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SOME PROPERTIES OF REPRODUCING KERNEL CARTAN SUBALGEBRA

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ABSTRACT. Let $\mathfrak{j}$ and $\mathfrak{j}'$ be the Cartan subalgebras of the complex semi-simple Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, $\mathfrak{j}^*$ and $(\mathfrak{j}')^*$ their duals, $\mathfrak{j}^\vee$ and $(\mathfrak{j}')^\vee$ the biduals of $\mathfrak{j}$ and $\mathfrak{j}'$ respectively. We consider $B(.,.)$, the restriction to $\mathfrak{j}$ and to $\mathfrak{j}'$ of the Killing form of $\mathfrak{g}$ and $\mathfrak{g}'$. In this work, using the kernel K of the reproducing kernel Cartan subalgebra $\mathfrak{j}^\vee$ and an operator Φ from $\mathfrak{j}^*$ to $(\mathfrak{j}')^*$, we construct another reproducing kernel Cartan subalgebra denoted by $\mathfrak{j}^\vee_\Phi$ obtained from the kernel $K \circ \Phi$ and study the relationships between $\mathfrak{j}^\vee$, $\mathfrak{j}^\vee_\Phi$ and $(\mathfrak{j}')^\vee$.

1. INTRODUCTION

The concept of reproducing kernel originated with the works of S. Bergmann and S. Szegö.

They presented the reproducing kernels of Szegö and Bergmann. Indeed, the reproducing kernels appeared in the first decade of the twentieth century. Their discovery was possible thanks to the work of S. Zaremba. His work concerned boundary value problems for harmonic and bi-harmonic functions. Moreover, the approach to reproducing kernels was originally and simultaneously developed in 1950 by Nachman Aronszajn and Stefen Bergmann. According to these authors:

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If we consider any set E. A reproducing kernel Hilbert space H on E is a Hilbert space of functions from E with values in $\mathbb{C}$ for which the evaluation function in each point of E is continuous (see [1, 2, 4, 6]).

Then, we can deduce that there exists a kernel $K : E \times E \Rightarrow \mathbb{C}$ such that for all $x \in E$,

$$f(x) = (f, K(., x))_H, \quad \text{for all} \quad f \in H.$$

Furthermore, Nachman Aronszajn proved that for any positive definite kernel K, there exists a unique Hilbert space admitting K as a reproducing kernel verifying the above equality (see [2, 8]).

The theory of reproducing kernel Hilbert spaces (RKHS) as defined by Aronszajn plays a very important role in mathematics. This theory is used in several fields such as: Deep and machine learning, statistics, signal processing, quantum mechanics, interpolation. Let's give a definition of an interpolation:

Let X and Y be sets, let $\{x_1, \dots, x_n\} \subseteq X$ be a collection of distinct points, and let $\{y_1, \dots, y_n\} \subseteq Y$.

We say that a function $h : X \Rightarrow Y$ interpolates these points if

$$h(x_i) = y_i, \quad \text{for all} \quad i = 1, \dots, n.$$

Let's now consider $\mathfrak{g}$ and $\mathfrak{g}'$ two complex semi-simple Lie algebras, $\mathfrak{j}$ and to $\mathfrak{j}'$ be their Cartan subalgebras respectively, $\mathfrak{j}^*$ to $(\mathfrak{j}')^*$ their duals.

We consider B(., .) the restriction to $\mathfrak{j}^*$ and $(\mathfrak{j}')^*$ of the Killing form of $\mathfrak{g}$ and $\mathfrak{g}'$. We also consider $\mathfrak{j}^\vee$ and $(\mathfrak{j}')^\vee$ the reproducing kernel Cartan subalgebras (RKCS) obtained using $\mathfrak{j}$ and $\mathfrak{j}'$ respectively (See [7]).

For all $\alpha, \beta \in \mathfrak{j}^*$, $(\alpha, \beta)_{\mathfrak{j}^*} = B(H_\alpha, H_\beta)$ where $H_\alpha, H_\beta \in \mathfrak{j}^*$. (See [3, 5, 14])

Let's consider $\mathcal{F}(\mathfrak{j}^*)$ the set of complex functions defined on $\mathfrak{j}^*$, L the map from $\mathfrak{j}^*$ into $\mathcal{F}(\mathfrak{j}^*)$. For $f \in \mathfrak{j}^*$ such that $\tilde{f} = Lf$,

$$\|\tilde{f}\|_{\mathfrak{j}^\vee} = \inf\{\|f\|_{\mathfrak{j}^*}; \tilde{f} = Lf\}.$$

If we consider two reproducing kernel spaces, we can think about the relationships between these spaces. On this way, Vern. I. Paulsen (See [13]) established the pull back in the case of reproducing kernel Hilbert spaces.

