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ON THE STEADY SOLUTIONS OF THE GENERAL FOUR VELOCITY BROADWELL MODEL

Pahon Lakou Defoou, Koundji Koffi Leroys Sossou, and Amah d'Almeida[1]

ABSTRACT. Existence and boundedness is proved for the solutions of boundary value problems resulting from the modelling of a flow in a rectangular box by the four velocity Broadwell model. The influence of the orientation of the model in relation to the sides of the rectangle on the form of the boundary value problem is analysed. The uniqueness of the maxwellian solutions is proved. The non uniqueness of the non maxwellian solutions is established by building different exact non maxwellian solutions for the same macroscopic density.

1. INTRODUCTION

The paper is devoted to the proof of the existence and the boundedness of the solutions to the boundary value problem for the general two-dimensional four velocity discrete model in a bounded domain. The plane four velocity discrete model of Broadwell is among the simplest discrete velocity models and it has been used to study initial and boundary value problems in one dimension [1,4]. The first papers on the boundary value problem for the Broadwell model in two dimensions established the existence and the boundedness of the solutions for

[1]*corresponding author*
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bounded boundary conditions [3] and in addition found exact solutions [8] of the following boundary value problem:

$$(1.1)\qquad \begin{cases} \dfrac{\partial N_1'}{\partial x'} = -\dfrac{\partial N_4'}{\partial x'} = Q' \\ \dfrac{\partial N_2'}{\partial y'} = -\dfrac{\partial N_3'}{\partial y'} = -Q' \\ N_1'(0,y') = \phi_1'(y') \\ N_2'(a,y') = \phi_2'(y') \\ N_3'(x',0) = \phi_3'(x') \\ N_4'(x',b) = \phi_4'(x') \end{cases}$$

$$Q' = 2s\left(N_2'N_3' - N_1'N_4'\right).$$

which models in an orthonormal reference $(O, \vec{e_1}, \vec{e_2})$ of the plane $\mathbb{R}^2$, the flow of a gas in a rectangular box, when the velocities of the discrete velocity model are $\vec{u_1} = c\vec{e_1}$, $\vec{u_2} = c\vec{e_2}$, $\vec{u_3} = -\vec{u_2}$, $\vec{u_4} = -\vec{u_1}$ and the origin O is chosen so that the edges of the box are located on the lines $x' = 0$, $x' = a$, $y' = 0$ and $y' = b$, $0 < b \leq a$. We denote as usual by $N_i(t', x', y')$ the number density of particles of velocity $\overrightarrow{u}_i$ in point $M(t', x', y')$ at time t'.

The velocities of the general four velocity planar discrete velocity model in the basis $(\vec{e_1}, \vec{e_2})$ of the reference are in fact $\vec{u_1} = c(cos\theta, sin\theta)$, $\vec{u_2} = c(-sin\theta, cos\theta)$, $\vec{u_3} = -\vec{u_2}$, $\vec{u_4} = -\vec{u_1}$, where $\theta = angle(\vec{e_1}, \vec{u_1})$ accounts of the orientation of the discrete velocity model with respect to the reference. Hence the boundary value problem has the form (1.1) if and only if $\theta \in \left\{0, \dfrac{\pi}{2}\right\}$. For $\theta \in \left]0, \dfrac{\pi}{2}\right[$, $cos\theta$ and $sin\theta$ are non zero and the system of the steady kinetic equations of the discrete model is:

$$(1.2)\qquad \begin{cases} c\cos\theta\dfrac{\partial N_1'}{\partial x'} + c\sin\theta\dfrac{\partial N_1'}{\partial y'} = Q' \\ -c\sin\theta\dfrac{\partial N_2'}{\partial x'} + c\cos\theta\dfrac{\partial N_2'}{\partial y'} = -Q' \\ c\sin\theta\dfrac{\partial N_3'}{\partial x'} - c\cos\theta\dfrac{\partial N_3'}{\partial y'} = -Q' \\ -c\cos\theta\dfrac{\partial N_4'}{\partial x'} - c\sin\theta\dfrac{\partial N_4'}{\partial y'} = Q'. \end{cases}$$

Obviously by solely comparing the kinetic equations of (1.1) and (1.2) we can see that the boundary value problem for $\theta = 0$ and $\theta = \dfrac{\pi}{2}$ is totally different from the ones for $\theta \in \left]0, \dfrac{\pi}{2}\right[$. The aim of this paper is to investigate the boundary value problem for the two-dimensional four velocity discrete model in the case where $\theta \in \left]0, \dfrac{\pi}{4}\right[\cup \left]\dfrac{\pi}{4}, \dfrac{\pi}{2}\right[$.

We emphasize the fact that for $\theta \in \left\{0, \dfrac{\pi}{4}, \dfrac{\pi}{2}\right\}$ the model is isotropic with respect to the reference in contrast with the cases $\theta \notin \left\{0, \dfrac{\pi}{4}, \dfrac{\pi}{2}\right\}$ although the result of such a fact is not discussed in the paper.

The paper is organized as follows. In section 2 we briefly describe the model and present the main result of the paper which is proved in section 3. We build in section 4 exact solutions and establish the non uniqueness of the non maxwellian ones.

2. Statement of the problem

2.1. The influence of the orientation of the model. We consider a gas flow described by the general four velocity discrete model in a rectangular box of lenght a and width b $(0 < b \leq a)$. Arranging as usual the velocities of the model into three groups corresponding to emerging, grazing and impinging particles in relation with each edge of the box [5], we derive, depending on the value of θ, the following boundary conditions:

(2.1) $N_1'(0, y') = \phi_1'(y');\ N_2'(x', 0) = \phi_2'(x');\ N_3'(x', b) = \phi_2'(x');\ N_4'(a, y') = \phi_4'(y')$

for $0 \leq \theta < \dfrac{\pi}{4}$,

(2.2) $N_1'(x', 0) = \phi_1'(x'); N_2'(a, y') = \phi_2'(y'); N_3'(0, y') = \phi_3'(y'); N_4'(x', b) = \phi_4'(x')$

for $\dfrac{\pi}{4} < \theta \leq \dfrac{\pi}{2}$, and

$$
\begin{aligned}
N_1'(0, y') &= \phi_1'(y'); \quad & N_1'(x', 0) &= \phi_5'(x') \\
N_2'(a, y') &= \phi_2'(y'); \quad & N_2'(x', 0) &= \phi_6'(x') \\
N_3'(0, y') &= \phi_3'(y'); \quad & N_3'(x', b) &= \phi_7'(x') \\
N_4'(a, y') &= \phi_4'(y'); \quad & N_4'(x', b) &= \phi_8'(x')
\end{aligned} \tag{2.3}
$$

for $\theta = \pi/4$.

The boundary value problems are the system (1.2) with respectively the boundary conditions (2.1), (2.2) and (2.3). The boundary value problems (1.2)-(2.1) and (1.2)-(2.2) are two point boundary value problems. The problem (1.2)-(2.3) is an overdetermined two point boundary value problem in the sense that it has more boundary conditions than unknowns. The orientation of the model with respect the reference can thus lead to two different kinds of boundary value problems. The existence of solutions of the boundary value problem is proved for $\theta \in \left\{0, \frac{\pi}{2}\right\}$ in [8] and for $\theta = \frac{\pi}{4}$ in [9]. The same task is done here for $\theta \in \left]0, \frac{\pi}{4}\right[\cup \left]\frac{\pi}{4}, \frac{\pi}{2}\right[$. As the boundary value problems (1.2)-(2.1) and (1.2)-(2.2) are identical apart from a permutation of the velocity indices, only a detailed account of the study of the existence of solutions of the problem (1.2)-(2.1) is given in the sequel.

2.2. **The non dimensional problem.** The problem is put in dimensionless form. The chosen reference values are: c for the velocity, n_0 for the densities, a and b for the length. We thus introduce the following non dimensional quantities:

$$N_i = N_i'/n_0, \; i = 1, 2, 3, 4, \quad x = x'/a, \quad y = y'/b,$$
$$\varepsilon = b/a, \quad Kn = (sn_0 a)^{-1}, \; \phi_j = \phi_j'/n_0, j = 1, \ldots, 4.$$

The Knudsen number Kn provides information on the degree of rarefaction of the flow while ε which is the channel aspect ratio provides information on the relative length. For $\theta < \frac{\pi}{4}$ the boundary value problem takes the form:

$$\left\{\begin{array}{l} \cos\theta \dfrac{\partial N_1}{\partial x} + \dfrac{1}{\varepsilon}\sin\theta \dfrac{\partial N_1}{\partial y} = Q \\ -\sin\theta \dfrac{\partial N_2}{\partial x} + \dfrac{1}{\varepsilon}\cos\theta \dfrac{\partial N_2}{\partial y} = -Q \\ \sin\theta \dfrac{\partial N_3}{\partial x} - \dfrac{1}{\varepsilon}\cos\theta \dfrac{\partial N_3}{\partial y} = -Q \\ -\cos\theta \dfrac{\partial N_4}{\partial x} - \dfrac{1}{\varepsilon}\sin\theta \dfrac{\partial N_4}{\partial y} = Q \\ N_1(0, y) = \phi_1(y); \quad N_2(x, 0) = \phi_2(x) \\ N_3(x, 1) = \phi_2(x); \quad N_4(1, y) = \phi_4(x) \end{array}\right. \tag{2.4}$$

with

$$Q = 2\left(N_2N_3 - N_1N_4\right)/Kn.$$

We prove in the sequel the following result:

Theorem 2.1. *The problem* (2.4) *has continuous, derivable and bounded solution if the boundary data* ϕ_i , $i = 1, \dots, 4$ *and their first derivatives are continuous and bounded.*

2.3. **The change of variables.** The system (2.4) is a two point boundary value problem. The N_i, $i = 1, 2, 3, 4$ are positive number densities. To simplify the form of (2.4), we make the following change of variables.

$$\mathscr{L} : (x, y) \mapsto (\alpha_1, \alpha_2) \text{ such that } \alpha_1 = x\cos\theta + \varepsilon y \sin\theta \ , \ \alpha_2 = -x\sin\theta + \varepsilon y \cos\theta.$$

