit of this sequence can only be n_i. Hence

$$\tilde{n}_{i\tau} \longrightarrow n_i \ \text{ in } \ \mathcal{C}\left([0,\ \mathsf{T}];\ L^2(\Omega)\right), \ \forall\ i \in \Lambda. \tag{4.44}$$

Thanks to (4.42) , (4.43) and (4.44) we can go to the limit and obtain the system (2.2) as the limiting form of equations (4.39). The conditions (2.2.8)-(2.2.11) and

(2.3) are easily verified and it is shown that $\mathrm{n} = \{n_1,\ n_2,\ n_3,\ n_4,\ n_9,\ n_{10}\}$ satisfies (2.2) and (2.3). The existence in **Theorem 2.1** is thus proved.

Note that since the solution of (2.2) satisfying (2.3) is unique, it is the entire sequence τ and not an extracted sequence which gives rise to the convergences (4.42) and (4.44).

In addition to **Theorem 2.1**, we have proved the following theorem:

Theorem 4.1 (Approximation theorem). *The functions* $n^1_{i\tau}$, $n^2_{i\tau}$ *and* $\tilde{n}_{i\tau}$, $\forall\ i \in \Lambda$, *defined by* (4.1), (4.6), (4.33) *and* (4.34) *converge when* τ *tends to* 0, *to the function* n_i, $i \in \Lambda$, *defined by* **Theorem 2.1**, *in* $\mathcal{C}\left([0,\ \mathsf{T}];\ L^2(\Omega)\right)$ *stong, in* $L^\infty\left([0,\ \mathsf{T}];\ H^1(\Omega)\right)$ *weak-star and in* $L^\infty([0,\ \mathsf{T}] \times \Omega)$ *weak-star.*

5. Conclusion

We use the fractional step method to prove the existence and the uniqueness of the solution of the initial boundary value problem associated to the system of the kinetic equations of the ten velocity discrete spatial model C_1 in one spatial dimension. We derive a numerical scheme to construct an approximate solution and show its convergence to the exact one. The boundary conditions prescribed are those of diffuse reflection. This work thus extend the results of Temam [12] to a more complex model having more velocities and two modulis of velocity. The fact that the model have two different modules of velocity and that we use the boundary conditions of diffuse reflection allows to properly study flow problems involving energetic processes.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOMÉ
LOMÉ, TOGO.
Email address: tinfili.natta@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOMÉ
LOMÉ, TOGO.
Email address: dal_me@yahoo.fr