In this work, we shall first give some results on interpolation in $RKCS$. After that, using the operator Φ from $(\mathfrak{j}')^*$ to $\mathfrak{j}^*$, we shall establish the relationships between the $RKCS$ $\mathfrak{j}^\vee$, $\mathfrak{j}^\vee_\Phi$ and $(\mathfrak{j}')^\vee$.

2. INTERPOLATION IN RKCS

Let $\mathfrak{j}^{\vee}$ be the RKCS with reproducing kernel K. Let's consider the fundamental system of roots $S = \{\alpha_i, i = 1, \dots. n\}$ of the pair $(\mathfrak{g}, \mathfrak{j})$ (See [9, 11, 12, 14]) and $\{c_1, \dots, c_n\} \subseteq \mathbb{C}$ a collection of elements of $\mathbb{C}$. We will show that there exists an interpolating function $\tilde{\beta} \in \mathfrak{j}^{\vee}$ between S and $\{c_1, \dots, c_n\}$ and then we will prove that there is a unique such function of minimum norm and we shall give its formula.

Let $(\mathfrak{j}^{\vee})_S \subseteq \mathfrak{j}^{\vee}$ denotes the subspace spanned by the kernel functions $\{K_{\alpha_1}, \dots, K_{\alpha_n}\}$. Let P_S denotes the orthogonal projection of $\mathfrak{j}^{\vee}$ onto $(\mathfrak{j}^{\vee})_S$.

Theorem 2.1. *Let's consider $\mathfrak{j}^{\vee}$, the RKCS with reproducing kernel K, the fundamental system of roots $S = \{\alpha_i, i = 1, \dots. n\}$ of the pair $(\mathfrak{g}, \mathfrak{j})$ and $\{c_1, \dots, c_n\} \subseteq \mathbb{C}$. Then, there exists $\tilde{\beta} \in \mathfrak{j}^{\vee}$, an interpolating function between these points if and only if $v = \{c_1, \dots, c_n\}^t$ is in the range of $(K(\alpha_i, \alpha_j))$, for all $i, j = 1 \dots, n$.*

Moreover, in this case if we choose $w = \{w_1, \dots, w_n\}^t$ to be any vector whose image is v, then the unique function of minimal norm in $\mathfrak{j}^{\vee}$ that interpolates these points is

$$\tilde{\zeta} = \sum_i w_i K_{\alpha_i}.$$

We have

$$\|\tilde{\zeta}\|_{\mathfrak{j}^{\vee}} = (v, w)_{\mathfrak{j}^*}.$$

Proof. Let's assume that there exists $\tilde{\beta} \in \mathfrak{j}^{\vee}$ such that $\tilde{\beta}(\alpha_i) = c_i$, for all $i = 1, \dots, n$. Then, the function of minimal norm is $P_S(\tilde{\beta}) = \sum_j d_j K_{\alpha_j}$ where $d_1, \dots, d_n \in \mathbb{C}$. Since

$$c_i = \tilde{\beta}(\alpha_i) = P_S(\tilde{\beta})(\alpha_i) = \sum_j d_j K_{\alpha_j}(\alpha_i),$$

we have that $y = (d_1, \dots, d_n)^t$ is a solution of $v = (K(\alpha_i, \alpha_j))w$. Conversely, if $w = \{w_1, \dots, w_n\}^t$ is any solution of the equation $v = (K(\alpha_i, \alpha_j))w$ and we set $\tilde{\zeta} = \sum_j w_j K_{\alpha_j}$, then $\tilde{\zeta}$ will be an interpolating function. Note that $w - y$ is in the kernel of $K(\alpha_i, \alpha_j)$, then $P_F(\tilde{\beta})$ and $\tilde{\zeta}$ are the same function (See [13]). So, $\tilde{\zeta}$ is the function of minimal norm that makes the interpolation between these points. Finally,

$$\sum_{i,j} \overline{w_i} w_j K(\alpha_i, \alpha_j) = (K(\alpha_i, \alpha_j)w, w) = (v, w).$$

□

We shall give a theorem that characterizes the functions that belong to an RKCS using the reproducing kernel.