$\mathscr{L}$ is an isomorphism of $[0, 1] \times [0, 1]$ onto $[0, \cos\theta + \varepsilon\sin\theta] \times [-\sin\theta, \varepsilon\cos\theta]$. The α_j, $j = 1, 2$ are the new variables and x, y are the old ones. The vertices $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$, $D = (0, 1)$ of the square in which the flow takes place are transformed into $A' = (0, 0)$, $B' = (\cos\theta, -\sin\theta)$, $C' = (\cos\theta + \varepsilon\sin\theta, \varepsilon\cos\theta - \sin\theta)$, $D' = (\varepsilon\sin\theta, \varepsilon\cos\theta)$ and the square $ABCD$ is transformed into the square $A'B'C'D'$ by the transformation $\mathscr{L}$. The local basis associated with the new coordinate system at any point M is $(M, \vec{u}_1, \vec{u}_2)$. The lines $x = 0$, $x = 1$, $y = 0$ and $y = 1$ are respectively transformed into $\cos\theta\alpha_1 - \sin\theta\alpha_2 = 0$, $\cos\theta\alpha_1 - \sin\theta\alpha_2 = 1$, $\sin\theta\alpha_1 + \cos\theta\alpha_2 = 0$, $\sin\theta\alpha_1 + \cos\theta\alpha_2 = \varepsilon$. So in the new coordinate system the velocities of the model are normal to the sides of the square and the boundary value problem (2.4) takes the form:

$$\begin{cases} \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} = -\dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} = \widetilde{Q} \\ \dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} = -\dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} = -\widetilde{Q} \\ \qquad \widetilde{N}_1\left(0, \alpha_2\right) = \widetilde{\phi}_1(\alpha_2) \\ \qquad \widetilde{N}_2\left(\alpha_1, 0\right) = \widetilde{\phi}_2(\alpha_1) \\ \qquad \widetilde{N}_3\left(\alpha_1, 1\right) = \widetilde{\phi}_3(\alpha_1) \\ \qquad \widetilde{N}_4\left(1, \alpha_2\right) = \widetilde{\phi}_4(\alpha_2) \end{cases} \tag{2.5}$$

with

$$\widetilde{Q} = 2\left(\widetilde{N}_2\widetilde{N}_3 - \widetilde{N}_1\widetilde{N}_4\right)/Kn.$$

3. Existence of solution of problem (2.5)

We put $J = [0, \cos\theta + \varepsilon\sin\theta] \times [-sin\theta, \varepsilon\cos\theta]$ and denote respectively by $\mathscr{C}(J)$ and $\mathscr{C}_+(J)$ the set of continuous functions defined on J and its subset of positive functions. We defined the following norms:

If $\alpha = (\alpha_1, \alpha_2) \in J$ and $M = (M_1, \dots, M_4) \in \mathscr{C}(J)^4$, then

$$\|\alpha\| = |\alpha_1| + |\alpha_2| \ , \ \|M_i\|_0 = \sup_{\alpha\in J} |M_i(\alpha)| \ , \ \|M\|_1 = \sup_{i\in\Lambda} \|M_i\|_0 \text{ with } \Lambda = \{1, 2, 3, 4\}.$$

We denote $|M| = (|M_1|, \dots, |M_4|)$.

3.1. Positivity of the solutions.

Proposition 3.1. *The solution* $(\widetilde{N}_1, \dots, \widetilde{N}_4)$ *of the problem* (2.5) *when it exists, belongs to* $\mathscr{C}_+(J)^4$.

Proof. Let

$$\begin{aligned}
\overline{N}_1(\alpha_1, \alpha_2) &= \exp\left[\int_0^{\alpha_1} \rho(\widetilde{N})(\alpha_1, s)ds\right] \widetilde{N}_1(\alpha_1, \alpha_2)\\
\overline{N}_2(\alpha_1, \alpha_2) &= \exp\left[\int_0^{\alpha_2} \rho(\widetilde{N})(s, \alpha_2)ds\right] \widetilde{N}_2(\alpha_1, \alpha_2)\\
\overline{N}_3(\alpha_1, \alpha_2) &= \exp\left[\int_1^{\alpha_2} \rho(\widetilde{N})(s, \alpha_2)ds\right] \widetilde{N}_3(\alpha_1, \alpha_2)\\
\overline{N}_4(\alpha_1, \alpha_2) &= \exp\left[\int_1^{\alpha_1} \rho(\widetilde{N})(\alpha_1, s)ds\right] \widetilde{N}_4(\alpha_1, \alpha_2)\\
\sigma_0 &= 2/Kn.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial\overline{N}_1}{\partial\alpha_1} &= \exp\left[\int_0^{\alpha_1} \rho(\widetilde{N})(s, \alpha_2)ds\right] \sigma_0\widetilde{N}_2\widetilde{N}_3 + \overline{N}_1\left[\rho(\widetilde{N}) - \sigma_0\widetilde{N}_4\right]\\
\frac{\partial\overline{N}_4}{\partial\alpha_1} &= -\exp\left[\int_1^{\alpha_1} \rho(\widetilde{N})(s, \alpha_2)ds\right] \sigma_0\widetilde{N}_2\widetilde{N}_3 + \overline{N}_4\left[\rho(\widetilde{N}) + \sigma_0\widetilde{N}_1\right]
\end{aligned}$$

$$\frac{\partial \overline{N}_2}{\partial \alpha_2} = \exp\left[\int_0^{\alpha_2} \rho(\widetilde{N})(\alpha_1, s)ds\right] \sigma_0 \widetilde{N}_1 \widetilde{N}_4 + \overline{N}_2 \left[\rho(\widetilde{N}) - \sigma_0 \widetilde{N}_3\right]$$
$$\frac{\partial \overline{N}_3}{\partial \alpha_2} = -\exp\left[\int_1^{\alpha_2} \rho(\widetilde{N})(\alpha_1, s)ds\right] \sigma_0 \widetilde{N}_1 \widetilde{N}_4 + \overline{N}_3 \left[\rho(\widetilde{N}) + \sigma_0 \widetilde{N}_2\right].$$

Putting $F(\widetilde{N}) = \widetilde{N}_1\widetilde{N}_4\sigma_0$ and $G(\widetilde{N}) = \widetilde{N}_2\widetilde{N}_3\sigma_0$ we get

$$\begin{aligned}
\overline{N}_1(\alpha_1,\alpha_2) &= \left(\widetilde{\phi}_1(\alpha_2) + \int_0^{\alpha_1} \exp\left[\int_0^{s} \sigma_0 \widetilde{N}_4(a,\alpha_2)da\right] G(\widetilde{N})(s,\alpha 2)ds\right) \\
&\times \exp\left[\int_0^{\alpha_1} \left[\rho(\widetilde{N}) - \sigma_0\widetilde{N}_4\right](s,\alpha_2)ds\right] \\
\overline{N}_2(\alpha_1,\alpha_2) &= \left(\widetilde{\phi}_2(\alpha_1) + \int_0^{\alpha_2} \exp\left[\int_0^{s} \sigma_0 \widetilde{N}_3(\alpha_1,a)da\right] F(\widetilde{N})(\alpha_1,s)ds\right) \\
&\times \exp\left[\int_0^{\alpha_2} \left[\rho(\widetilde{N}) - \sigma_0\widetilde{N}_3\right](\alpha_1,s)ds\right] \\
\overline{N}_3(\alpha_1,\alpha_2) &= \left(\widetilde{\phi}_3(\alpha_1) + \int_1^{\alpha_2} \exp\left[\int_1^{\alpha_2} \sigma_0 \widetilde{N}_2(\alpha_1,a)da\right] F(\widetilde{N})(\alpha_1,s)ds\right) \\
&\times \exp\left[\int_1^{\alpha_2} \left[\rho(\widetilde{N}) - \sigma_0\widetilde{N}_2\right](\alpha_1,s)ds\right] \\
\overline{N}_4(\alpha_1,\alpha_2) &= \left(\widetilde{\phi}_4(\alpha_2) + \int_1^{\alpha_1} \exp\left[\int_1^{s} \sigma_0 \widetilde{N}_1(a,\alpha_2)da\right] G(\widetilde{N})(s,\alpha_2)ds\right) \\
&\times \exp\left[\int_1^{\alpha_1} \left[\rho(\widetilde{N}) - \sigma_0\widetilde{N}_1\right](s,\alpha_2)ds\right].
\end{aligned}$$

As $\widetilde{\phi}_i$, $i \in \Lambda$ are positive then $\overline{N}_k$, $k \in \Lambda$ are positive and so are $\widetilde{N}_k$, $k \in \Lambda$. Hence if a solution of (2.5) exists then it is positive. □

3.2. **Definition of an auxiliary problem.** We put $\rho^+(\widetilde{N}) = \widetilde{N}_1 + \widetilde{N}_4$ and $\rho^-(\widetilde{N}) = \widetilde{N}_2 + \widetilde{N}_3$ and consider for $\sigma > 0$ the following problem:

$$(3.1) \qquad \begin{cases}
\dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} + \sigma\widetilde{N}_1\rho^+(\widetilde{N}) = \widetilde{Q} + \sigma\widetilde{N}_1\rho^+(\widetilde{N}) &= Q_1^\sigma(\widetilde{N}) \\
\dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} + \sigma\widetilde{N}_2\rho^-(\widetilde{N}) = -\widetilde{Q} + \sigma\widetilde{N}_2\rho^-(\widetilde{N}) &= Q_2^\sigma(\widetilde{N}) \\
\dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} + \sigma\widetilde{N}_3\rho^-(\widetilde{N}) = \widetilde{Q} + \sigma\widetilde{N}_3\rho^-(\widetilde{N}) &= Q_3^\sigma(\widetilde{N}) \\
\dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} + \sigma\widetilde{N}_4\rho^+(\widetilde{N}) = -\widetilde{Q} + \sigma\widetilde{N}_4\rho^+(\widetilde{N}) &= Q_4^\sigma(\widetilde{N})
\end{cases}$$

$$\begin{cases} \widetilde{N}_1\,(0,\alpha_2) &=& \widetilde{\phi}_1(\alpha_2) \\ \widetilde{N}_2\,(\alpha_1,0) &=& \widetilde{\phi}_2(\alpha_1) \\ \widetilde{N}_3\,(\alpha_1,1) &=& \widetilde{\phi}_3(\alpha_1) \\ \widetilde{N}_4\,(1,\alpha_2) &=& \widetilde{\phi}_4(\alpha_2). \end{cases}$$

The problem (3.1) is obtained from problem (2.5) by adding $\sigma\widetilde{N}_i\rho^{\pm}(\widetilde{N})$ to the two members of the kinetic equation for $\widetilde{N}_i$, $i \in \Lambda$ so the two systems of equations are equivalent.