Theorem 2.2. *Let* $\mathfrak{j}^{\vee}$ *be the RKCS with reproducing kernel* K *and let* $\tilde{\beta} : \mathfrak{j}^* \Rightarrow \mathbb{C}$ *be a function. Then the following are equivalent:*

(1) $\tilde{\beta} \in \mathfrak{j}^{\vee}$*;*

(2) *For the fundamental system of roots* $S = \{\alpha_i, i = 1, \dots .n\}$ *of the pair* $(\mathfrak{g}, \mathfrak{j})$*, there exists a constant,* $c \geq 0$ *and a function* $\tilde{\eta} \in \mathfrak{j}^{\vee}$ *verifying* $\|\tilde{\eta}\| \leq c$ *such that*

$$\tilde{\beta}(\alpha_i) = \tilde{\eta}(\alpha_i), \quad i = 1, \dots n;$$

(3) *There exists a constant,* $c \geq 0$*, such that the function*

$$c^2 K(\alpha, \gamma) - \tilde{\beta}(\alpha)\overline{\tilde{\beta}(\gamma)}$$

is a kernel function and if $\tilde{\beta} \in \mathfrak{j}^{\vee}$ *then* $\|\tilde{\beta}\|$ *corresponds to the least* c *that satisfies the above inequality.*

Proof.

(1) $\Rightarrow$ (3) Let $c_1, \dots, c_n$ be complex numbers. Let's consider the fundamental system of roots $S = \{\alpha_i, i = 1, \dots .n\}$ of the pair $(\mathfrak{g}, \mathfrak{j})$ and set $\tilde{\delta} = \sum_j c_j K_{\alpha_j}$. Then,

$$\begin{aligned}\sum_{i,j} \overline{c_i} c_j \tilde{\beta}(\alpha_i)\overline{\tilde{\beta}(\alpha_j)} &= |\sum_i \overline{c_i}\tilde{\beta}(\alpha_i)|^2 = \|(\tilde{\beta}, \tilde{\delta})\| \\ &\leq \|\tilde{\beta}\|^2 \|\tilde{\delta}\|^2 \\ &\leq \|\tilde{\beta}\|^2 \sum_{i,j} \overline{c_i} c_j K(\alpha_i, \alpha_j).\end{aligned}$$

Since the choice of the scalars was arbitrary, we have that

$$(\tilde{\beta}(\alpha_i)\tilde{\beta}(\alpha_j)) \leq \|\tilde{\beta}^2\| K(\alpha_i, \alpha_j)$$

and if we set $c = \|\tilde{\beta}\|$, we get 3).

(3) $\Rightarrow$ (2) Let's consider the fundamental system of roots $S = \{\alpha_i, i = 1, \dots .n\}$ of the pair $(\mathfrak{g}, \mathfrak{j})$. We can deduce that the vector v whose entries are $c_i = \tilde{\beta}(\alpha_i)$ is in the range of $K(\alpha_i, \alpha_j)$ (See [13] Proposition 3.10 p.66). Using the Interpolation Theorem (Theorem 2.1), we deduce that there exists $\tilde{\eta} = \sum_j c_j K_{\alpha_j}$ in $(\mathfrak{j}^{\vee})_S$ such that $\tilde{\eta}(\alpha_i) = \tilde{\beta}(\alpha_i)$. Let set w, the vector whose components are the $c_i, i = 1, \dots .n$, then $\|\tilde{\eta}\|^2 = (v, w) \leq c^2$ (See [13] Proposition 3.10 p.66).

$(2) \Rightarrow (1)$ By assumption, for the fundamental system of roots $S = \{\alpha_i, i = 1, \dots .n\}$ of the pair $(\mathfrak{g}, \mathfrak{j})$, there exists a constant $c \geq 0$ and a function $\tilde{\eta} \in \mathfrak{j}^\vee$ verifying $\|\tilde{\eta}\| \leq c^2$ such that

$$\tilde{\beta}(\alpha_i) = \tilde{\eta}(\alpha_i), i = 1, \dots .n.$$

Thus, we get 1) and the proof is complete. □

3. Some properties of reproducing kernel Cartan subalgebra

Let $\mathfrak{j}$ and $\mathfrak{j}'$ be the Cartan subalgebras of the complex semi-simple Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, $\mathfrak{j}^*$ and $(\mathfrak{j}')^*$ their duals, $\mathfrak{j}^\vee$ and $(\mathfrak{j}')^\vee$ the biduals of $\mathfrak{j}$ and $\mathfrak{j}'$ respectively. We consider $B(.,.)$ the restriction to $\mathfrak{j}$ and to $\mathfrak{j}'$ of the Killing form of $\mathfrak{g}$ and $\mathfrak{g}'$.