3.2.1. *Existence of solutions of* (3.1).

Proposition 3.2. *The problem* (3.1) *has a solution which belongs to* $\mathscr{C}_+(J)^4$ *for sufficiently large* σ.

Proof. Consider for $M \in \mathscr{C}(J)^4$ the following boundary value problem:

$$\begin{cases} \dfrac{\partial\widetilde{N}_1}{\partial\alpha_1} + \sigma\widetilde{N}_1\rho^{+}(|M|) &=& Q_1^{\sigma}(|M|) \\ \dfrac{\partial\widetilde{N}_2}{\partial\alpha_2} + \sigma\widetilde{N}_2\rho^{-}(|M|) &=& Q_2^{\sigma}(|M|) \\ \dfrac{\partial\widetilde{N}_3}{\partial\alpha_2} + \sigma\widetilde{N}_3\rho^{-}(|M|) &=& Q_3^{\sigma}(|M|) \\ \dfrac{\partial\widetilde{N}_4}{\partial\alpha_1} + \sigma\widetilde{N}_4\rho^{+}(|M|) &=& Q_4^{\sigma}(|M|) \\ \widetilde{N}_1\,(0,\alpha_2) &=& \widetilde{\phi}_1(\alpha_2) \\ \widetilde{N}_2\,(\alpha_1,0) &=& \widetilde{\phi}_2(\alpha_1) \\ \widetilde{N}_3\,(\alpha_1,1) &=& \widetilde{\phi}_3(\alpha_1) \\ \widetilde{N}_4\,(1,\alpha_2) &=& \widetilde{\phi}_4(\alpha_2)). \end{cases} \tag{3.2}$$

Lemma 3.1. *The problem* (3.2) *has for given* $M \in \mathscr{C}(J)^4$ *an unique solution which belongs to* $\mathscr{C}_+(J)^4$ *for sufficiently large* σ.

Proof. The problem (3.2) is a linear problem associated with the problem (3.1) and it is solved by splitting it into the two following boundary value problems:

$$(3.3) \qquad \begin{cases} \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} + \sigma \widetilde{N}_1 \rho^+(|M|) & = & Q_1^\sigma(|M|) \\ \dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} + \sigma \widetilde{N}_4 \rho^+(|M|) & = & Q_4^\sigma(|M|) \\ \widetilde{N}_1\,(0, \alpha_2) & = & \widetilde{\phi}_1(\alpha_2) \\ \widetilde{N}_4\,(1, \alpha_2) & = & \widetilde{\phi}_4(\alpha_2)) \end{cases}$$

$$(3.4) \qquad \begin{cases} \dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} + \sigma \widetilde{N}_2 \rho^-(|M|) & = & Q_2^\sigma(|M|) \\ \dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} + \sigma \widetilde{N}_3 \rho^-(|M|) & = & Q_3^\sigma(|M|) \\ \widetilde{N}_2\,(\alpha_1, 0) & = & \widetilde{\phi}_2(\alpha_1) \\ \widetilde{N}_3\,(\alpha_1, 1) & = & \widetilde{\phi}_3(\alpha_1). \end{cases}$$

The unique solution of (3.2) is:

$$(3.5) \qquad \begin{aligned} \widetilde{N}_1(\alpha_1, \alpha_2) &= \widetilde{\phi}_1(\alpha_2) g^+(\alpha_1, \alpha_2) \\ &\quad + \int_0^{\alpha_1} Q_1^\sigma(|M|)(s, \alpha_2) f^+(\alpha_1 - s, \alpha_2) ds \\ \widetilde{N}_2(\alpha_1, \alpha_2) &= \widetilde{\phi}_2(\alpha_1) g^-(\alpha_1, \alpha_2) \\ &\quad + \int_0^{\alpha_2} Q_2^\sigma(|M|)(\alpha_1, s) f^-(\alpha_1, \alpha_2 - s) ds \\ \widetilde{N}_3(\alpha_1, \alpha_2) &= \widetilde{\phi}_3(\alpha_1) f^-(\alpha_1, \alpha_2 - 1) \\ &\quad + \int_1^{\alpha_2} Q_3^\sigma(|M|)(\alpha_1, s) f^-(\alpha_1, \alpha_2 - s) ds \\ \widetilde{N}_4(\alpha_1, \alpha_2) &= \widetilde{\phi}_4(\alpha_2) f^+(\alpha_1 - 1, \alpha_2) \\ &\quad + \int_1^{\alpha_1} Q_4^\sigma(|M|)(s, \alpha_1) f^+(\alpha_1 - s, \alpha_2) ds \end{aligned}$$

with

$$\begin{aligned} g^+(\alpha_1, \alpha_2) &= \exp\left[-\sigma \int_{\alpha_2}^{\alpha_1} \rho^+(|M|)(s, \alpha_2) ds\right] \\ g^-(\alpha_1, \alpha_2) &= \exp\left[-\sigma \int_{\alpha_1}^{\alpha_2} \rho^-(|M|)(\alpha_1, s) ds\right] \\ f^+(\alpha_1 - s, \alpha_2) &= \frac{g^+(\alpha_1, \alpha_2)}{g^+(s, \alpha_2)} \\ f^-(\alpha_1, \alpha_2 - s) &= \frac{g^-(\alpha_1, \alpha_2)}{g^-(\alpha_1, s)}. \end{aligned}$$

For sufficiently large σ, Q_i^σ is positive $\forall i \in \Lambda$. Hence as $\widetilde{\phi}_i$ is positive $\forall i \in \Lambda$, $\widetilde{N}_i(\alpha_1,\alpha_2) > 0$, $i = 1,2$, $\forall(\alpha_1,\alpha_2) \in J$ and $\widetilde{N}_i(\alpha_1,\alpha_2) > 0$, $i = 3,4$, $\forall(\alpha_1,\alpha_2) \in J$ if and only if:

$$\begin{aligned} &\int_{\alpha_2}^{1} Q_3^\sigma(|M|)(\alpha_1,s)f^-(\alpha_1,\alpha_2-s)ds \leqslant \sup_{(\alpha_1,\alpha_2)\in J} Q_3^\sigma(|M|) \\ &\int_{\alpha_1}^{1} Q_4^\sigma(|M|)(s,\alpha_1)f^+(\alpha_1-s,\alpha_2)ds \leqslant \sup_{(\alpha_1,\alpha_2)\in J} Q_4^\sigma(|M|) \end{aligned} \tag{3.6}$$

and it is sufficient that $\widetilde{\phi}_3 > \sup_{(\alpha_1,\alpha_2)\in J} Q_3^\sigma(|M|)$ and $\widetilde{\phi}_4 > \sup_{(\alpha_1,\alpha_2)\in J} Q_4^\sigma(|M|)$ to have $\widetilde{N} \in \mathscr{C}_+(J)^4$. □

Lemma 3.2. *$\mathscr{T}$ is continuous and compact on $\mathscr{C}(J)^4$.*

Proof. We have $\mathscr{T}(M) = \widetilde{N}$ if and only if $\widetilde{N}$ is given by the relations (3.5) from which we deduce:

$$\begin{aligned} \left|\widetilde{N}_1(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_1(\alpha_2)\right| |g^+(\alpha_1,\alpha_2)| + \left|\int_0^{\alpha_1} Q_1^\sigma(|M|)(s,\alpha_2)f^+(\alpha_1-s,\alpha_2)ds\right| \\ \left|\widetilde{N}_2(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_2(\alpha_1)\right| |g^-(\alpha_1,\alpha_2)| + \left|\int_0^{\alpha_2} Q_2^\sigma(|M|)(\alpha_1,s)f^-(\alpha_1,\alpha_2-s)ds\right| \\ \left|\widetilde{N}_3(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_3(\alpha_1)\right| |f^-(\alpha_1,\alpha_2-1)| + \left|\int_1^{\alpha_2} Q_3^\sigma(|M|)(\alpha_1,s)f^-(\alpha_1,\alpha_2-s)ds\right| \\ \left|\widetilde{N}_4(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_4(\alpha_2)\right| |f^+(\alpha_1-1,\alpha_2)| + \left|\int_1^{\alpha_1} Q_4^\sigma(|M|)(s,\alpha_1)f^+(\alpha_1-s,\alpha_2)ds\right|. \end{aligned}$$

In one hand using the Generalized Mean Value Theorem, as f^+ and f^- are strictly positive functions, we can find $c_1 \in]0,\alpha_1[$, $c_2 \in]0,\alpha_2[$, $c_3 \in]\alpha_2,1[$, $c_4 \in]\alpha_1,1[$, such that

$$\begin{aligned} \int_0^{\alpha_1} Q_1^\sigma(|M|)(s,\alpha_2)f^+(\alpha_1-s,\alpha_2)ds &= Q_1^\sigma(|M|)(c_1,\alpha_2)\int_0^{\alpha_1} f^+(\alpha_1-s,\alpha_2)ds \\ \int_0^{\alpha_2} Q_2^\sigma(|M|)(\alpha_1,s)f^-(\alpha_1,\alpha_2-s)ds &= Q_2^\sigma(|M|)(\alpha_1,c_2)\int_0^{\alpha_2} f^-(\alpha_1,\alpha_2-s)ds \\ \int_{\alpha_2}^{1} Q_3^\sigma(|M|)(\alpha_1,s)f^-(\alpha_1,\alpha_2-s)ds &= Q_3^\sigma(|M|)(\alpha_1,c_3)\int_{\alpha_2}^{1} f^-(\alpha_1,\alpha_2-s)ds \\ \int_{\alpha_1}^{1} Q_4^\sigma(|M|)(s,\alpha_2)f^+(\alpha_1-s,\alpha_2)ds &= Q_4^\sigma(|M|)(c_4,\alpha_2)\int_{\alpha_1}^{1} f^+(\alpha_1-s,\alpha_2)ds. \end{aligned}$$