Let $K : \mathfrak{j}^* \times \mathfrak{j}^* \Rightarrow \mathbb{C}$ be a kernel function. If we have $(\mathfrak{j}')^* \subseteq \mathfrak{j}^*$, then the restriction of K to $(\mathfrak{j}')^* \times (\mathfrak{j}')^*$ is also a kernel function and we can use K to form an $RKCS$ on $(\mathfrak{j}')^*$. Let's consider an operator Φ from $(\mathfrak{j}')^*$ to $\mathfrak{j}^*$, then we let $K \circ \Phi : (\mathfrak{j}')^* \times (\mathfrak{j}')^* \Rightarrow \mathbb{C}$ denotes the function given by

$$K \circ \Phi(\alpha', \beta') = K(\Phi(\alpha'), \Phi(\beta')).$$

Theorem 3.1. *If we consider $\Phi : (\mathfrak{j}')^* \Rightarrow \mathfrak{j}^*$ and let $K : \mathfrak{j}^* \times \mathfrak{j}^* \Rightarrow \mathbb{C}$ be a kernel function. Then $K \circ \Phi$ is a kernel function on $(\mathfrak{j}')^* \times (\mathfrak{j}')^*$ denoted by $\mathfrak{j}_\Phi^\vee$ such that $\mathfrak{j}_\Phi^\vee = \{\tilde{u} \circ \Phi : \tilde{u} \in \mathfrak{j}^\vee\}$. For $u \in \mathfrak{j}_\Phi^\vee$, we have*

$$\|u\|_{\mathfrak{j}_\Phi^\vee} = \min\{\|\tilde{u}\|_{\mathfrak{j}^\vee}; u = \tilde{u} \circ \Phi\}.$$

Proof. Let's consider the fundamental system of roots $S = \{\alpha_i, i = 1, \dots .n\}$ of the pair $(\mathfrak{g}, \mathfrak{j})$; $\alpha_1', \dots, \alpha_m' \in (\mathfrak{j}')^*$, let $c_1, \dots, c_n \in \mathbb{C}$ and let

$$\{\alpha_1, \dots, \alpha_n\} = \{\Phi(\alpha_1'), \dots, \Phi(\alpha_n')\},$$

such that $n \leq m$.

Set $A_k = \{i : \Phi(\alpha_i') = \alpha_i\}$ and let $b_k = \sum_{i \in A_k} c_i$. Then,

$$\sum_{i,j=1}^{n} \overline{c_i} c_j K(\Phi(\alpha_i'), \Phi(\alpha_j')) = \sum_{k,l=1}^{p} \sum_{i \in A_k} \sum_{j \in A_l} \overline{c_i} c_j K(\alpha_k, \alpha_l)$$
$$= \sum_{k,l=1}^{p} \overline{b_k} b_l K(\alpha_k, \alpha_l) \geq 0.$$

Hence, $K \circ \Phi$ is a kernel function on $(\mathfrak{j}')^* \times (\mathfrak{j}')^*$.

Let $\tilde{u} \in \mathfrak{j}^\vee$, with $\|\tilde{u}\|_{\mathfrak{j}^\vee} = c$, then $\tilde{u}(\alpha)\tilde{u}(\beta) \leq c^2 K(\alpha, \beta)$. Since we have this inequality, we see that

$$\tilde{u} \circ \Phi(\alpha')\overline{\tilde{u} \circ \Phi(\beta')} \leq c^2 K(\Phi(\alpha'), \Phi(\beta')).$$

Hence, using Theorem 2.2, $\tilde{u} \circ \Phi \in \mathfrak{j}^\vee_\Phi$ with $\|\tilde{u} \circ \Phi\| \leq c$. That means there exists a contractive, linear map $C_\Phi : \mathfrak{j}^\vee \Rightarrow \mathfrak{j}^\vee_\Phi$ given by $C_\Phi(\tilde{u}) = \tilde{u} \circ \Phi$.

Let's set for $\beta' \in (\mathfrak{j})'$ $l_{\beta'}(.) = K(\Phi(.), \Phi(\beta'))$, the kernel functions for $\mathfrak{j}^\vee_\Phi$.