In the other hand in accordance with the Mean Value Theorem we can find $d_1 \in]0,\alpha_1[$, $d_2 \in]0,\alpha_2[$, $d_3 \in]\alpha_2,1[$ and $d_4 \in]\alpha_1,1[$ such that

$$\begin{aligned}
\int_0^{\alpha_1} f^+(\alpha_1 - s,\alpha_2)ds &= \alpha_1 f^+(\alpha_1 - d_1,\alpha_2)\\
\int_0^{\alpha_2} f^-(\alpha_1,\alpha_2 - s)ds &= \alpha_2 f^-(\alpha_1,\alpha_2 - d_2)\\
\int_{\alpha_2}^{1} f^-(\alpha_1,\alpha_- s)ds &= (1-\alpha_2)\, f^-(\alpha_1,\alpha_2 - d_3)\\
\int_{\alpha_1}^{1} f^+(\alpha_1 - s,\alpha_2)ds &= (1-\alpha_1)\, f^+(\alpha_1 - d_4,\alpha_2).
\end{aligned}$$

Hence letting:

$$A^+(\alpha_2) = \exp\left(\sigma \int_0^1 \rho^+(|M|)(s-1,\alpha_2)ds\right),$$
$$A^-(\alpha_1) = \exp\left(\sigma \int_0^1 \rho^+(|M|)(\alpha_1,s-1)ds\right)$$

we get:

$$\begin{aligned}
\left|\widetilde{N}_1(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_1(\alpha_2)\right| + |Q_1^\sigma(|M|)(,c_1,\alpha_2)|\\
\left|\widetilde{N}_2(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_2(\alpha_1)\right| + |Q_2^\sigma(|M|)(\alpha_1,c_2)|\\
\left|\widetilde{N}_3(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_3(\alpha_1)\right| |A^-(\alpha_1)| + |Q_3^\sigma(|M|)(\alpha_1,c_3)|\\
\left|\widetilde{N}_4(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_4(\alpha_2)\right| |A^+(\alpha_2)| + |Q_4^\sigma(|M|)(c_4,\alpha_2)|.
\end{aligned}$$

since $|g^\pm(\alpha_1,\alpha_2)| > 1$. From which we infer:

(3.7) $$\|\mathscr{T}(M)\|_1 \leqslant \max\left(\left\|\widetilde{\phi}_1\right\|_0, \left\|\widetilde{\phi}_2\right\|_0, \left\|\widetilde{\phi}_3\right\|_0 \|A^-\|_0, \left\|\widetilde{\phi}_4\right\|_0 \|A^+\|_0\right) + \|Q^\sigma(|M|)\|_1$$

Thus $\mathscr{T}$ is continuous and bounded since $\widetilde{\phi}_i$, $i \in \Lambda$ $A^\pm$ and Q^σ are continuous and bounded. Hence if $M \in \mathscr{C}(J)^4$ is bounded then $\widetilde{N} = \mathscr{T}(M)$ is bounded.

Otherwise if $\widetilde{N}$ is the solution of the problem (3.2) then $\forall M \in \mathscr{C}(J)^4$

$$\begin{cases}
\dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} = Q_1^\sigma(|M|) - \sigma \widetilde{N}_1 \rho^+(|M|)\\
\dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} = Q_2^\sigma(|M|) - \sigma \widetilde{N}_2 \rho^-(|M|)\\
\dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} = Q_3^\sigma(|M|) - \sigma \widetilde{N}_3 \rho^-(|M|)\\
\dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} = Q_4^\sigma(|M|) - \sigma \widetilde{N}_4 \rho^+(|M|).
\end{cases}$$

Thus

$$(3.8)\qquad \begin{cases} \left|\dfrac{\partial \widetilde{N}_1}{\partial \alpha_1}\right| \leqslant |Q_1^\sigma(|M|)| + \sigma \left|\widetilde{N}_1 \rho^+(|M|)\right| \\ \left|\dfrac{\partial \widetilde{N}_2}{\partial \alpha_2}\right| \leqslant |Q_2^\sigma(|M|)| + \sigma \left|\widetilde{N}_2 \rho^-(|M|)\right| \\ \left|\dfrac{\partial \widetilde{N}_3}{\partial \alpha_2}\right| \leqslant |Q_3^\sigma(|M|)| + \sigma \left|\widetilde{N}_3 \rho^-(|M|)\right| \\ \left|\dfrac{\partial \widetilde{N}_4}{\partial \alpha_1}\right| \leqslant |Q_4^\sigma(|M|)| + \sigma \left|\widetilde{N}_4 \rho^+(|M|)\right|. \end{cases}$$

Thus for bounded $M \in \mathscr{C}(J)^4$ $\dfrac{\partial \widetilde{N}_1}{\partial \alpha_1}, \dfrac{\partial \widetilde{N}_2}{\partial \alpha_2}, \dfrac{\partial \widetilde{N}_3}{\partial \alpha_2}$ and $\dfrac{\partial \widetilde{N}_4}{\partial \alpha_1}$ are uniformly bounded in J. Otherwise, from the kinetic equations of (3.2), we derive the conservation equations

$$(3.9)\qquad \begin{cases} \dfrac{\partial}{\partial \alpha_2}(\widetilde{N}_2 - \widetilde{N}_3) + \dfrac{\partial}{\partial \alpha_1}(\widetilde{N}_1 - \widetilde{N}_4) = 0 \\ \dfrac{\partial}{\partial \alpha_2}(\widetilde{N}_2 + \widetilde{N}_3) = 0 \\ \dfrac{\partial}{\partial \alpha_1}(\widetilde{N}_1 + \widetilde{N}_4) = 0. \end{cases}$$

From (3.9) we deduce the system

$$(3.10)\qquad \begin{cases} \dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} + \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} = 0 \\ \dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} + \dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} = 0. \end{cases}$$

We differentiate the equations (3.10) with respect to α_2 and get as the $\widetilde{N}_i$, $i \in \Lambda$ are differentiable functions of α_1 and α_2 the system:

$$(3.11)\qquad \begin{cases} \dfrac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2} + \dfrac{\partial^2 \widetilde{N}_1}{\partial \alpha_2 \partial \alpha_1} = 0 \\ \dfrac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2} + \dfrac{\partial^2 \widetilde{N}_4}{\partial \alpha_2 \partial \alpha_1} = 0. \end{cases}$$

We integrate the Eqs. (3.11) with respect to α_1 and get:

$$\begin{aligned}
\frac{\partial \widetilde{N}_1}{\partial \alpha_2} &= -\int_0^{\alpha_1} \frac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}(s,\alpha_2)ds + \Theta_1(\alpha_2) \\
\frac{\partial \widetilde{N}_4}{\partial \alpha_2} &= -\int_1^{\alpha_1} \frac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(s,\alpha_2)ds + \Theta_4(\alpha_2).
\end{aligned}$$

Then we integrate with respect to α_2 and we have:

$$\begin{aligned}
\widetilde{N}_1(\alpha_1,\alpha_2) &= -\int_0^{\alpha_2}\int_0^{\alpha_1} \frac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}(s,t)dsdt + \int_0^{\alpha_2} \Theta_1(t)dt + \chi_1(\alpha_1) \\
\widetilde{N}_4(\alpha_1,\alpha_2) &= -\int_0^{\alpha_2}\int_1^{\alpha_1} \frac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(s,t)dsdt + \int_0^{\alpha_2} \Theta_4(t)dt + \chi_4(\alpha_1).
\end{aligned}$$

Using the boundary conditions we have the system:

$$\begin{aligned}
\widetilde{N}_1(0,\alpha_2) &= \int_0^{\alpha_2} \Theta_1(t)\,dt + \chi_1(0) = \widetilde{\phi}_1(\alpha_2) \\
\widetilde{N}_4(1,\alpha_2) &= \int_0^{\alpha_2} \Theta_4(t)dt + \chi_4(1) = \widetilde{\phi}_4(\alpha_2).
\end{aligned}$$

From which we get by differentiation $\Theta_j = \dfrac{d\widetilde{\phi}_j}{d\alpha_2}$, $j = 1,4$. We thus have:

$$\begin{aligned}
\frac{\partial \widetilde{N}_1}{\partial \alpha_2}(\alpha_1,\alpha_2) &= -\int_0^{\alpha_1} \frac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}(s,\alpha_2)ds + \frac{d\widetilde{\phi}_1}{d\alpha_2}(\alpha_2) \\
\frac{\partial \widetilde{N}_4}{\partial \alpha_2}(\alpha_1,\alpha_2) &= -\int_1^{\alpha_1} \frac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(s,\alpha_2)ds + \frac{d\widetilde{\phi}_4}{d\alpha_2}(\alpha_2).
\end{aligned}$$

Similarly we obtain:

$$\begin{aligned}
\frac{\partial \widetilde{N}_2}{\partial \alpha_1}(\alpha_1,\alpha_2) &= -\int_0^{\alpha_2} \frac{\partial^2 \widetilde{N}_1}{\partial \alpha_1^2}(\alpha_1,s)ds + \frac{d\widetilde{\phi}_2}{d\alpha_1}(\alpha_1) \\
\frac{\partial \widetilde{N}_3}{\partial \alpha_1}(\alpha_1,\alpha_2) &= -\int_1^{\alpha_2} \frac{\partial^2 \widetilde{N}_4}{\partial \alpha_1^2}(\alpha_1,s)ds + \frac{d\widetilde{\phi}_3}{d\alpha_1}(\alpha_1).
\end{aligned}$$

Using the expressions (3.5) of $\widetilde{N}_i$, $i \in \Lambda$ we have

$$\begin{aligned}
\frac{\partial^2 \widetilde{N}_1}{\partial \alpha_1^2}(\alpha_1,\alpha_2) &= \widetilde{\phi}_1(\alpha_2)\frac{\partial^2 g^+}{\partial \alpha_1^2}(\alpha_1,\alpha_2) + \frac{\partial Q_1^\sigma(|M|)}{\partial \alpha_1}(\alpha_1,\alpha_2) \\
\frac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}(\alpha_1,\alpha_2) &= \widetilde{\phi}_2(\alpha_1)\frac{\partial^2 g^-}{\partial \alpha_2^2}(\alpha_1,\alpha_2) + \frac{\partial Q_2^\sigma(|M|)}{\partial \alpha_2}(\alpha_1,\alpha_2)
\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(\alpha_1,\alpha_2) &= \widetilde{\phi}_3(\alpha_1)\frac{\partial^2 f^-}{\partial \alpha_2^2}(\alpha_1,\alpha_2-1)+\frac{\partial Q_3^\sigma(|M|)}{\partial \alpha_2}(\alpha_1,\alpha_2)\\ \frac{\partial^2 \widetilde{N}_4}{\partial \alpha_1^2}(\alpha_1,\alpha_2) &= \widetilde{\phi}_4(\alpha_1)\frac{\partial^2 f^+}{\partial \alpha_1^2}(\alpha_1-1,\alpha_2)+\frac{\partial Q_4^\sigma(|M|)}{\partial \alpha_1}(\alpha_1,\alpha_2).\end{aligned}$$

As

$$\frac{\partial \rho^+}{\partial \alpha_1}=\frac{\partial \rho^-}{\partial \alpha_2}=0$$

we have

$$\begin{aligned}\frac{\partial^2 g^+}{\partial \alpha_1^2}(\alpha_1,\alpha_2) &= \sigma^2 {\rho^+}^2(|M|)g^+(\alpha_1,\alpha_2)\\ \frac{\partial^2 g^-}{\partial \alpha_2^2}(\alpha_1,\alpha_2) &= \sigma^2 {\rho^-}^2(|M|)g^-(\alpha_1,\alpha_2)\\ \frac{\partial^2 f^+}{\partial \alpha_1^2}(\alpha_1-1,\alpha_2) &= \sigma^2 {\rho^+}^2(|M|)f^+(\alpha_1-1,\alpha_2)\\ \frac{\partial^2 f^-}{\partial \alpha_2^2}(\alpha_1,\alpha_2-1) &= \sigma^2 {\rho^-}^2(|M|)f^-(\alpha_1,\alpha_2-1).\end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{\partial^2 \widetilde{N}_1}{\partial \alpha_1^2}(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_1(\alpha_2)\right|\sigma^2 {\rho^+}^2(|M|)(\alpha_1,\alpha_2)g^+(\alpha_1,\alpha_2)\\ &\quad+\left|\frac{\partial Q_1^\sigma}{\partial \alpha_1}(\alpha_1,\alpha_2)\right|\\ \left|\frac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_2(\alpha_1)\right|\sigma^2 {\rho^-}^2(|M|)(\alpha_1,\alpha_2)g^-(\alpha_1,\alpha_2)\\ &\quad+\left|\frac{\partial Q_2^\sigma}{\partial \alpha_2}(\alpha_1,\alpha_2)\right|\\ \left|\frac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_3(\alpha_1)\right|\sigma^2 {\rho^-}^2(|M|)(\alpha_1,\alpha_2)g^-(\alpha_1,\alpha_2)\\ &\quad+\left|\frac{\partial Q_3^\sigma(|M|)}{\partial \alpha_2}(\alpha_1,\alpha_2)\right|\\ \left|\frac{\partial^2 \widetilde{N}_4}{\partial \alpha_1^2}(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_4(\alpha_2)\right|\sigma^2 {\rho^+}^2(|M|)(\alpha_1,\alpha_2)g^+(\alpha_1,\alpha_2)\\ &\quad+\left|\frac{\partial Q_4^\sigma(|M|)}{\partial \alpha_1}(\alpha_1,\alpha_2)\right|\end{aligned} \tag{3.12}$$

and

$$(3.13)\qquad \begin{array}{lcl} \left|\dfrac{\partial \widetilde{N}_1}{\partial \alpha_2}(\alpha_1,\alpha_2)\right| & \leqslant & \left|\dfrac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}(\alpha_1,\alpha_2)\right| + \left|\dfrac{d\widetilde{\phi}_1}{d\alpha_2}(\alpha_2)\right| \\ \left|\dfrac{\partial \widetilde{N}_2}{\partial \alpha_1}(\alpha_1,\alpha_2)\right| & \leqslant & \left|\dfrac{\partial^2 \widetilde{N}_1}{\partial \alpha_1^2}(\alpha_1,\alpha_2)\right| + \left|\dfrac{d\widetilde{\phi}_2}{d\alpha_1}(\alpha_1)\right| \\ \left|\dfrac{\partial \widetilde{N}_3}{\partial \alpha_1}(\alpha_1,\alpha_2)\right| & \leqslant & \left|\dfrac{\partial^2 \widetilde{N}_4}{\partial \alpha_1^2}(\alpha_1,\alpha_2)\right| + \left|\dfrac{d\widetilde{\phi}_3}{d\alpha_1}(\alpha_1)\right| \\ \left|\dfrac{\partial \widetilde{N}_4}{\partial \alpha_2}(\alpha_1,\alpha_2)\right| & \leqslant & \left|\dfrac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(\alpha_1,\alpha_2)\right| + \left|\dfrac{d\widetilde{\phi}_4}{d\alpha_2}(\alpha_2)\right|. \end{array}$$

The inequalities (3.12) and (3.13) show that $\left|\dfrac{\partial \widetilde{N}_1}{\partial \alpha_1}\right|$, $\left|\dfrac{\partial \widetilde{N}_2}{\partial \alpha_2}\right|$, $\left|\dfrac{\partial \widetilde{N}_3}{\partial \alpha_2}\right|$ and $\left|\dfrac{\partial \widetilde{N}_4}{\partial \alpha_1}\right|$ are bounded for bounded $M \in \mathscr{C}(J)^4$ provided that $\left|\dfrac{d\widetilde{\phi}_1}{d\alpha_2}\right|$, $\left|\dfrac{d\widetilde{\phi}_2}{d\alpha_1}\right|$, $\left|\dfrac{d\widetilde{\phi}_3}{d\alpha_1}\right|$ and $\left|\dfrac{d\widetilde{\phi}_4}{d\alpha_2}\right|$ are bounded.

We thus prove that if $M \in \mathscr{C}(J)^4$ is bounded then $\dfrac{\partial \widetilde{N}_i}{\partial \alpha_1}$ and $\dfrac{\partial \widetilde{N}_i}{\partial \alpha_2}$, $i \in \Lambda$ are uniformly bounded if $\dfrac{d\widetilde{\phi}_j}{d\alpha_2}$ $j = 1, 4$ and $\dfrac{d\widetilde{\phi}_k}{d\alpha_1}$ $k = 2, 3$ are bounded. Then it exists $\beta > 0$ and $\gamma > 0$ such that $\forall i \in \Lambda$

$$\left|\frac{\partial \widetilde{N}_i}{\partial \alpha_2}\right| < \beta \quad \text{in } [0, \cos\theta + \varepsilon \sin\theta]$$

and

$$\left|\frac{\partial \widetilde{N}_i}{\partial \alpha_1}\right| < \gamma \quad \text{in } [-\sin\theta, \varepsilon\cos\theta]\,.$$

Given $\alpha^1 = (\alpha_1^1, \alpha_2^1) \in J$ and $\alpha^2 = (\alpha_1^2, \alpha_2^2) \in J$. We deduce from the Mean Value Theorem, that its exists $\alpha^0 = (\alpha_1^0, \alpha_2^0) \in [\alpha^1, \alpha^2] \subset J$ such that

$$\widetilde{N}_i\left(\alpha^1\right) - \widetilde{N}_i\left(\alpha^2\right) = d\widetilde{N}_i\left(\alpha^0\right)\left(\alpha^1 - \alpha^2\right), \quad i \in \Lambda$$

with

$$\left[\alpha^1, \alpha^2\right] = \left\{\alpha \in \mathbb{R}^2 / \alpha = t\left(\alpha^1 - \alpha^2\right) + \alpha^2\,,\ t \in [0,1]\right\}$$

and

$$d\widetilde{N}_i\left(\alpha^0\right)(h) = \frac{\partial \widetilde{N}_i}{\partial \alpha_1}\left(\alpha^0\right) h_1 + \frac{\partial \widetilde{N}_i}{\partial \alpha_2}\left(\alpha^0\right) h_2 \quad \forall h = (h_1, h_2) \in \mathbb{R}^2.$$

Hence

$$\begin{aligned}\left|\widetilde{N}_i\left(\alpha^1\right)-\widetilde{N}_i\left(\alpha^2\right)\right| &= \left|d\widetilde{N}_i\left(\alpha^0\right)\left(\alpha^1-\alpha^2\right)\right| \\ &\leqslant \left\|d\widetilde{N}_i\left(\alpha^0\right)\right\|_0 \|\alpha^1-\alpha^2\|\end{aligned}$$

with

$$\begin{aligned}\left\|d\widetilde{N}_i\left(\alpha^0\right)\right\|_0 &= \sup_{\|h\|\leqslant 1}\frac{\left|d\widetilde{N}_i\left(\alpha^0\right)h\right|}{\|h\|} \\ &= \sup_{\|h\|\leqslant 1}\frac{\left|\dfrac{\partial\widetilde{N}_i}{\partial\alpha_1}\left(\alpha^0\right)h_1+\dfrac{\partial\widetilde{N}_i}{\partial\alpha_2}\left(\alpha^0\right)h_2\right|}{|h_1|+|h_2|}.\end{aligned}$$

But

$$\begin{aligned}&\left|\frac{\partial\widetilde{N}_i}{\partial\alpha_1}\left(\alpha^0\right)h_1+\frac{\partial\widetilde{N}_i}{\partial\alpha_2}\left(\alpha^0\right)h_2\right| \\ \leqslant\ & \left\|\frac{\partial\widetilde{N}_i}{\partial\alpha_1}\left(\alpha^0\right)\right\|_0|h_1|+\left\|\frac{\partial\widetilde{N}_i}{\partial\alpha_2}\left(\alpha^0\right)\right\|_0|h_2| \\ \leqslant\ & \max\left(\left\|\frac{\partial\widetilde{N}_i}{\partial\alpha_1}\left(\alpha^0\right)\right\|_0,\left\|\frac{\partial\widetilde{N}_i}{\partial\alpha_2}\left(\alpha^0\right)\right\|_0\right)\|h\| \\ \leqslant\ & \max(\beta,\gamma)\|h\|.\end{aligned}$$

That is $\left\|d\widetilde{N}_i\left(\alpha^0\right)\right\|_0\leqslant\max(\beta,\gamma)$. Then $\left|\widetilde{N}_i\left(\alpha^1\right)-\widetilde{N}_i\left(\alpha^2\right)\right|\leqslant\max(\beta,\gamma)\,\|\alpha^1-\alpha^2\|$.

Let $\varepsilon>0$, it is sufficient that $\|\alpha^1-\alpha^2\|<\dfrac{\varepsilon}{\max(\beta,\alpha)}$ to have $\left|\widetilde{N}_i\left(\alpha^1\right)-\widetilde{N}_i\left(\alpha^2\right)\right|<\varepsilon$, $\forall\, i\in\wedge$.