For $c_1, \dots, c_n \in \mathbb{C}$, if $u = \sum_i c_i l_{\beta'_i}$, then,

$$\|u\|_{\mathfrak{j}^\vee_\Phi} = \| \sum_i c_i K_{\Phi(\beta'_i)} \|_{\mathfrak{j}^\vee}.$$

That means there exists an isometry, $D : \mathfrak{j}^\vee_\Phi \Rightarrow \mathfrak{j}^\vee$, satisfying

$$D(l_{\beta'}) = K_{\Phi(\beta')},$$

then $C_\Phi \circ D$ is the identity on $\mathfrak{j}^\vee_\Phi$. Hence, for any $u \in \mathfrak{j}^\vee_\Phi$, $\tilde{u} = D(u)$ satisfies $u = \tilde{u} \circ \Phi$ with $\|u\| = \|\tilde{u}\|$ and the theorem is proved. □

By considering $\mathfrak{j}^*$ and $(\mathfrak{j}')^*$ the duals of the Cartan subalgebras of the complex semi-simple Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, a function $\Phi : (\mathfrak{j}')^* \Rightarrow \mathfrak{j}^*$, and a kernel function $K : \mathfrak{j}^* \times \mathfrak{j}^* \Rightarrow \mathbb{C}$, we call the RKCS $\mathfrak{j}^\vee_\Phi$ the pull-back of $\mathfrak{j}^\vee$ along Φ and we call the linear map, $C_\Phi : \mathfrak{j}^\vee \Rightarrow \mathfrak{j}^\vee_\Phi$ the pull-back Cartan map.

Now, Given the sets $\mathfrak{j}^*$ and $(\mathfrak{j}')^*$, the kernel functions K on $\mathfrak{j}^* \times \mathfrak{j}^*$, K' on $(\mathfrak{j}^*)' \times (\mathfrak{j}^*)'$. Our goal now is to identify those functions $\Psi : \mathfrak{j}^* \Rightarrow (\mathfrak{j}')^*$ such that there exists a bounded linear map $C_\Psi : (\mathfrak{j}')^\vee \Rightarrow \mathfrak{j}^\vee$ defined by $C_\Psi(\tilde{u}) = \tilde{u} \circ \Psi$.

Theorem 3.2. *Let $\mathfrak{j}$ and $\mathfrak{j}'$ be the Cartan subalgebras of the complex semi-simple Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, $\mathfrak{j}^*$ and $(\mathfrak{j}')^*$ their duals, $\mathfrak{j}^\vee$ and $(\mathfrak{j}')^\vee$ the biduals of $\mathfrak{j}$ and $\mathfrak{j}'$ respectively. If we consider a function $\Psi : (\mathfrak{j}')^* \Rightarrow \mathfrak{j}^*$, the kernel functions K on $\mathfrak{j}^* \times \mathfrak{j}^*$, K' on $(\mathfrak{j}')^* \times (\mathfrak{j}')^*$, then the following are equivalent:*

1) $\{(\tilde{u})' \circ \Psi : (\tilde{u})' \in (\mathfrak{j}')^\vee\} \subseteq \mathfrak{j}^\vee$;
2) $C_\Psi : (\mathfrak{j}')^\vee \Rightarrow \mathfrak{j}^\vee$ *is a bounded, linear operator;*
3) *There exists $c > 0$, such that $K' \circ \Psi \leq c^2 K$.*

Moreover, $\|C_\Psi\|$ is the least such constant c.

Proof. We see that (2) implies (1).

$(3) \Rightarrow (2)$

To get (3) from (2), let $(\tilde{u})' \in (\mathfrak{j}')^{\vee}$ with $\|(\tilde{u})'\| = M$. Then

$$[(\tilde{u})'](\alpha')\overline{[(\tilde{u})'](\beta')} \leq M^2 K'(\alpha', \beta'),$$

which implies that

$$[(\tilde{u})'](\Psi(\alpha))\overline{[(\tilde{u})'](\Psi(\beta))} \leq M^2 K'(\Psi(\alpha), \Psi(\beta)) \leq M^2 c^2 K(\alpha, \beta).$$

Thus, we get that $C_{\Psi}(\tilde{u}') = \tilde{u}' \circ \Psi \in \mathfrak{j}^{\vee}$ with $\|C_{\Psi}(\tilde{u}')\| \leq \|(\tilde{u}')\|$. Hence, C_{Ψ} is bounded and $\|C_{\Psi}\| \leq c$.

$(1) \Rightarrow (3)$ Using Theorem 3.1, 1) is equivalent to

$$(\mathfrak{j}')^{\vee}_{\Psi} \subseteq \mathfrak{j}^{\vee},$$

which is also equivalent to 3) by Aronszajn's inclusion (see [10] Theorem 6 p.37). Hence, (3) is proved. □

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