We prove that for all solution $\widetilde{N}$ of (3.2)

$$\forall\,\varepsilon>0\,,\ \exists\eta>0\,,\ \left\|\alpha^1-\alpha^2\right\|<\eta\Longrightarrow\left|\widetilde{N}_i\left(\alpha^1\right)-\widetilde{N}_i\left(\alpha^2\right)\right|<\varepsilon\quad\forall\,\alpha^1\,,\,\alpha^2\in J.$$

The set of the solutions of (3.2) is thus equicontinuous so $\mathscr{T}$ is compact on every bounded subset of $\mathscr{C}_+(J)^4$. □

Lemma 3.3. *Every solution of the equation* $\widetilde{N}=\lambda\mathscr{T}(\widetilde{N})$, $0<\lambda<1$, *is bounded.*

Proof. $\widetilde{N}$ is a solutions of $\widetilde{N} = \lambda \mathscr{T}(\widetilde{N})$ if and only if

$$(3.14) \quad \begin{cases} \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} + \sigma \widetilde{N}_1 \rho^+(\widetilde{N}) = \lambda Q_1^\sigma(\widetilde{N}) & (3.14)-(1) \\ \dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} + \sigma \widetilde{N}_2 \rho^-(\widetilde{N}) = \lambda Q_2^\sigma(\widetilde{N}) & (3.14)-(2) \\ \dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} + \sigma \widetilde{N}_3 \rho^-(\widetilde{N}) = \lambda Q_3^\sigma(\widetilde{N}) & (3.14)-(3) \\ \dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} + \sigma \widetilde{N}_4 \rho^+(\widetilde{N}) = \lambda Q_4^\sigma(\widetilde{N}) & (3.14)-(4) \\ \qquad \widetilde{N}_1(0, \alpha_2) = \lambda \widetilde{\phi}_1(\alpha_2) \\ \qquad \widetilde{N}_2(\alpha_1, 0) = \lambda \widetilde{\phi}_2(\alpha_1) \\ \qquad \widetilde{N}_3(\alpha_1, 1) = \lambda \widetilde{\phi}_3(\alpha_1) \\ \qquad \widetilde{N}_4(1, \alpha_2) = \lambda \widetilde{\phi}_4(\alpha_2). \end{cases}$$

Making the sums (3.14)-(1)+(3.14)-(4) and (3.14)-(2)+(3.14)-(3), we obtain for the determination of the partial macroscopic densities ρ^+ and ρ^- the following system of partial differential equations:

$$(3.15) \quad \begin{cases} \dfrac{\partial \left[\rho^+(\widetilde{N})\right]}{\partial \alpha_1} + (1-\lambda)\sigma \left[\rho^+(\widetilde{N})\right]^2 = 0 \\ \dfrac{\partial \left[\rho^-(\widetilde{N})\right]}{\partial \alpha_2} + (1-\lambda)\sigma \left[\rho^-(\widetilde{N})\right]^2 = 0. \end{cases}$$

The unique solution of system (3.15) is obviously

$$(3.16) \quad \begin{cases} \rho^+(\widetilde{N})(\alpha_1, \alpha_2) = \dfrac{1}{(1-\lambda)\sigma\alpha_1 + h^+(\alpha_2)} \\ \\ \rho^-(\widetilde{N})(\alpha_1, \alpha_2) = \dfrac{1}{(1-\lambda)\sigma\alpha_2 + h^-(\alpha_1)}. \end{cases}$$

The problem (3.14) is a two point boundary value problem and only a part of the data are given at each boundary. Hence we have $\widetilde{N}_1(0, \alpha_2)$ on the line $\alpha_1 = 0$, $\widetilde{N}_4(1, \alpha_2)$ on the line $\alpha_1 = 1$, $\widetilde{N}_2(\alpha_1, 0)$ on the line $\alpha_2 = 0$ and $\widetilde{N}_3(\alpha_1, 1)$ on the line $\alpha_2 = 1$.

We thus introduce the positive functions of α_2, λ_k^+ and the positive functions of de α_1, λ_k^- , $k = 0, 1$ such that

$$\begin{array}{rcl} \widetilde{N}_4\,(0,\alpha_2) & = & \lambda_0^+(\alpha_2)\widetilde{N}_1\,(0,\alpha_2) \\ \widetilde{N}_1\,(1,\alpha_2) & = & \lambda_1^+(\alpha_2)\widetilde{N}_4\,(1,\alpha_2) \\ \widetilde{N}_3\,(\alpha_1,0) & = & \lambda_0^-(\alpha_1)\widetilde{N}_2\,(\alpha_1,0) \\ \widetilde{N}_2\,(\alpha_1,1) & = & \lambda_1^-(\alpha_2)\widetilde{N}_3\,(\alpha_1,1)\,. \end{array} \tag{3.17}$$

The relations (3.17) which are by no means reflection laws and are obtained just by comparing functions of the same variables at the boundaries of the domain J allow to compute the values of ρ^+ and ρ^- at the boundaries:

$$\begin{array}{rcccl} \rho^+(\widetilde{N})\,(\alpha_2) & = & \left[1+\lambda_0^+(\alpha_2)\right]\lambda\widetilde{\phi}_1(\alpha_2) & = & \left[1+\lambda_1^+(\alpha_2)\right]\lambda\widetilde{\phi}_4(\alpha_2) \\ \rho^-(\widetilde{N})\,(\alpha_1) & = & \left[1+\lambda_0^-(\alpha_1)\right]\lambda\widetilde{\phi}_2(\alpha_1) & = & \left[1+\lambda_1^-(\alpha_1)\right]\lambda\widetilde{\phi}_3(\alpha_1). \end{array}$$

From which we infer:

$$h^+(\alpha_2) = \frac{1}{\left[1+\lambda_0^+(\alpha_2)\right]\lambda\widetilde{\phi}_1(\alpha_2)} + \sigma(\lambda-1)$$
$$h^-(\alpha_1) = \frac{1}{\left[1+\lambda_0^-(\alpha_1)\right]\lambda\widetilde{\phi}_2(\alpha_1)} + \sigma(\lambda-1).$$

Hence

$$\left\{ \begin{array}{l} \rho^+(\widetilde{N})(\alpha_1,\alpha_2) = \dfrac{1}{(1-\lambda)\sigma\alpha_1 + \dfrac{1}{\left[1+\lambda_0^+(\alpha_2)\right]\lambda\widetilde{\phi}_1(\alpha_2)}} \\ \rho^-(\widetilde{N})(\alpha_1,\alpha_2) = \dfrac{1}{(1-\lambda)\sigma\alpha_2 + \dfrac{1}{\left[1+\lambda_0^-(\alpha_1)\right]\lambda\widetilde{\phi}_2(\alpha_1)}}. \end{array} \right. \tag{3.18}$$

Thus for $0 < \lambda < 1$, ρ^+ and ρ^- are continuous and bounded as $\widetilde{\phi}_i$, $i = 1, 2$ and $\lambda_0^\pm$ and so are the number densities $\widetilde{N}_i$, $i \in \Lambda$. □

Remark 3.1. *For $\lambda = 1$ the solutions* (3.18) *are not singular. Moreover they satisfy the partial conservation equations*

$$\frac{\partial\left[\rho^+(\widetilde{N})\right]}{\partial\alpha_1} = \frac{\partial\left[\rho^-(\widetilde{N})\right]}{\partial\alpha_2} = 0.$$

and consequently depend upon one variable.

Finally we conclude to the existence of the solutions of problem (3.1) by using the fixed point theorem of Schaefer [10]

Theorem 3.1. *Let T be a continuous and compact mapping of a Banach X into itself, such that the set*
$\{x \in X\ ,\ x = \lambda T(x)\}$ *is bounded* $\forall\,\lambda,\ 0 < \lambda < 1$. *Then T has a fixed point.*

The $\widetilde{N}_i\,, i \in \Lambda$ which are positive functions of α_1 and α_2 exist, are continuous and bounded. Thus the problem (2.4) possesses a solution N positive, continuous and bounded. □

4. Exact solutions

The problem (2.5) admits two kinds of solutions: non maxwellian and maxwellian solutions corresponding to non zero and vanishing collision terms. In this section it is shown that the non maxwellian solutions are not unique in general in contrast to the maxwellian ones which are as usual unique.

For $\lambda = 1$, ρ^+ and ρ^- are known and we have:

$$\rho^+(\widetilde{N})(\alpha_2) = \left(\widetilde{N}_1 + \widetilde{N}_4\right)(\alpha_2) \quad \text{and} \quad \rho^-(\widetilde{N})(\alpha_1) = \left(\widetilde{N}_2 + \widetilde{N}_3\right)(\alpha_1).$$

Then

$$\begin{cases} \widetilde{N}_4(\alpha_1,\alpha_2) &= \rho^+(\widetilde{N})(\alpha_2) - \widetilde{N}_1(\alpha_1,\alpha_2) \\ \widetilde{N}_3(\alpha_1,\alpha_2) &= \rho^-(\widetilde{N})(\alpha_1) - \widetilde{N}_2(\alpha_1,\alpha_2) \end{cases} \tag{4.1}$$

and the system (2.5) becomes:

$$\begin{cases} \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} &= \sigma_0\left[\left(\widetilde{N}_1 - \dfrac{\rho^+}{2}\right)^2 - \left(\widetilde{N}_2 - \dfrac{\rho^-}{2}\right)^2 + \dfrac{\rho^{-2} - \rho^{+2}}{4}\right] \\ \dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} &= -\sigma_0\left[\left(\widetilde{N}_1 - \dfrac{\rho^+}{2}\right)^2 - \left(\widetilde{N}_2 - \dfrac{\rho^-}{2}\right)^2 + \dfrac{\rho^{-2} - \rho^{+2}}{4}\right] \\ \widetilde{N}_4(\alpha_1,\alpha_2) &= \rho^+(\widetilde{N})(\alpha_2) - \widetilde{N}_1(\alpha_1,\alpha_2) \\ \widetilde{N}_3(\alpha_1,\alpha_2) &= \rho^-(\widetilde{N})(\alpha_1) - \widetilde{N}_2(\alpha_1,\alpha_2) \\ \widetilde{N}_1\,(0,\alpha_2) &= \widetilde{\phi}_1(\alpha_2) \\ \widetilde{N}_2\,(\alpha_1,0) &= \widetilde{\phi}_2(\alpha_1) \\ \widetilde{N}_3\,(\alpha_1,1) &= \widetilde{\phi}_3(\alpha_1) \\ \widetilde{N}_4\,(1,\alpha_2) &= \widetilde{\phi}_4(\alpha_2) \end{cases} \tag{4.2}$$

The boundary value problem for the numbers densities $\widetilde{N}_i$, $i = 1, 2$ is thus:

$$(4.3)\quad \begin{cases} \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} = -\dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} & = \sigma_0 \left[\left(\widetilde{N}_1 - \dfrac{\rho^+}{2} \right)^2 - \left(\widetilde{N}_2 - \dfrac{\rho^-}{2} \right)^2 + \dfrac{\rho^{-2} - \rho^{+2}}{4} \right] \\ & = Q_1(\widetilde{N}) \\ \widetilde{N}_1\,(0, \alpha_2) & = \widetilde{\phi}_1(\alpha_2) \\ \widetilde{N}_1\,(1, \alpha_2) & = \rho^+(\alpha_2) - \widetilde{\phi}_4(\alpha_2) \\ \widetilde{N}_2\,(\alpha_1, 0) & = \widetilde{\phi}_2(\alpha_1) \\ \widetilde{N}_2\,(\alpha_1, 1) & = \rho^-(\alpha_1) - \widetilde{\phi}_3(\alpha_1) \end{cases}$$

Letting

$$F_1(\alpha_1, \alpha_2) = \widetilde{N}_1(\alpha_1, \alpha_2) - \frac{\rho^+(\alpha_2)}{2} \quad \text{and} \quad F_2(\alpha_1, \alpha_2) = \widetilde{N}_2(\alpha_1, \alpha_2) - \frac{\rho^-(\alpha_1)}{2}$$

the system (4.3) takes the form:

$$(4.4)\quad \begin{cases} \dfrac{\partial F_1}{\partial \alpha_1} = -\dfrac{\partial F_2}{\partial \alpha_2} & = \sigma_0 \left[F_1^2 - F_2^2 + \dfrac{\rho^{-2} - \rho^{+2}}{4} \right] = Q_1(\widetilde{N}) \\ F_1\,(0, \alpha_2) & = \widetilde{\phi}_1(\alpha_2) - \dfrac{\rho^+(\alpha_2)}{2} \\ F_1\,(1, \alpha_2) & = -\widetilde{\phi}_4(\alpha_2) + \dfrac{\rho^+(\alpha_2)}{2} \\ F_2\,(\alpha_1, 0) & = \widetilde{\phi}_2(\alpha_1) - \dfrac{\rho^-(\alpha_1)}{2} \\ F_2\,(\alpha_1, 1) & = -\widetilde{\phi}_3(\alpha_1) + \dfrac{\rho^-(\alpha_1)}{2} \end{cases}$$

The system (4.4) has a simpler form but its exact resolution is complicated. However it shows the dependence of the solutions in the partial mean densities $\rho^{\pm}$ and permits to find exact solutions of the problem (4.3) in particular cases.

4.1. **The maxwellian solutions.** The maxwellian densities of the discrete velocity model associated to the macroscopic variables ρ, U and V are the strictly positive number densities $\widetilde{N}_{iM}$, $i = 1 \dots 4$ which satisfy the following system of equations:

$$(4.5)\quad \begin{cases} \rho & = \widetilde{N}_{1M} + \widetilde{N}_{2M} + \widetilde{N}_{3M} + \widetilde{N}_{4M} \\ \rho U & = c \left(\cos\theta \widetilde{N}_{1M} - sin\theta \widetilde{N}_{2M} + \sin\theta \widetilde{N}_{3M} - \cos\theta \widetilde{N}_{4M} \right) \\ \rho V & = c \left(\sin\theta \widetilde{N}_{1M} + \cos\theta \widetilde{N}_{2M} - \cos\theta \widetilde{N}_{3M} - \sin\theta \widetilde{N}_{4M} \right) \\ 0 & = \widetilde{N}_{1M} \widetilde{N}_{4M} - \widetilde{N}_{2M} \widetilde{N}_{3M} \end{cases}$$

The unique positive solutions of (4.5) are:

$$
(4.6)\qquad
\begin{aligned}
\widetilde{N}_{1M} &= \frac{\rho\,[1+\cos 2\theta(u^2-v^2)+2uv\sin 2\theta+2u\cos\theta+2v\sin\theta]}{4}\\
\widetilde{N}_{2M} &= \frac{\rho\,[1-\cos 2\theta(u^2-v^2)-2uv\sin 2\theta+2v\cos\theta-2u\sin\theta]}{4}\\
\widetilde{N}_{3M} &= \frac{\rho\,[1-\cos 2\theta(u^2-v^2)-2uv\sin 2\theta-2v\cos\theta+2u\sin\theta]}{4}\\
\widetilde{N}_{4M} &= \frac{\rho\,[1+\cos 2\theta(u^2-v^2)+2uv\sin 2\theta-2u\cos\theta-2v\sin\theta]}{4}
\end{aligned}
$$

Proposition 4.1. *The maxwellian solutions of* (4.3) *are unique.*

Proof. The collision terms of the system (4.3) vanish for a maxwellian solution. We thus have:

$$
(4.7)\qquad
\begin{cases}
\dfrac{\partial F_1}{\partial \alpha_1} = -\dfrac{\partial F_2}{\partial \alpha_2} = \sigma_0\left[F_1^2 - F_2^2 + \dfrac{\rho^{-2}-\rho^{+2}}{4}\right] = 0\\
F_1\,(0,\alpha_2) = \widetilde{\phi}_1(\alpha_2) - \dfrac{\rho^+(\alpha_2)}{2}\\
F_1\,(1,\alpha_2) = -\widetilde{\phi}_4(\alpha_2) + \dfrac{\rho^+(\alpha_2)}{2}\\
F_2\,(\alpha_1,0) = \widetilde{\phi}_2(\alpha_1) - \dfrac{\rho^-(\alpha_1)}{2}\\
F_2\,(\alpha_1,1) = -\widetilde{\phi}_3(\alpha_1) + \dfrac{\rho^-(\alpha_1)}{2}
\end{cases}
$$

From which we deduce

$$
(4.8)\qquad
\begin{cases}
F_1 \equiv F_1(\alpha_2), \qquad F_2 \equiv F_2(\alpha_1)\\
\dfrac{\rho^{+2}}{4} - F_1^2 = \dfrac{\rho^{-2}}{4} - F_2^2 = c_1\\
\rho^+(\alpha_2) = \widetilde{\phi}_1(\alpha_2) + \widetilde{\phi}_4(\alpha_2)\\
\rho^-(\alpha_1) = \widetilde{\phi}_2(\alpha_1) + \widetilde{\phi}_3(\alpha_1)
\end{cases}
$$

The solutions of (4.8) are:

$$
F_1(\alpha_2) = \pm\frac{1}{2}\sqrt{\rho^{+2}(\alpha_2) - 4c_1}, \quad F_2(\alpha_1) = \pm\frac{1}{2}\sqrt{\rho^{-2}(\alpha_1) - 4c_1}
$$

We thus have:

(1) For $c_1 < 0$, $\sqrt{\rho^{\pm 2} - 4c_1} > \rho^{\pm}$ and some of the $\widetilde{N}_i$ are negative.

(2) For $c_1 = 0$, $F_1(\alpha_2) = \dfrac{\rho^+(\alpha_2)}{2}$ and $F_2(\alpha_1) = \dfrac{\rho^-(\alpha_1)}{2}$ which leads to $\widetilde{\phi}_1(\alpha_2) = \rho^+(\alpha_2)$, $\widetilde{\phi}_2(\alpha_1) = \rho^-(\alpha_1)$ and $\widetilde{\phi}_3 = \widetilde{\phi}_4 = 0$.

Hence as the number densities are strictly positive the solution is not admissible for $c_1 \leq 0$. We only have strictly positive number densities for $c_1 > 0$:

$$\begin{aligned} \widetilde{N}_1(\alpha_1,\alpha_2) &= \frac{\rho^+(\alpha_2)}{2} + \frac{1}{2}\sqrt{\rho^{+2}(\alpha_2) - 4c_1} \\ \widetilde{N}_2(\alpha_1,\alpha_2) &= \frac{\rho^-(\alpha_1)}{2} + \frac{1}{2}\sqrt{\rho^{-2}(\alpha_1) - 4c_1} \\ \widetilde{N}_3(\alpha_1,\alpha_2) &= \frac{\rho^-(\alpha_1)}{2} - \frac{1}{2}\sqrt{\rho^{-2}(\alpha_1) - 4c_1} \\ \widetilde{N}_4(\alpha_1,\alpha_2) &= \frac{\rho^+(\alpha_2)}{2} - \frac{1}{2}\sqrt{\rho^{+2}(\alpha_2) - 4c_1}. \end{aligned} \tag{4.9}$$

Taking into account the boundary conditions, we get:

$$\begin{gathered} \rho^+(\alpha_2) = \widetilde{\phi}_1(\alpha_2) + \widetilde{\phi}_4(\alpha_2) \\ \rho^-(\alpha_1) = \widetilde{\phi}_2(\alpha_1) + \widetilde{\phi}_3(\alpha_1) \\ c_1 = \widetilde{\phi}_1(\alpha_2).\widetilde{\phi}_4(\alpha_2) = \widetilde{\phi}_2(\alpha_1).\widetilde{\phi}_3(\alpha_1). \end{gathered} \tag{4.10}$$

The validity of the third relation (4.10) imposes the dependence of the boundary data in the form:

$$\begin{cases} \widetilde{\phi}_4(\alpha_2) &= \dfrac{c_1}{\widetilde{\phi}_1(\alpha_2)} \\ \widetilde{\phi}_3(\alpha_1) &= \dfrac{c_1}{\widetilde{\phi}_2(\alpha_1)}. \end{cases} \tag{4.11}$$

The Maxwellian solutions are thus:

$$\begin{aligned} \widetilde{N}_1(\alpha_1,\alpha_2) &= \widetilde{\phi}_1(\alpha_2) \\ \widetilde{N}_2(\alpha_1,\alpha_2) &= \widetilde{\phi}_2(\alpha_1) \\ \widetilde{N}_3(\alpha_1,\alpha_2) &= \frac{c_1}{\widetilde{\phi}_2(\alpha_1)} \\ \widetilde{N}_4(\alpha_1,\alpha_2) &= \frac{c_1}{\widetilde{\phi}_1(\alpha_2)}. \end{aligned} \tag{4.12}$$

The solutions (4.12) are associated to the macroscopic variables:

$$\begin{aligned} \rho &= \widetilde{\phi}_1 + \widetilde{\phi}_2 + \frac{c_1}{\widetilde{\phi}_1} + \frac{c_1}{\widetilde{\phi}_2} \\ \rho U &= c\left[\cos\theta\widetilde{\phi}_1 - \sin\theta\widetilde{\phi}_2 + \sin\theta\frac{c_1}{\widetilde{\phi}_2} - \cos\theta\frac{c_1}{\widetilde{\phi}_1}\right] \\ \rho V &= c\left[\sin\theta\widetilde{\phi}_1 + \cos\theta\widetilde{\phi}_2 - \cos\theta\frac{c_1}{\widetilde{\phi}_2} - \sin\theta\frac{c_1}{\widetilde{\phi}_1}\right] \end{aligned} \tag{4.13}$$

Letting $u = \dfrac{U}{c}$ and $v = \dfrac{V}{c}$ we deduce from the resolution of (4.13):

$$\widetilde{\phi}_1 = \frac{\rho\left[1 + \cos 2\theta(u^2 - v^2) + 2uv \sin 2\theta + 2u\cos\theta + 2v \sin\theta\right]}{4}$$
$$\widetilde{\phi}_2 = \frac{\rho\left[1 - \cos 2\theta(u^2 - v^2) - 2uv \sin 2\theta + 2v\cos\theta - 2u \sin\theta\right]}{4}$$
$$c1 = \frac{\rho^2\left[1 - 2(u^2 + v^2) + (u^2 - v^2)^2 \cos^2 2\theta + 4u^2v^2 \sin^2 2\theta + 2uv(u^2 - v^2)\sin 4\theta\right]}{16}$$

So the densities (4.12) are the unique maxwellian solutions of the model associated to the macroscopic variables ρ, U and V defined by (4.6). □

4.2. **Non uniqueness of non maxwellian solutions.** It is difficult to discuss of the uniqueness of the solution in general as the kinetic equations of (4.4) depend explicitly on the partial mean densities $\rho^{\pm}$, the form of which determines the type of the solutions. As a result, we investigate the possible existence of several solutions for the same set of the partial densities $\rho^{\pm}$ and find that for linear and constant partial mean densities $\rho^{\pm}$ there exist at least two different kinds of solutions of the kinetic equations of (4.4).

4.2.1. *The case of linear mean densities.* We take advantage of the fact that the partial mean densities are functions of one variable and the kinetic equations are of Riccati type to put $\rho^-(\alpha_1) = c_0\alpha_1 + c_1$ and $\rho^+(\alpha_2) = c_0\alpha_2 + c_2$, c_i, $i = 0, 1, 2$ constants, and search the solutions in the forms prescribed below.

First kind of solutions. We seek solutions of the form:

$$\begin{aligned} F_1(\alpha_1, \alpha_2) &= \frac{1}{m(\alpha_1, \alpha_2)} + \frac{\rho^+}{2}(\alpha_2) \\ F_2(\alpha_1, \alpha_2) &= \frac{1}{m(\alpha_1, \alpha_2)} + \frac{\rho^-}{2}(\alpha_1). \end{aligned} \tag{4.14}$$

We find after computations:

$$m(\alpha_1, \alpha_2) = \lambda \exp\left\{\frac{\sigma_0}{2}\left[c_0\left(\alpha_1 - \alpha_2\right)^2 + 2(c_1 - c_2)(\alpha_1 - \alpha_2)\right]\right\}. \tag{4.15}$$

Second kind of solutions. Now we seek solutions of the form:

$$F_1(\alpha_1, \alpha_2) = \frac{1}{m(\alpha_1, \alpha_2)} + \frac{\rho^+}{2}(\alpha_2);\ F_2(\alpha_1, \alpha_2) = \frac{\rho^+(\alpha_2)}{\rho^-(\alpha_1) m(\alpha_1, \alpha_2)} + \frac{\rho^-}{2}(\alpha_1). \tag{4.16}$$

We find after computations:

$$m(\alpha_1,\alpha_2) = -\sigma_0\left[\alpha_1 + \frac{c_1}{c_0} + \frac{\left(\alpha_2+\frac{c_2}{c_0}\right)^2}{\alpha_1+\frac{c_1}{c_0}}\right] + c_1\left(c_0\alpha_2+c_2\right). \tag{4.17}$$

The solutions (4.15) are transcendental while the solutions (4.17) are algebraic so they are of different kinds and are not equal although they correspond to the same partial mean densities.

4.2.2. *The case of constant mean densities.* The solution (4.17) do not exist for $c_0 = 0$ thus the solution for constant $\rho^{\pm}$ is not simply a particular case of the linear case.

A general solution for constant partial mean densities. Given $\rho^+ = \rho_0^+$, $\rho^- = \rho_0^-$ with $\rho_0^{\pm} \in \mathbb{R}_+^*$ constant and $\rho_0^- \neq \rho_0^+$, we seek the solutions in the form:

$$\begin{aligned} F_1(\alpha_1,\alpha_2) &= \frac{\rho_0^+}{2} + \frac{k}{n(\alpha_1,\alpha_2)} \\ F_2(\alpha_1,\alpha_2) &= \frac{\rho_0^-}{2} + \frac{l}{n(\alpha_1,\alpha_2)}. \end{aligned} \tag{4.18}$$

We find:

$$n(\alpha_1,\alpha_2) = c_1\exp\left[\left(\rho_0^- l - \rho_0^+ k\right)\left(\frac{\alpha_1}{k} - \frac{\alpha_2}{l}\right)\right] + \frac{k^2-l^2}{\rho_0^- l - \rho_0^+ k} \tag{4.19}$$

$\forall k, l \in \mathbb{R}_+^*$, such that $\rho_0^- l - \rho_0^+ k \neq 0$.

Other particular solutions. The solutions (4.18), (4.19) exist $\forall \rho_0^-, \rho_0^+, k, l \in \mathbb{R}_+^*$ such that $\rho_0^- l - \rho_0^+ k \neq 0$. However other solutions exist for particular value of $\rho_0^{\pm}$. For instance $\forall k,\ l \in \mathbb{R}_+^*$, for $\rho^- = \dfrac{c_0}{k\sigma_0(k-l)^2(k+l)^2}$ and $\rho^+ = \dfrac{c_0}{l\sigma_0(k-l)^2(k+l)^2}$ we have the general solution

(4.20)

$$F_1(\alpha_1,\alpha_2) = \frac{c_0}{2l\sigma_0(k^2-l^2)^2} + \frac{c_0 k}{c_1\exp\left[\frac{c_0}{kl(k^2-l^2)}\left(\frac{\alpha_2}{l}-\frac{\alpha_1}{k}\right)\right]c_0 - \sigma_0 kl(k^2-l^2)^2}$$

$$F_2(\alpha_1,\alpha_2) = \frac{c_0}{2k\sigma_0(k^2-l^2)^2} + \frac{c_0 l}{c_1\exp\left[\frac{c_0}{kl(k^2-l^2)}\left(\frac{\alpha_2}{l}-\frac{\alpha_1}{k}\right)\right]c_0 - \sigma_0 kl(k^2-l^2)^2}$$

and the second solution

$$(4.21) \quad \begin{aligned} F_1(\alpha_1, \alpha_2) &= \frac{c_0}{2l\sigma_0(k^2 - l^2)^2} + \frac{l^2 k}{c_1 kl + \sigma_0(k^2 - l^2)(\alpha_1 k - \alpha_2 l)} \\ F_2(\alpha_1, \alpha_2) &= \frac{c_0}{2k\sigma_0(k^2 - l^2)^2} + \frac{k^2 l}{c_1 kl + \sigma_0(k^2 - l^2)(\alpha_1 k - \alpha_2 l)}. \end{aligned}$$

When the partial mean densities are equal, $\rho^- = \rho^+ = \rho_0$, we have also two solutions at least. The general solution:

$$\begin{aligned} F_1(\alpha_1, \alpha_2) &= \frac{\rho_0}{2} + \frac{\rho_0 k}{c_1 \exp\left[\sigma_0(k - l)\rho_0 \left(\frac{\alpha_2}{l} - \frac{\alpha_1}{k}\right)\right]} \\ F_2(\alpha_1, \alpha_2) &= \frac{\rho_0}{2} + \frac{\rho_0 l}{c_1 \exp\left[\sigma_0(k - l)\rho_0 \left(\frac{\alpha_2}{l} - \frac{\alpha_1}{k}\right)\right]}, \end{aligned}$$

and the solution:

$$(4.22) \quad \begin{aligned} F_1(\alpha_1, \alpha_2) &= \frac{\rho_0}{2} - \frac{c_2^2 c_3}{\sigma_0\left(c_2^2 - c_3^2\right)} + \frac{c_2 c_3^2 \tanh\left(c_1 + c_2\alpha_1 + c_3\alpha_2\right)}{\sigma_0\left(c_2^2 - c_3^2\right)} \\ F_2(\alpha_1, \alpha_2) &= \frac{\rho_0}{2} + \frac{c_2 c_3^2}{\sigma_0\left(c_2^2 - c_3^2\right)} - \frac{c_2^2 c_3 \tanh\left(c_1 + c_2\alpha_1 + c_3\alpha_2\right)}{\sigma_0\left(c_2^2 - c_3^2\right)}. \end{aligned}$$

5. Conclusion

We show that the boundary value problem for the two dimensional Broadwell four velocity discrete model has bounded solution for $\theta \in \left]0, \frac{\pi}{4}\right[\cup \left]\frac{\pi}{4}, \frac{\pi}{2}\right[$. This study extends and completes the work done in [8] and [9]. The orientation of the model with respect the coordinate axes is important in the statement of the mathematical problem and can determinate the type of the boundary value problem. The solution is not unique in general and two kinds of solutions are brought to the fore: maxwellian and non maxwellian solutions. As usual, the maxwellian solutions associated to given macroscopic variables are unique. In contrast, the non maxwellian ones are not unique in general and their form depend explicitly upon the partial mean densities and one can build for the same set of the latters at least two different exact analytic solutions. This is an illustration of the fact that for the Broadwell model different microscopic states can lead to the same macroscopic behavior for the total density.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOME
LOME, TOGO.
Email address: laknoa20@yahoo.fr

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOME
LOME, TOGO.
Email address: sossouk.k.leroys@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOME
LOME, TOGO.
Email address: dal_me@yahoo.fr